Number of states with given spin *I* and isospin *T* for *n* fermions in a *j* orbit*

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Abstract: In this study, we investigate formulas of the number of states with a given total spin I and isospin T for n nucleons in a single-j shell denoted by $D_{IT}(j,n)$. Talmi's recursion formulas for the number of states with a given z-axis projection of total spin are generalized by further considering the isospin couplings and are applied to derive explicit formulas of $D_{IT}(j,n)$.

Keywords: nuclear structure, shell model, nuclear spin and isospin

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I. INTRODUCTION

Determining the number of states with a given spin I for identical nucleons in a single-j shell [denoted by $D_I(j,n)$] is a common practice in nuclear structure theory and atomic physics. In the nuclear shell model, $D_I(j,n)$ is usually tabulated for relatively small j and n. In cases with larger j or n, $D_I(j,n)$ might be obtained by subtracting the number of states with total angular momentum projection $M=I+1$ from that with $M=I$ [1], where M equals $m_1+m_2+\cdots+m_n$, and m_i $[i=1,2,\cdots,n]$ is the projection of j on the z-axis for the i-th nucleon. Other recipes to extract $D_I(j,n)$ include Racha's formulas in terms of the seniority scheme [2] and generating function method studied extensively by Katriel *et al.* [3] and Sunko and collaborators [4–6].

The first explicit and compact formula of $D_I(j,n)$ was obtained specifically for $I=0$ and $n=4$ by Ginocchio and Haxton while studying the fractional quantum Hall effect [7]. In Ref. [8], $D_I(j,n)$ was constructed empirically for $n=3$ and 4, and for a few I values with $n=5$. The formulas for $n=3$ was soon proved by Talmi, who introduced his recursive formulas [9]. These formulas were later derived based on the reduction rule from $SU(4)$ to $SO(3)$ group [10]; the formula for $n=4$ was derived by the reduction rule from $SU(5)$ to the $SO(3)$ group [11], with a demonstration that $D_I(j,n)$ can be actually derived based on the reduction rule from $SU(n+1)$ to $SO(3)$ group. The Ginocchio-Haxton formula of $D_0(j,4)$ was also revisited by Zamick and Escuderos [12]. An explicit recursion formula from $n-1$ particles to n particles was obtained in Ref. [13] by introducing "pseudo" particles which allow fermions take integer spins. In Refs. [14, 15], the number of states, denoted by $D_I(j,n)$, was applied to derive sum rules of six-j and nine-j symbols, some of which were also revisited in Ref. [16]. In the last decade, Pain and collaborators extensively studied the odd-even staggering of $D_I(j,n)$ [17, 18], and compact formulas of $D_I(j,n)$ for $n=3,4$, and 5 [19]. The enumeration of the number of states with given spin was also extended to boson systems in Ref. [20]. The study of $D_I(j,n)$ motivated a number of studies related to a single-j shell; here, we mention Refs. [21–34] without providing further details for completeness.

Given that the nuclear shell model Hamiltonian respects the isospin symmetry, it is natural to generalize the enumeration of $D_I(j,n)$ to the number of states with given spin I and isospin T, denoted by $D_{IT}(j,n)$, for nucleons in a single-j shell. In Ref. [35], Zamick and Escuderos found a few simple relations between $D_{IT}(j,4)$ with $T=0$ and $D_{IT}(j,4)$ with $T=2$. In Ref. [36], compact and explicit formulas of $D_{IT}(j,n)$ for $n=3$ and 4 were derived in terms of sum rules of six-j and nine-j symbols involving the expression of $D_I(j,n)$ given in Ref. [11].

Similar to the enumeration of $D_I(j,n)$, $D_{IT}(j,n)$ can be obtained in terms of the number of states with given spin projection M and isospin projection M_T. This number is denoted by $N_{MM_T}(j,n)$. For n fermions in a single-j shell,

$$\begin{aligned} D_{IT}(j,n) = &[\, N_{M=I,M_T=T}(j,n) - N_{M=I+1,M_T=T}(j,n)\,] \\ &- [\, N_{M=I,M_T=T+1}(j,n) - N_{M=I+1,M_T=T+1}(j,n)\,] \\ = &N_{M=I,M_T=T}(j,n) + N_{M=I+1,M_T=T+1}(j,n) \\ &- N_{M=I+1,M_T=T}(j,n) - N_{M=I,M_T=T+1}(j,n). \end{aligned} \tag{1}$$

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In this study, we investigate $D_{IT}(j,n)$ in terms of $N_{MM_T}(j,n)$, which is obtained by generalizing Talmi's recursion formula of $N_M(j,n)$, in which isospin is not considered. Compact formulas of $D_{IT}(j,n)$ are derived for $n=3$ as an exemplification. The rest of this paper is organized as follows. In Sec. II, we generalize Talmi's formulas for $N_M(j,n)$ by further considering the isospin symmetry, and present recursion formulas of N_{MM_T}. In Sec. III, we apply our generalized recursion formulas of N_{MM_T} and extract $N_{MM_T}(n,j)$ for $n=3$. In Sec. IV, $D_{IT}(j,3)$ is derived by using Eq. (1). In Sec. V, we summarize our study. In Appendix A, we provide an alternative mathematical proof of our generalized Talmi's recursion formula by using the generating function method. In Appendix B, we tabulate $D_{IT}(j,n)$ with $n=3$ and 4, $j=5/2, 7/2$, and 9/2.

II. NUMBER OF STATES WITH GIVEN *z*-COMPONENT PROJECTIONS OF SPIN AND ISOSPIN

We use the convention that the *z*-axis projection of the isospin for a neutron and a proton equals 1/2 and $-1/2$, respectively. For n nucleons in a single-j shell, we denote the projections of total spin I and total isospin T, respectively, as M and M_T. With this convention, the neutron number equals $n/2+M_T$, and the proton number equals $n/2-M_T$. Let us denote the *z*-axis spin projections of neutrons and protons by using $(m_{\nu_1}, m_{\nu_2}, ..., m_{\nu_{2j+1}})$ and $(m_{\pi_1}, m_{\pi_2}, ..., m_{\pi_{2j+1}})$, respectively, with the convention that $j=m_{\nu_1}>m_{\nu_2}>...>m_{\nu_{2j+1}}=-j$ for neutrons and $j=m_{\pi_1}>m_{\pi_2}>...>m_{\pi_{2j+1}}=-j$ for protons. We define $n_{\nu_i}=1$ (or $n_{\pi_i}=1$) for $i=1,...,2j+1$ if the orbit of m_{ν_i} (or m_{π_i}) is filled; otherwise, it equals zero.

According to the above conventions, we have

$$\begin{aligned} M =& n_{\nu_1}m_{\nu_1}+n_{\nu_2}m_{\nu_2}+...+n_{\nu_{2j+1}}m_{\nu_{2j+1}} \\ &+n_{\pi_1}m_{\pi_1}+n_{\pi_2}m_{\pi_2}+...+n_{\pi_{2j+1}}m_{\pi_{2j+1}} \ , \end{aligned}$$

and the maximum of I equals

$$\begin{aligned} I_{\max} =& M_{\max} = \left(j+(j-1)+...+(j-n/2-M_T+1)\right) \\ &+\left(j+(j-1)+...+(j-n/2+M_T+1)\right) \\ =& n(2j+1)/2-n^2/4-M_T^2. \end{aligned} \quad (2)$$

We denote the number of states with given M and M_T for n nucleons in a single-j shell by using $N_{MM_T}(j,n)$. Clearly, we can divide the values of $(n_{\nu_1}, n_{\nu_{2j+1}}, n_{\pi_1}, n_{\pi_{2j+1}})$ into sixteen cases, as tabulated in Table 1. Here, we exemplify them by using ($n_{\nu_1}=1, n_{\nu_{2j+1}}=1, n_{\pi_1}=1, n_{\pi_{2j+1}}=1$). In this case, there are $(n-4)$ nucleons distributed in orbits for which the absolute values of m_i values equal or are below $(j-1)$. Consequently, the number of states with given M and M_T, i.e., $N_{MM_T}(j,n)$, equals the number of states of the *same* M and M_T, but with $(n-4)$ nucleons in a $(j-1)$ shell, namely, $N_{MM_T}(j-1,n-4)$. According to this classification, we are able to obtain $N_{MM_T}(j,n)$ by using the values of $N_{MM_T}(j',n')$, where either j' is smaller than j, or n' is smaller than n, or both (j',n') are smaller than (j,n). In other words, from Table 1 we have

Table 1. Number of states corresponding to the available values of $n_{\nu_1}, n_{\nu_{2j+1}}, n_{\pi_1}, n_{\pi_{2j+1}}$.

n_{ν_1}	$n_{\nu_{2j+1}}$	n_{π_1}	$n_{\pi_{2j+1}}$	Number of states
1	1	1	1	$N_{M,M_T}(j-1,n-4)$
1	1	1	0	$N_{M-j,M_T-1/2}(j-1,n-3)$
1	1	0	1	$N_{M+j,M_T-1/2}(j-1,n-3)$
1	1	0	0	$N_{M,M_T-1}(j-1,n-2)$
1	0	1	1	$N_{M-j,M_T+1/2}(j-1,n-3)$
1	0	1	0	$N_{M-2j,M_T}(j-1,n-2)$
1	0	0	1	$N_{M,M_T}(j-1,n-2)$
1	0	0	0	$N_{M-j,M_T-1/2}(j-1,n-1)$
0	1	1	1	$N_{M+j,M_T+1/2}(j-1,n-3)$
0	1	1	1	$N_{M,M_T}(j-1,n-2)$
0	1	0	1	$N_{M+2j,M_T}(j-1,n-2)$
0	1	0	0	$N_{M+j,M_T-1/2}(j-1,n-1)$
0	0	1	1	$N_{M,M_T+1}(j-1,n-2)$
0	0	1	0	$N_{M-j,M_T+1/2}(j-1,n-1)$
0	0	0	1	$N_{M+j,M_T+1/2}(j-1,n-1)$
0	0	0	0	$N_{M,M_T}(j-1,n)$

$$\begin{aligned} N_{MM_T}(j,n) =& \sum_{\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\}} \\ & N_{M-j\left(n_{\nu_1}-n_{\nu_{2j+1}}+n_{\pi_1}-n_{\pi_{2j+1}}\right), M_T-\frac{1}{2}\left(n_{\nu_1}+n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}}\right)} \\ & \left(j-1, n-n_{\nu_1}-n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}}\right). \end{aligned} \quad (3)$$

Here, the summation over $\{n_{\nu_1}, n_{\nu_{2j+1}}, n_{\pi_1}, n_{\pi_{2j+1}}\}$ refers to the sixteen cases listed in Table 1. Note that the above formula holds also for $n=1,2,3,4$, with the convention that $N_{MM_T}(j,n)=0$ if $n<0$ and $N_{MM_T}(j,0)=\delta_{M,0}\delta_{M_T,0}$. Note also that Eq. (3) can be proved alternatively based on the generating-function technique described in Ref. [3], as presented in Appendix A.

III. RECURSION FORMULA OF $N_{MM_T}(j,3)$

The value of $N_{MM_T}(j,n)$ with $n=1$ or 2 is trivially determined as follows. For $M_T=\pm 1/2$ and $|M|\le j$, $N_{M,M_T=\pm\frac{1}{2}}(j,1)=1$; otherwise, it equals zero. For $n=2$

with $M_T=\pm 1$, the highest value of M equals $M_{\max}=2j-1$ and for $M_T=0$, $M_{\max}=2j$. For $M_T=\pm 1$ and $M\le 2j-1$,

$$N_{M,M_T=\pm 1}(j,2)=\left\lfloor\frac{2j+1-|M|}{2}\right\rfloor, \tag{4}$$

where "⌊" and "⌋" means taking the largest integer without exceeding the value inside; for $M_T=0$ and $M\le 2j$,

$$N_{M,M_T=0}(j,2)=2j+1-|M|. \tag{5}$$

Otherwise, $N_{MM_T}(j,2)=0$. With these results, we can construct explicit formulas of $N_{MM_T}(j,3)$ by using Eq. (3).

A. $M_T=1/2$ and $j<M\le M_{\max}$

We first discuss $N_{MM_T}(j,3)$ with the requirement that $M_T=\frac{1}{2}$ and $j<M\le M_{\max}$. We recursively apply Eq. (3) to this term and obtain the value of $N_{MM_T}(j,n)$:

$$\begin{aligned}N_{MM_T}(j,n)=&N_{MM_T}(j-1,n)+\sum_{\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\}}{}^{\prime} N_{M-j(n_{\nu_1}-n_{\nu_{2j+1}}+n_{\pi_1}-n_{\pi_{2j+1}}),M_T-\frac{1}{2}(n_{\nu_1}+n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}})}\left(j-1,n-n_{\nu_1}-n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}}\right)\\ =&N_{MM_T}(j-l,n)+\sum_{i=0}^{l-1}\sum_{\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\}}{}^{\prime}\\ &N_{M-(j-i)(n_{\nu_1}-n_{\nu_{2j+1}}+n_{\pi_1}-n_{\pi_{2j+1}}),M_T-\frac{1}{2}(n_{\nu_1}+n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}})}\left(j-i-1,n-n_{\nu_1}-n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}}\right),\end{aligned} \tag{6}$$

where $\sum'$ denotes summations that exclude the case $\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\}=\{0,0,0,0\}$ and decouple the term $N_{M,M_T}(j-1,n)$ [the last term in the summation of the sixteen terms of Table 1]; l is fixed with the requirement that $N_{MM_T}(j-l,n)=0$ while $N_{MM_T}(j-l+1,n)\neq 0$, which means that

$$\begin{aligned}M&>\frac{n(2j+1)}{2}-\frac{n^2}{4}-M_T^2-ln,\\ M&\le\frac{n(2j+1)}{2}-\frac{n^2}{4}-M_T^2-(l-1)n,\end{aligned}$$

or equivalently,

$$l=\left\lfloor\frac{2j+1}{2}-\frac{n}{4}-\frac{M_T^2+M}{n}\right\rfloor+1. \tag{7}$$

For $n=3$ and $M_T=1/2$, the value of l becomes

$$l=\left\lfloor j-\frac{M+1}{3}\right\rfloor+1. \tag{8}$$

According to Eq. (6), with the conditions $N_{MM_T}(j-l,n)=0$ and $N_{MM_T}(j-l+1,n)\neq 0$, we have

$$N_{M,M_T=\frac{1}{2}}(j,3)=\sum_{i=0}^{l-1}\sum_{\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\}}{}^{\prime} N_{M-(j-i)(n_{\nu_1}-n_{\nu_{2j+1}}+n_{\pi_1}-n_{\pi_{2j+1}}),\frac{1}{2}-\frac{1}{2}(n_{\nu_1}+n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}})}\left(j-i-1,3-n_{\nu_1}-n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}}\right). \tag{9}$$

On the right hand side of the above formula, there are many terms that break the requirement of Eq. (2) for $n\le 2$. Let us exemplify this by using two cases: $\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\}=\{0,0,0,1\}$ and $\{1,1,1,0\}$. For the first case, $\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\}=\{0,0,0,1\}$, the corresponding contribution to $N_{M,M_T=\frac{1}{2}}(j,3)$ on the right hand of Eq. (10) is $N_{M+(j-i),M_T=1}(j-i-1,2)$. Given that $M>j$ here, the value of $M+(j-i)$ is larger than the maximum $M_{\max}$ of two nucleons in a single $j-i-1$ shell, i.e., $M_{\max}=2j-2i-2$. Similarly, for $\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\}=\{1,1,1,0\}$, the corresponding contribution on the right hand side is $N_{M-(j-i),M_T=0}(j-i-1,0)$. Given that $M-(j-i)>0$, $N_{M-(j-i),M_T=0}(j-i-1,0)=\delta_{M-(j-i),0}=0$. As a result, we obtain

$$\begin{aligned}N_{M,M_T=\frac{1}{2}}(j,3)&=\sum_{i=0}^{l-1}\left(N_{M-(j-i),M_T=0}(j-i-1,2)+N_{M-(j-i),M_T=1}(j-i-1,2)+N_{M-2(j-i),M_T=\frac{1}{2}}(j-i-1,1)\right)\\ &=\sum_{i=0}^{l-1}\left(3j-3i-1-M+\left\lfloor\frac{3j-3i-1-M}{2}\right\rfloor+1\right)=(3j-1-M)l-\frac{3l(l-1)}{2}+\sum_{i=0}^{l-1}\left\lfloor\frac{3j-3i-1-M}{2}\right\rfloor+l.\end{aligned} \tag{10}$$

The summation of $\left\lfloor\frac{3j-3i-1-M}{2}\right\rfloor$ is tedious but straightforward. One has

$$l-1=\begin{cases}2k & \text{if when } 3j-M=6k+1,6k+2,6k+3,\\ 2k+1 & \text{if when } 3j-M=6k+4,6k+5,6k+6.\end{cases}$$

When $3j-M=6k+1$, the summation becomes

$$\sum_{i=0}^{l-1}\left\lfloor\frac{3j-3i-1-M}{2}\right\rfloor=\sum_{i=0}^{2k}\left(3k-2i+\left\lfloor\frac{i}{2}\right\rfloor\right)$$
$$=\sum_{i=0}^{2k}(3k-2i)+2\sum_{i=0}^{k-1}i+k=3k^2+k.$$

This summation for the case $3j-M=6k+2,3,4,5,6$ can be obtained similarly, as follows:

$$\sum_{i=0}^{l-1}\left\lfloor\frac{3j-3i-1-M}{2}\right\rfloor=\begin{cases}3k^2+k & \text{if } 3j-M=6k+1,\\ 3k^2+2k & \text{if } 3j-M=6k+2,\\ 3k^2+3k+1 & \text{if } 3j-M=6k+3,\\ 3k^2+4k+1 & \text{if } 3j-M=6k+4,\\ 3k^2+5k+2 & \text{if } 3j-M=6k+5,\\ 3k^2+6k+3 & \text{if } 3j-M=6k+6.\end{cases} \tag{11}$$

Substituting these results into Eq. (10), we obtain

$$N_{M,M_T=\frac{1}{2}}(j,3)=\begin{cases}9k^2+6k+1 & \text{if } 3j-M=6k+1,\\ 9k^2+9k+2 & \text{if } 3j-M=6k+2,\\ 9k^2+12k+4 & \text{if } 3j-M=6k+3,\\ 9k^2+15k+6 & \text{if } 3j-M=6k+4,\\ 9k^2+18k+9 & \text{if } 3j-M=6k+5,\\ 9k^2+21k+12 & \text{if } 3j-M=6k+6\end{cases}=\frac{1}{4}(3j-M)^2+\frac{1}{2}(3j-M)+\eta(3j-M), \tag{12}$$

where $\eta(3j-M)=\left(0,\frac{1}{4}\right)$ if $\mathrm{mod}(3j-M,2)=(0,1)$, respectively.

B. $T_z=1/2$ and $0<M\leq j$

Let us now address $N_{M,M_T=\frac{1}{2}}(j,3)$ with $T_z=1/2$ and $0<M\leq j$. In this case, Eq. (3) is applied recursively for $j-M+1$ times. As a result, we have

$$\begin{aligned}N_{MM_T}(j,n)=&N_{MM_T}(j-1,n)+\sum_{\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\}}{}'\\ &N_{M-j\left(n_{\nu_1}-n_{\nu_{2j+1}}+n_{\pi_1}-n_{\pi_{2j+1}}\right),M_T-\frac{1}{2}\left(n_{\nu_1}+n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}}\right)}\left(j-1,n-n_{\nu_1}-n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}}\right)\\ =&N_{MM_T}(M-1,n)+\sum_{i=0}^{j-M}\sum_{\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\}}{}'\\ &N_{M-(j-i)\left(n_{\nu_1}-n_{\nu_{2j+1}}+n_{\pi_1}-n_{\pi_{2j+1}}\right),M_T-\frac{1}{2}\left(n_{\nu_1}+n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}}\right)}\left(j-i-1,n-n_{\nu_1}-n_{\nu_{2j+1}}-n_{\pi_1}-n_{\pi_{2j+1}}\right),\end{aligned} \tag{13}$$

where, again, $\sum'$ denotes summation among the sixteen sets of $\left\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\right\}$ excluding $\left\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}}\right\}=\{0,0,0,0\}$. For $M_T=\frac{1}{2}$ and $n=3$, the above formula is reduced to

$$\begin{aligned}N_{M,M_T=\frac{1}{2}}(j,3)=&1+N_{M,M_T=\frac{1}{2}}(M-1,3)+\sum_{i=0}^{j-M}\Big(N_{M-(j-i),M_T=0}(j-i-1,2)+N_{M+(j-i),M_T=0}(j-i-1,2)\\ &+N_{M-(j-i),M_T=1}(j-i-1,2)+N_{M+(j-i),M_T=1}(j-i-1,2)\\ &+N_{M,M_T=-\frac{1}{2}}(j-i-1,1)+N_{M-2(j-i),M_T=\frac{1}{2}}(j-i-1,1)+N_{M,M_T=\frac{1}{2}}(j-i-1,1)\\ &+N_{M,M_T=\frac{1}{2}}(j-i-1,1)+N_{M+2(j-i),M_T=\frac{1}{2}}(j-i-1,1)+N_{M,M_T=\frac{3}{2}}(j-i-1,1)\Big)\\ =&1+N_{M,M_T=\frac{1}{2}}(M-1,3)+\sum_{i=0}^{j-M}(j-i+M-1)+\sum_{i=0}^{j-M-2}(j-i-M-1)\end{aligned}$$

$$+\sum_{i=0}^{j-M}\left\lfloor\frac{j-i+M-1}{2}\right\rfloor+\sum_{i=0}^{j-M-3}\left\lfloor\frac{j-i-M-1}{2}\right\rfloor+3\sum_{i=0}^{j-M-1}1, \tag{14}$$

where the first term " 1" is given by $N_{M=0,M_T=0}(j-i-1,0)=1$ with $\{n_{\nu_1},n_{\nu_{2j+1}},n_{\pi_1},n_{\pi_{2j+1}},i\}=\{1,1,1,0,j-M\}$; the second term $N_{M,M_T=\frac{1}{2}}(M-1,3)$ is given in Eq. (12) with $j=M-1$, i.e.,

$$N_{M,M_T=\frac{1}{2}}(M-1,3)=M^2-2M+\frac{3}{4}. \tag{15}$$

The summations in Eq. (14) equal zero when the upper limit of the given summation index is larger than its lower limit. On the right hand side, the summations of $\left\lfloor\frac{j-i+M-1}{2}\right\rfloor$ and $\left\lfloor\frac{j-i-M-1}{2}\right\rfloor$ are classified into two cases, $j-M=2k$ and $2k+1$. For $j-M=2k$, we obtain

$$\begin{aligned}\sum_{i=0}^{j-M}\left\lfloor\frac{j-i+M-1}{2}\right\rfloor&=\sum_{i=0}^{2k}\left\lfloor\frac{2k-i+2M-1}{2}\right\rfloor\\&=\sum_{i=0}^{2k}\left(k-i+\frac{2M-1}{2}+\left\lfloor\frac{i}{2}\right\rfloor\right)\\&=\sum_{i=0}^{2k}\left(k-i+\frac{2M-1}{2}\right)+2\sum_{i=0}^{k-1}i+k\\&=(j-M+1)\left(M-\frac{1}{2}\right)+\left(\frac{j-M}{2}\right)^2.\end{aligned}$$

The summation with $j-M=2k+1$ is obtained similarly. The results of these two cases are unified as follows:

$$\sum_{i=0}^{j-M}\left\lfloor\frac{j-i+M-1}{2}\right\rfloor=\left(j-M+\frac{1}{4}\right)\left(M-\frac{1}{2}\right)+\left(\frac{j-M}{2}\right)^2+\begin{cases}0 & \text{if}\bmod(j-M,2)=0,\\-\frac{1}{4} & \text{if}\bmod(j-M,2)=1.\end{cases} \tag{16}$$

Following a similar procedure, we obtain

$$\sum_{i=0}^{j-M-3}\left\lfloor\frac{j-i-M-1}{2}\right\rfloor=\left(\frac{j-M}{2}\right)^2-\frac{j-M}{2}+\begin{cases}0 & \text{if}\bmod(j-M,2)=0,\\+\frac{1}{4} & \text{if}\bmod(j-M,2)=1.\end{cases} \tag{17}$$

Substituting Eqs. (15)–(17) into Eq. (14), we obtain $N_{M,M_T=\frac{1}{2}}(j,3)$ with $0<M\le j$ in a compact form:

$$N_{M,M_T=\frac{1}{2}}(j,3)=\frac{3}{2}j^2+j+\frac{1}{4}-\frac{M^2}{2}. \tag{18}$$

C. $M_T=3/2$ and $0<M$

The results for $N_{MM_T}(j,n)$ are reduced to $N_M(j,n)$ for identical particles, and Eq. (3) is reduced to Talmi's recursion formula. For this case, $N_{M,M_T=\frac{3}{2}}(j,3)$ was obtained in Ref. [19], which we cite for completeness. For $j<M\le M_{\max}$, according to Eqs. (2.11a)–(2.11b) in Ref. [19],

$$N_{M,M_T=\frac{3}{2}}(j,3)=\frac{1}{12}(3j-M)^2+\alpha(3j-M), \tag{19}$$

where

$$\alpha(3j-M)=\quad 0,-\frac{1}{12},-\frac{1}{3},\frac{1}{4},-\frac{1}{3},-\frac{1}{12},$$

for $\bmod(3j-M,6)=0,1,2,3,4,5$, respectively. For $0<M\le j$, according to Eqs. (2.19a)-(2.19b) in Ref. [19],

$$N_{M,M_T=\frac{3}{2}}(j,3)=\frac{1}{2}j^2-\frac{1}{6}M^2+\beta\left(M-\frac{1}{2}\right), \tag{20}$$

where

$$\beta(3j-M)=\quad -\frac{1}{12},\frac{1}{4},-\frac{1}{12},$$

for $\bmod(M-\frac{1}{2},3)=0,1,2$, respectively.

IV. NUMBER OF STATES WITH GIVEN I AND T

Provided the results of $N_{MM_T}(j,3)$, i.e., Eqs. (12), (18), (19), and (20), we can readily substitute them into Eq. (1) to obtain the number of states with given I and T, i.e., $D_{IT}(j,3)$.

We first consider the case $T=1/2$. When $I>j$ and $T=1/2$, we obtain

$$\begin{aligned}D_{I,T=\frac{1}{2}}(j,3)=&N_{I,\frac{1}{2}}(j,3)+N_{I+1,\frac{3}{2}}(j,3)\\&-N_{I+1,\frac{1}{2}}(j,3)-N_{I,\frac{3}{2}}(j,3).\end{aligned} \tag{21}$$

According to Eqs. (12) and (19), the above formula is reduced to

$$D_{I,T=\frac{1}{2}}(j,3) = -\frac{1}{3}I + j + \frac{1}{3} + \eta(3j-I) - \eta(3j-I-1) + \alpha(3j-I-1) - \alpha(3j-I)$$
$$= -\frac{1}{3}I + j + \begin{cases} 0 & \text{if} \bmod(3j-I,3)=0, \\ \frac{2}{3} & \text{if} \bmod(3j-I,3)=1, \\ \frac{1}{3} & \text{if} \bmod(3j-I,3)=2 \end{cases}$$
$$= \left\lfloor \frac{3j-I+2}{3} \right\rfloor. \tag{22}$$

For $I = j$ and $T = 1/2$,

$$D_{I,T=\frac{1}{2}}(j,3) = N_{I,\frac{1}{2}}(j,3) + N_{I+1,\frac{3}{2}}(j,3) - N_{I+1,\frac{1}{2}}(j,3) - N_{I,\frac{3}{2}}(j,3). \tag{23}$$

By using Eqs. (12), (18), (19), and (20), we have

$$D_{I,T=\frac{1}{2}}(j,3) = \frac{2}{3}j + \frac{7}{12} + \alpha(2j-1) - \eta(2j-1) - \beta\left(j-\frac{1}{2}\right)$$
$$= \frac{2}{3}j + \begin{cases} \frac{2}{3} & \text{if} \bmod(2j,3)=1, \\ 0 & \text{if} \bmod(2j,3)=3, \\ \frac{1}{3} & \text{if} \bmod(2j,3)=5 \end{cases}$$
$$= \left\lfloor \frac{2j+2}{3} \right\rfloor. \tag{24}$$

Similarly, for $0 < I < j$ and $T = 1/2$,

$$D_{I,T=\frac{1}{2}}(j,3) = N_{I,\frac{1}{2}}(j,3) + N_{I+1,\frac{3}{2}}(j,3) - N_{I+1,\frac{1}{2}}(j,3) - N_{I,\frac{3}{2}}(j,3). \tag{25}$$

By using Eqs. (18) and (20), the above formula is reduced to

$$D_{I,T=\frac{1}{2}}(j,3) = \frac{2}{3}J + \frac{1}{3} + \beta\left(I+\frac{1}{2}\right) - \beta\left(I-\frac{1}{2}\right)$$
$$= \begin{cases} \frac{2}{3}I + \frac{2}{3} & \text{if} \bmod(2I,3)=1, \\ \frac{2}{3}I & \text{if} \bmod(2I,3)=3, \\ \frac{2}{3}I + \frac{1}{3} & \text{if} \bmod(2I,3)=5 \end{cases}$$
$$= \left\lfloor \frac{2I+2}{3} \right\rfloor. \tag{26}$$

Eqs. (24)–(26) can be rewritten as follows. For $I \geq j$ and $T = 1/2$,

$$D_{I,T=\frac{1}{2}}(j,3) = \left\lfloor \frac{3j-I+2}{3} \right\rfloor, \tag{27}$$

and for $0 < I \leq j$ and $T = 1/2$,

$$D_{I,T=\frac{1}{2}}(j,3) = \left\lfloor \frac{2I+2}{3} \right\rfloor. \tag{28}$$

These results [Eqs. (27)–(28) in this paper] are consistent with Eqs. (25)–(26) in Ref. [36], which were obtained in terms of sum rules of six-j symbols, except that the formulas derived in this paper are expressed in a more transparent and understandable form.

The $T = 3/2$ case is much simpler than the $T = 1/2$ case. One uses Talmi's recursion formulas for $N_{M=I,\frac{3}{2}}(j,n)$ and obtains $D_{I,T=\frac{3}{2}}(j,3)$ straightforwardly. The resulting formulas are consistent with those given in Ref. [8] and Ref. [19]. We show them next for completeness. According to Eq. (1) in Ref. [8], for $0 < I \leq j$ and $T = 3/2$,

$$D_{I,T=\frac{3}{2}}(j,3) = \left\lfloor \frac{2I+3}{6} \right\rfloor; \tag{29}$$

and according to Eq. (2) in Ref. [8], for $I \geq j$ and $T = 3/2$,

$$D_{I,T=\frac{3}{2}}(j,3) = \left\lfloor \frac{3j-3-I}{6} \right\rfloor + \delta_I. \tag{30}$$

where

$$\delta_I = \begin{cases} 0 & \text{if } \bmod(3j-3-I,6)=1, \\ 1 & \text{otherwise.} \end{cases}$$

In principle, one can follow the above procedure as explained in Eqs. (1)–(3) for larger n values. However, this procedure becomes formidably tedious with lengthy formulas and tables for larger n values. Instead of those formulas, we calculate $D_{IT}(j,n)$ for $n=3$ and 4 and $j = 5/2$, 7/2, and 9/2 and tabulate them in Table B1 and Table B2 in Appendix B.

V. SUMMARY

To summarize, in this paper we generalize Talmi's recursion formula of the number of states with given spin I by further consideration of the isospin symmetry. This generalization is also proved alternatively based on the generating function method. We also exemplify the generalized Talmi's recursion formulas obtained in this paper by applying our generalized Talmi's recursion formulas in constructing the number of states with given spin and isospin of three nucleons in a single-j shell.

APPENDIX A: GENERATING-FUNCTION TECHNIQUE

In this Appendix, we present an alternative proof of the recursion formula presented in Eq. (3) based on the generating-function method.

A generating function is a polynomial function for which the expansion coefficients are related to the number of states with a given requirement. We define the generating function corresponding to $N_{MM_T}(j,n)$, similar to that in Ref. [17],

$$f_j(x,y,z)=\sum_{n=0}^{\infty}\sum_{M=-\infty}^{\infty}\sum_{M_T=-\infty}^{\infty}z^n x^M y^{M_T} N_{MM_T}(j,n), \tag{A1}$$

which yields

$$\begin{aligned} f_j(x,y,z)=&\sum_{n=0}^{\infty}\sum_{M=-\infty}^{\infty}\sum_{M_T=-\infty}^{\infty}z^n x^M y^{M_T}\\ &\sum_{\{n_{\nu_1},\ldots,n_{\nu_{2j+1}},n_{\pi_1},\ldots,n_{\pi_{2j+1}}\}}\\ &\left(\delta_{n,n_{\nu_1}+\ldots+n_{\pi_{2j+1}}}\delta_{M_T,\frac{1}{2}\left(n_{\nu_1}+\ldots-n_{\pi_{2j+1}}\right)}\right.\\ &\left.\delta_{M,n_{\nu_1}m_{\nu_1}+\ldots+n_{\pi_{2j+1}}m_{\pi_{2j+1}}}\right)\\ =&\prod_{i=1}^{2j+1}\left(1+zx^{m_{\nu_i}}y^{\frac{1}{2}}\right)\left(1+zx^{m_{\pi_i}}y^{-\frac{1}{2}}\right)\\ =&f_{j-1}(x,y,z)g_i(x,y,z) \end{aligned} \tag{A2}$$

with

$$\begin{aligned} f_j(x,y,z)=&\prod_{i=2}^{2j}\left(1+zx^{m_{\nu_i}}y^{\frac{1}{2}}\right)\left(1+zx^{m_{\pi_i}}y^{-\frac{1}{2}}\right)\\ g_j(x,y,z)=&\left(1+zx^{-j}y^{\frac{1}{2}}\right)\left(1+zx^{j}y^{\frac{1}{2}}\right)\\ &\times\left(1+zx^{-j}y^{-\frac{1}{2}}\right)\left(1+zx^{j}y^{-\frac{1}{2}}\right). \end{aligned} \tag{A3}$$

The generating function is expanded in powers of z as follows:

$$f_j(x,y,z)=\sum_{n=0}^{\infty}z^n f_{j,n}(x,y) \tag{A3}$$

with

$$\begin{aligned} f_{j,n}(x,y)=&\frac{1}{n!}\frac{\partial^n}{\partial z^n}f_j(x,y,z)\Big|_{z=0}\\ =&\sum_{M=-\infty}^{\infty}\sum_{M_T=-\infty}^{\infty}N_{MM_T}(j,n)x^M y^{M_T}. \end{aligned} \tag{A5}$$

Using the Leibniz formula, one obtains

$$\begin{aligned} f_{j,n}(x,y)=&\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^{n-k}}{\partial z^{n-k}}f_{j-1}(x,y,z)\\ &\times\frac{\partial^k}{\partial z^k}g_j(x,y,z)\Big|_{z=0}. \end{aligned} \tag{A6}$$

By using Eqs. (A1), (A5), the above relation yields

$$\begin{aligned} &\sum_{M=-\infty}^{\infty}\sum_{M_T=-\infty}^{\infty}N_{MM_T}(j,n)x^M y^{M_T}\\ =&\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\sum_{M=-\infty}^{\infty}\sum_{M_T=-\infty}^{\infty}N_{MM_T}(j-1,n-k)\\ &x^M y^{M_T}\times\frac{\partial^k}{\partial z^k}g_i(x,y,z)\Big|_{z=0}. \end{aligned} \tag{A7}$$

Given that $g_j(x,y,z)$ is a fourth-order polynomial, the only derivatives that are nonzero correspond to

$$\begin{aligned} k=0:&\ \frac{\partial^0}{\partial z^0}g_j(x,y,z)\Big|_{z=0}=1,\\ k=1:&\ \frac{\partial^1}{\partial z^1}g_j(x,y,z)\Big|_{z=0}=x^{-j}y^{\frac{1}{2}}+x^{j}y^{\frac{1}{2}}\\ &\qquad+x^{-j}y^{-\frac{1}{2}}+x^{j}y^{-\frac{1}{2}},\\ k=2:&\ \frac{\partial^2}{\partial z^2}g_j(x,y,z)\Big|_{z=0}=y+y^{-1}+x^{-2j}+x^{2j}\\ &\qquad+1+1,\\ k=3:&\ \frac{\partial^3}{\partial z^3}g_j(x,y,z)\Big|_{z=0}=x^{-j}y^{\frac{1}{2}}+x^{j}y^{\frac{1}{2}}\\ &\qquad+x^{-j}y^{-\frac{1}{2}}+x^{j}y^{-\frac{1}{2}}, \end{aligned}$$

$$k=4:\ \frac{\partial^4}{\partial z^4}g_j(x,y,z)\Big|_{z=0}=1. \tag{A8}$$

Eq. (A7) holds for any independent variables x and y; therefore, the coefficients of homogeneous terms on both sides are equal, which immediately leads to Eq. (3).

APPENDIX B: $D_{IT}(J,N)$ WITH $J=3/2-9/2$ AND $N=3-4$

In this Appendix, we tabulate $D_{IT}(j,n)$ with $j=3/2$–9/2 and n=3–4, for convenience.

Table B1. $D_{IT}(j,n)$ for j=5/2-9/2, with n=3. I^D in this Table represents that there are D states with spin I for the j^n configuration.

j	T	I^D
$\frac{5}{2}$	$\frac{1}{2}$	$(\frac{1}{2})^1,(\frac{3}{2})^1,(\frac{5}{2})^2,(\frac{7}{2})^2,(\frac{9}{2})^1,(\frac{11}{2})^1,(\frac{13}{2})^1$
$\frac{5}{2}$	$\frac{3}{2}$	$(\frac{3}{2})^1,(\frac{5}{2})^2,(\frac{9}{2})^1$
$\frac{7}{2}$	$\frac{1}{2}$	$(\frac{1}{2})^1,(\frac{3}{2})^1,(\frac{5}{2})^2,(\frac{7}{2})^3,(\frac{9}{2})^2,$
		$(\frac{11}{2})^2,(\frac{13}{2})^2,(\frac{15}{2})^1,(\frac{17}{2})^1,(\frac{19}{2})^1$
$\frac{7}{2}$	$\frac{3}{2}$	$(\frac{3}{2})^1,(\frac{5}{2})^2,(\frac{7}{2})^3,(\frac{9}{2})^1,(\frac{11}{2})^1,(\frac{15}{2})^1$
$\frac{9}{2}$	$\frac{1}{2}$	$(\frac{1}{2})^1,(\frac{3}{2})^1,(\frac{5}{2})^2,(\frac{7}{2})^3,(\frac{9}{2})^3,(\frac{11}{2})^3,$
		$(\frac{13}{2})^3,(\frac{15}{2})^2,(\frac{17}{2})^2,(\frac{19}{2})^2,(\frac{21}{2})^1,(\frac{23}{2})^1,(\frac{25}{2})^1$
$\frac{9}{2}$	$\frac{3}{2}$	$(\frac{3}{2})^1,(\frac{5}{2})^1,(\frac{7}{2})^1,(\frac{9}{2})^2,(\frac{11}{2})^1,(\frac{13}{2})^1,(\frac{15}{2})^1,(\frac{17}{2})^1,(\frac{21}{2})^1$

Table B2. The same as Table B1 except for $n=4$.

j	T	I^D
$\frac{5}{2}$	0	$0^2,2^3,3^1,4^3,5^1,6^2,8^1$
$\frac{5}{2}$	1	$1^2,2^2,3^3,4^2,5^2,6^1,7^1$
$\frac{5}{2}$	2	$0^1,2^1,4^1$
$\frac{7}{2}$	0	$0^3,2^4,3^2,4^5,5^2,6^5,7^2,8^3,9^1,10^2,12^1$
$\frac{7}{2}$	1	$1^3,2^3,3^5,4^4,5^5,6^4,7^4,8^2,9^2,10^1,11^1$
$\frac{7}{2}$	2	$0^1,2^2,4^2,5^1,6^1,8^1$
$\frac{9}{2}$	0	$0^3,2^6,3^2,4^7,5^4,6^7,7^4,$
		$8^7,9^3,10^5,11^2,12^3,13^1,14^2,16^1$
$\frac{9}{2}$	1	$1^4,2^4,3^7,4^6,5^8,6^7,7^8,$
		$8^6,9^6,10^4,11^4,12^2,13^2,14^1,15^1$
$\frac{9}{2}$	2	$0^2,2^2,3^1,4^3,5^1,6^3,7^1,8^2,9^1,10^1,12^1$

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