General one-loop reduction in generalized Feynman parametrization form*

Hongbin Wang(王宏斌)†

Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, China

Abstract: The search for an effective reduction method is one of the main topics in higher loop computation. Recently, an alternative reduction method was proposed by Chen in [1, 2]. In this paper, we test the power of Chen's new method using one-loop scalar integrals with propagators of higher power. More explicitly, with the improved version of the method, we can cancel the dimension shift and terms with unwanted power shifting. Thus, the obtained integrating-by-parts relations are significantly simpler and can be solved easily. Using this method, we present explicit examples of a bubble, triangle, box, and pentagon with one doubled propagator. With these results, we complete our previous computations in [3] with the missing tadpole coefficients and show the potential of Chen's method for efficient reduction in higher loop integrals.

Keywords: Feynman integral reduction, parametrization form, one loop integral

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I. INTRODUCTION

The calculation of multi-loop integrals is essential when theoretically predicting the scatting amplitude of a given process. For these calculations, the PV-reduction method [4] is a widely used approach, and one way to implement the reduction method is to use the integrating-by-parts (IBP) relation [5–7]. As one of the most powerful techniques for loop integral reduction, IBP gives a large number of recurrence relations, and the reduction can be represented by a combination of simpler integrals via Gauss elimination. However, as the propagator number and power increase, the IBP method becomes inefficient; hence, more efficient reduction methods must be found.

The unitarity cut method is an alternative reduction method and has been proven to be useful for one-loop integrals [8–17]. In a physical one-loop process, the power of the propagator is just one; however, if the method is complete, it should be able to reduce integrals with higher power propagators. Such a situation is not simply a theoretical curiosity but appears in higher loop diagrams as a sub-diagram. Furthermore, although the scalar basis is natural for one-loop integrals, in general, the choice of basis can be different, depending on the physical input. For example, in the topology of a one-loop bubble, the basis, in which one propagator has a power of two, could be used as part of the UT-basis [18, 19].

In a previous study [3], we successfully obtained an analytical reduction result for one-loop integrals with high power propagators by combining the tricks of differential operators and unitarity cut. We gave coefficients to all bases except the tadpoles'; however, the unitarity method could not be used because the tadpole has only one propagator. To complete this investigation, the missing tadpole coefficients must be found using other efficient methods.

Other than the unitarity cut method, there are proposals to overcome the difficulties of IBP using tricks and other representations of integrals, such as the Baikov representation [20, 21] and Feynman parametrization representation [22, 23] for loop integrals. In recent years, Chen proposed a new representation for loop integrals [1, 2]. His method is based on *the generalized Feynman parametrization representation*, that is, an extra parameter x_{n+1} is introduced to combine $\mathcal{U},\mathcal{F}$ in the standard Feynman parametrization representation. Such a generalization will offer several benefits when deriving the IBP recurrence relation, as shown in this paper.

As a common feature, the IBP recurrence relation derived using the generalized Feynman parametrization rep-

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† E-mail: 21836003@zju.edu.cn

resentation will naturally have terms in different spacetime dimensions. Because we are always concerned with reducing a particular dimension D, which is typically set to $4-2\epsilon$ for renormalization, we wish to cancel these terms in different dimensions. In general, this is not easy. In [24], Gluza, Kajda, and Kosower showed how to avoid the change in the power of propagators in standard momentum space. Larsen and Zhang considered the Baikov representation and demonstrated how to eliminate both dimensional shifting and the change in the power of propagators [25–30]. These methods require a solution to the syzygy equations, which is generally not easy. In Chen's second paper [2], he proposed a new technique to simplify the recurrence relation based on non-commutative algebra.

Motivated by the above discussion and preparing Chen's method for high-loop computations, in this paper, we use Chen's method to find the missing tadpole coefficients from our previous study. Furthermore, we use the idea of removing terms with dimensional shifting in the derived IBP relation to construct a simpler reduction method, with the analytic results expressed by the elements of the coefficient matrix $\hat{A}$.

This paper is organized as follows: In section II, we review and illustrate Chen's new method with a simple example in section II.A. In the example, integrals naturally emerge in different dimensions. We discuss the physical meaning of the boundary terms, which contribute to the sub-topologies. To cancel dimensional shifting in the parametrization form and simplify the IBP relation, a new trick is proposed in section II.B in which free auxiliary parameters are added based on the fact that F in the integrand is a homogeneous function of x_i with degree $L+1$. Using this trick, we successfully cancel dimensional shifting and drop the terms that we are not concerned with. Moreover, we present a simplified IBP relation in which all the integrals are in the particular dimension D and integrals other than the target have a lower total propagator power. The analytic result is presented as a determinant of the cofactor of the matrix $\hat{A}$, which is entirely determined by a graph. In section III, we calculate a triangle $I_3(1,1,2)$, box $I_4(1,1,1,2)$, and pentagon $I_5(1,1,1,1,2)$ in parametric form using this trick and present the analytic results of all coefficients to the master basis, especially the tadpole parts, to complement our previous study.

II. CHEN'S REDUCTION METHOD IN PARAMETRIC FORM

In this section, we introduce a new reduction method proposed by Chen in [1]. The general form of a loop integral is given by

$$I[N(l)](k)=\int \mathrm{d}^D l_1 \mathrm{d}^D l_2\cdots \mathrm{d}^D l_L \frac{N(l)}{D_1^{k_1}D_2^{k_2}D_3^{k_3}\cdots D_n^{k_n}}, \tag{1}$$

where, for simplicity, we denote $l=(l_1,l_2,l_3,\cdots,l_L)$ and $k=(k_1,k_2,k_3,\cdots,k_n)$. Because we consider only scalar integrals with $N(l)=1$ in this paper, let us label

$$I(L;\lambda_1+1,\cdots,\lambda_n+1)=\int \mathrm{d}^D l_1\cdots \mathrm{d}^D l_L \frac{1}{D_1^{\lambda_1+1}\cdots D_n^{\lambda_n+1}}. \tag{2}$$

Using the Feynman parametrization procedure,

$$\sum_i^L \alpha_i D_i=\sum_{i,j}^L A_{ij} l_i\cdot l_j+2\sum_{i=1}^L B_i\cdot l_i+C, \tag{3}$$

and thus loop integrals can be found as

$$\int \mathrm{d}^D l_1\cdots \mathrm{d}^D l_L \mathrm{e}^{\mathrm{i}(\sum\alpha_i D_i)} = \mathrm{e}^{\mathrm{i}\pi L(1-\frac{D}{2})/2}\pi^{LD/2}(\mathrm{Det}\,A)^{-D/2} \times \mathrm{e}^{\mathrm{i}(C-\sum A_{ij}^{-1}B_i\cdot B_j)}. \tag{4}$$

Defining $U(\alpha)=\mathrm{Det}\,A$ and $C-\sum A_{ij}^{-1}B_i\cdot B_j \equiv V(\alpha)/U(\alpha)-\sum m_i^2\alpha_i$ [1)], we can see that $U(\alpha)$ is a homogeneous function of α_i with degree L, whereas $V(\alpha)$ is a homogeneous function of α_i with degree $L+1$. The loop integral becomes

$$\begin{aligned}&I(L;\lambda_1+1,\cdots,\lambda_n+1)\\ =&\frac{\mathrm{e}^{-\sum((\lambda_i+1)/2)\mathrm{i}\pi}}{\Pi_{i=1}^n\Gamma(\lambda_i+1)}\mathrm{e}^{\mathrm{i}\pi L(1-D/2)/2}\pi^{LD/2}\\ &\times\int \mathrm{d}\alpha_1\cdots \mathrm{d}\alpha_n U(\alpha)^{-D/2}\mathrm{e}^{\mathrm{i}[V(\alpha)/U(\alpha)-\sum m_i^2\alpha_i]}\alpha_1^{\lambda_1}\cdots\alpha_n^{\lambda_n}.\end{aligned} \tag{5}$$

To derive the parametric form suggested by Chen, we perform the following: Using the α-representation of general propagators,

$$\frac{1}{(l^2-m^2)^{\lambda+1}}=\frac{\mathrm{e}^{-((\lambda+1)/2)\mathrm{i}\pi}}{\Gamma(\lambda+1)}\int_0^\infty \mathrm{d}\alpha \mathrm{e}^{\mathrm{i}\alpha(l^2-m^2)}\alpha^\lambda,\ \mathrm{Im}\{l^2-m^2\}>0, \tag{6}$$

where "$i\epsilon$" is neglected, we obtain

1) The relation has been verified in many places based on the method in graph theory.

$$I(L;\lambda_1+1,\cdots,\lambda_n+1)=\frac{e^{-\sum_i^n((\lambda_i+1)/2)i\pi}}{\Pi_{i=1}^n\Gamma(\lambda_i+1)}\int d^Dl_1\cdots d^Dl_L \times\int_0^\infty d\alpha_1\cdots d\alpha_n e^{i\sum_{i=1}^n\alpha_i D_i}\alpha_1^{\lambda_1}\cdots\alpha_n^{\lambda_n}. \tag{7}$$

To go further, we change the integral variables to $\alpha_i=\eta x_i$. Because there is a total of n independent variables, we must insert another constraint condition. In general, we could let

$$\sum_{i\in S\,(1,2,3,\cdots n)} x_i=1, \tag{8}$$

where S is an arbitrary non-trivial subset of $\{1,2,3,\cdots n\}$. After carrying out the integration over η, the second line of Eq. (5) becomes

$$\begin{aligned}&(-i)^{(n+\lambda-DL/2)}\Gamma\left(n+\lambda-\frac{DL}{2}\right)\times\int dx_1\cdots dx_n\delta\left(\sum_{j\in S}x_j-1\right)\frac{U(x)^{n+\lambda-(D/2)(L+1)}}{[-V(x)+U(x)\sum m_i^2x_i]^{n+\lambda-DL/2}}x_1^{\lambda_1}\cdots x_n^{\lambda_n}\\ =&(-i)^{n+\lambda-DL/2}\Gamma\left(n+\lambda-\frac{DL}{2}\right)\int dx_1\cdots dx_n\delta(\sum_{j\in S}x_j-1)U^{\lambda_u}f^{\lambda_f}x_1^{\lambda_1}\cdots x_n^{\lambda_n},\end{aligned} \tag{9}$$

where

$$\begin{aligned}&U(x)=\eta^{-L}U(\alpha)=\eta^{-L}U(\eta x_i),\quad V(x)=\eta^{-L-1}V(\alpha)=\eta^{-L-1}V(\eta x),\quad f(x)=-V(x)+U(x)\sum m_i^2x_i\\ &\lambda=\sum_{i=1}^n\lambda_i,\quad \lambda_u=n+\lambda-\frac{D}{2}(L+1),\quad \lambda_f=-n-\lambda+\frac{DL}{2}.\end{aligned} \tag{10}$$

Finally, via Mellin transformation[1)]

$$A^{\lambda_1}B^{\lambda_2}=\frac{\Gamma(-\lambda_1-\lambda_2)}{\Gamma(-\lambda_1)\Gamma(-\lambda_2)}\int_0^\infty dx(A+Bx)^{\lambda_1+\lambda_2}x^{-\lambda_2-1}, \tag{11}$$

we can express (9) as

$$\begin{aligned}&(-i)^{n+\lambda-DL/2}\Gamma\left(n+\lambda-\frac{DL}{2}\right)\frac{\Gamma(-\lambda_u-\lambda_f)}{\Gamma(-\lambda_u)\Gamma(-\lambda_f)}\int dx_1\cdots dx_n\delta\left(\sum_{j\in S}x_j-1\right)\int_0^\infty dx_{n+1}\times(Ux_{n+1}+f)^{\lambda_u+\lambda_f}x_{n+1}^{-\lambda_u-1}x_1^{\lambda_1}\cdots x_n^{\lambda_n}\\ \equiv&(-i)^{n+\lambda-DL/2}\frac{\Gamma(n+\lambda-DL/2)\Gamma(-\lambda_u-\lambda_f)}{\Gamma(-\lambda_u)\Gamma(-\lambda_f)}\int d\Pi^{(n+1)}F^{\lambda_0}x_1^{\lambda_1}\cdots x_n^{\lambda_n}x_{n+1}^{\lambda_{n+1}}\\ \equiv&(-i)^{n+\lambda-DL/2}\frac{\Gamma(n+\lambda-\frac{DL}{2})\Gamma(-\lambda_u-\lambda_f)}{\Gamma(-\lambda_u)\Gamma(-\lambda_f)}i_{\lambda_0;\lambda_1,\cdots\lambda_n},\end{aligned} \tag{12}$$

where

$$\begin{aligned}d\Pi^{(n+1)}=&dx_1\cdots dx_{n+1}\delta(\sum_{j\in S}x_j-1),\\ F=&Ux_{n+1}+f,\quad \lambda=\sum_{i=1}^n\lambda_i,\\ \lambda_0=&\lambda_u+\lambda_f=-\frac{D}{2},\\ \lambda_{n+1}=&-\lambda_u-1=\frac{D}{2}(L+1)-\lambda-1-n.\end{aligned} \tag{13}$$

Combined, we finally obtain the parametric form of the scalar loop integrals in (5),

$$I(L;\lambda_1+1,\cdots,\lambda_n+1)=(-1)^{n+\lambda}i^L\pi^{LD/2}\frac{\Gamma(-\lambda_0)}{\Pi_{i=1}^{n+1}\Gamma(\lambda_i+1)}i_{\lambda_0;\lambda_1,\cdots\lambda_n}. \tag{14}$$

A. IBP identity in parametric form

The parametric form of (14) is the starting point of

1) Different from the traditional Feynman parametrization, here we should add a new auxiliary parameter x_{n+1} to transform the integral into symmetric form.

Chen's proposal. The IBP relations in this form are given by[1)2)]

$$\int \mathrm{d}\Pi^{(n+1)} \frac{\partial}{\partial x_i}\left\{F^{\lambda_0} x_1^{\lambda_1} \cdots x_n^{\lambda_n} x_{n+1}^{\lambda_{n+1}+1}\right\} + \delta_{\lambda_i,0} \int \mathrm{d}\Pi^{(n)}\left\{F^{\lambda_0} x_1^{\lambda_1} \cdots x_n^{\lambda_n} x_{n+1}^{\lambda_{n+1}+1}\right\}\Big|_{x_i=0} = 0 \tag{15}$$

where $i = 1,...,n+1$, and $\mathrm{d}\Pi^{(n)}$ in the second term is

$$\mathrm{d}\Pi^{(n)} = \mathrm{d}x_1 \cdots \mathrm{d}\hat{x}_i \cdots \mathrm{d}x_n \mathrm{d}x_{n+1} \delta\left(\sum_{j\in S} x_j - 1\right). \tag{16}$$

The second term in (15) contributes to a boundary term, which leads to the sub-topologies of the former term.

To illustrate the IBP relation (15), we present the reduction in $I_2(1,2)$ as an example. The general form of one-loop bubble integrals is given by

$$I_2(m+1,n+1) = \int \frac{\mathrm{d}^D l}{(l^2-m_1^2)^{m+1}((l-p_1)^2-m_2^2)^{n+1}}, \tag{17}$$

and the corresponding parametric form is (in this paper, we ignore the former factor $\pi^{LD/2}$)

$$I_2(m+1,n+1) = \mathrm{i}(-1)^{m+n+2} \times \frac{\Gamma\left(\frac{D}{2}\right)}{\Gamma(m+1)\Gamma(n+1)\Gamma(D-2-m-n)} \times \int \mathrm{d}\Pi^{(3)} F^{\lambda_0} x_1^m x_2^n x_3^{\lambda_3}, \tag{18}$$

where

$$F = (x_1+x_2)(m_1^2 x_1 + m_2^2 x_2 + x_3) - p_1^2 x_1 x_2, \tag{19}$$

and

$$i_{\lambda_0;m,n} = \int \mathrm{d}\Pi^{(3)} F^{\lambda_0} x_1^m x_2^n x_3^{\lambda_3}, \tag{20}$$

with $\lambda_0 = -\frac{D}{2}$, and $\lambda_3 = -3-m-n-2\lambda_0$. Using Eq. (15), we can obtain three IBP recurrence relations. First, taking $\frac{\partial}{\partial x_1}$, the first term in (15) gives

$$\lambda_0 i_{\lambda_0-1;m,n} + 2m_1^2 \lambda_0 i_{\lambda_0-1;m+1,n} + \Delta\lambda_0 i_{\lambda_0-1;m,n+1}, \tag{21}$$

where $\Delta = m_1^2 + m_2^2 - p_1^2$. The second term gives

$$\delta_{m,0} \int \mathrm{d}\Pi^{(2)} (x_3 + m_2^2 x_2)^{\lambda_0} x_2^{n+\lambda_0} x_3^{-2-n-2\lambda_0} = \delta_{m,0} i_{\lambda_0;-1,n}. \tag{22}$$

Here, the notation $i_{\lambda_0;-1,n}$ must be explained. From the middle expression of (22), we see that it is the parametric form of the tadpole $\int \mathrm{d}^D l/(l^2-m_2^2)^{n+1}$. To emphasize its origin, that is,from a bubble by removingthe first propagator, we extend the definition of $i_{\lambda_0;\lambda_1,...,\lambda_n}$ given in (12) by setting $\lambda_1 = -1$ [3)]. Using the extended notation, we obtain the first IBP relation

$$\lambda_0 i_{\lambda_0-1;m,n} + 2m_1^2 \lambda_0 i_{\lambda_0-1;m+1,n} + \Delta\lambda_0 i_{\lambda_0-1;m,n+1} + \delta_{m,0} i_{\lambda_0;-1,n} = 0. \tag{23}$$

When we set $m = n = 0$ in (23), this reads

$$\lambda_0 i_{\lambda_0-1;0,0} + 2m_1^2 \lambda_0 i_{\lambda_0-1;1,0} + \Delta\lambda_0 i_{\lambda_0-1;0,1} + i_{\lambda_0;-1,0} = 0. \tag{24}$$

Similarly, we can take the differential $\frac{\partial}{\partial x_2}$ and obtain the second IBP relation

$$\lambda_0 i_{\lambda_0-1;0,0} + \Delta\lambda_0 i_{\lambda_0-1;1,0} + 2m_2^2 \lambda_0 i_{\lambda_0-1;0,1} + i_{\lambda_0;0,-1} = 0. \tag{25}$$

We should solve $i_{\lambda_0;0,1}$ by $i_{\lambda_0;0,0}$ from (24) and (25). However, for the bubble part, we have $\lambda_0 - 1$ instead of λ_0. This could be fixed by rewriting $\lambda_0 \to \lambda_0 + 1$ because λ_0 is a free parameter. However, the boundary tadpole part $i_{\lambda_0;0,-1}$ will become $i_{\lambda_0+1;0,-1}$, that is, it will have the dimensional shifting, which is a common feature in the parametric IBP relation.

To deal with this, using the parametric form of tadpoles

$$i_{\lambda_0;m,-1} = \int \mathrm{d}\Pi^{(2)} (x_1 x_3 + m_1^2 x_1^2)^{\lambda_0} x_1^m x_3^{-2-m-2\lambda_0} \tag{26}$$

and taking the $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_3}$, we can obtain two IBP relations,

1) In some sense, the parametric form can be considered as the **generalized Feynman parametrization form**. Thus the IBP relation (15) could be called the IBP relation in the generalized Feynman parametrization form.

2) The IBP relation requires the term in the bracket of the first term to be degree $(-n)$, which can be obtained by multiplying any monomial of degree one. Here in (15) we have multiplied x_{n+1} by our experiences from later examples, but one can make other choices.

3) Same notation has also been used in [2] (see Eq. (5a)).

$$\lambda_0 i_{\lambda_0-1;m,-1}+2m_1^2\lambda_0 i_{\lambda_0-1;m+1,-1}+m i_{\lambda_0,m-1,-1}=0,$$
$$\lambda_0 i_{\lambda_0-1;m+1,-1}+(-1-m-2\lambda_0)i_{\lambda_0;m,-1}=0, \tag{27}$$

from which we solve

$$i_{\lambda_0;0,-1}=\frac{-\lambda_0}{2m_1^2(2\lambda_0+1)}i_{\lambda_0-1;0,-1},$$
$$i_{\lambda_0;-1,0}=\frac{-\lambda_0}{2m_2^2(2\lambda_0+1)}i_{\lambda_0-1;-1,0}. \tag{28}$$

Inserting (28) into (24) and (25), we can solve $i_{\lambda_0-1;0,1}$. After shifting $\lambda_0\to\lambda_0+1$, we finally get

$$i_{\lambda_0;0,1}=\frac{2m_1^2-\Delta}{\Delta^2-4m_1^2m_2^2}i_{\lambda_0;0,0}+\frac{-1}{(2\lambda_0+3)(\Delta^2-4m_1^2m_2^2)}i_{\lambda_0;0,-1}+\frac{\Delta}{2m_2^2(2\lambda_0+3)(\Delta^2-4m_1^2m_2^2)}i_{\lambda_0;-1,0}. \tag{29}$$

Translating back to the scalar basis, we obtain the reduction in $I_2(1,2)$ as

$$I_2(1,2)=c_{2\to2}I_2(1,1)+c_{2\to1;\bar{2}}I_2(1,0)+c_{2\to1;\bar{1}}I_2(0,1), \tag{30}$$

with the coefficients

$$c_{2\to2}=\frac{(D-3)(\Delta-2m_1^2)}{\Delta^2-4m_1^2m_2^2},$$
$$c_{2\to1;\bar{2}}=\frac{D-2}{\Delta^2-4m_1^2m_2^2},$$
$$c_{2\to1;\bar{1}}=\frac{(D-2)\Delta}{2m_2^2(4m_1^2m_2^2-\Delta^2)}. \tag{31}$$

This result is confirmed with FIRE6 [31, 32].

B. Improvement of parametric IBP

As shown in the previous subsection, the IBP relation given in (15) will contain integrals with dimension shift, which makes the reduction program slightly troublesome. As reviewed in the introduction, there are several references dealing with this or related problems. Based on these studies, an improved version of the IBP relation has been given in [2] (see Eq. (12) and (13)). All these methods require a solution to the syzygy equations, which is not generally an easy task. However, for our one-loop integrals, the function $F(x)$ is a homogeneous function of x_i with degree of two[1)]. This good property simplifies the related syzygy equations, which can then be directly solved[2)]. In this paper, we develop a direct algorithm to express the IBP relations without dimension shift and terms with unwanted higher power propagators.

In the generalized parametric representation, our improved IBP relation involves multiplying Eq. (15) by a degree zero coefficient z_i, for example, $z_i=x_1^{\alpha}x_2^{\beta}x_3^{-\alpha-\beta}$. Because the degree of the new integrand does not change, the IBP identity still holds. Summing them together we get[3)]

$$\sum_{i=1}^{n+1}\int d\Pi^{(n+1)}\frac{\partial}{\partial x_i}\left\{z_iF^{\lambda_0}x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_{n+1}^{\lambda_{n+1}+1}\right\}+\sum_{i=1}^{n+1}\delta_{\lambda_i,0}\int d\Pi^{(n)}z_iF^{\lambda_0}x_1^{\lambda_1}\cdots x_n^{\lambda_n}x_{n+1}^{\lambda_{n+1}}|_{x_i=0}=0. \tag{32}$$

Because the second boundary term involves integrals with sub-topologies, we focus on the first term. Expanding it, we get

$$\int d\Pi^{(n+1)}\left[\sum_{i=1}^{n+1}\left(\frac{\partial z_i}{\partial x_i}+\lambda_0\frac{z_i\frac{\partial F}{\partial x_i}}{F}+\lambda_i\frac{z_i}{x_i}\right)+\frac{z_{n+1}}{x_{n+1}}\right]\times F^{\lambda_0}x_1^{\lambda_i}x_2^{\lambda_2}\cdots x_n^{\lambda_n}x_{n+1}^{\lambda_{n+1}+1}. \tag{33}$$

From (13), we can see that the power λ_0 of F is related to dimension. To cancel the dimension shift, we must choose the proper coefficients z_i so that $\sum_{i=1}^{n+1}z_i\frac{\partial F}{\partial x_i}$ is a multiple of the function F, that is,

$$\sum_{i=1}^{n+1}z_i\frac{\partial F}{\partial x_i}+BF=0\,. \tag{34}$$

Because the coefficients z_i are not polynomials, (34) is not the "normal syzygy equation," and we cannot directly use the technique developed for the polynomial ring. In [2], Chen developed a method based on the lift and down operators. Here, for the one-loop integrals, we can solve it directly with free auxiliary parameters, as shown later in this paper. When reinserting the solutions to the IBP recurrence relation, we can choose these free parameters to cancel both the dimension shift and un-

1) Note the $F(x)$ is a homogeneous function of degree $L+1$ where L is the number of loops.

2) In general, this trick could be extended to high loops to avoid the troublesome calculation of syzygy equations.

3) Notice the summation of i is form 1 to $n+1$, where we have included the auxiliary parameter x_{n+1}, which is an apparent different from the tradition Feynman parametrization.

wanted terms with higher power propagators, which leads to a simpler recurrence relation.

Now, the idea is explained in detail. Note that in the one loop case, the homogeneous function F is a degree two function of x_i; therefore, we can write F as

$$F = \frac{1}{2}A_{ij}x_i x_j, \tag{35}$$

where A is a symmetric matrix[1)]. Thus, we have

$$f_i \equiv \frac{\partial F}{\partial x_i}, \quad \hat{f} = \hat{A}\hat{x}, \quad \hat{f} \equiv \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \\ f_{n+1} \end{bmatrix}, \quad \hat{x} \equiv \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix},$$

Solving $\hat{x} = \hat{A}^{-1}\hat{f}$, we have

$$\begin{aligned} F =& \frac{1}{2}\hat{x}^T A\hat{x} = \frac{1}{2}\hat{f}^T(\hat{A}^{-1})^T\hat{A}\hat{A}^{-1}\hat{f} \\ =& \frac{1}{2}\hat{f}^T(\hat{A}^{-1})^T\hat{f} \equiv \hat{f}^T\hat{K}\hat{f}, \\ K =& \frac{1}{2}A^{-1}, \end{aligned} \tag{36}$$

where the coefficient matrix $\hat{K}$ is a real symmetry matrix. In fact, we can go further. Using

$$0 = \quad \hat{f}^T\hat{K}_A\hat{f} \tag{37}$$

with any antisymmetric matrix K_A, we can add (37) to (36) to obtain a more general form[2)]

$$\begin{aligned} F =& \hat{f}^T\hat{K}\hat{f} + \hat{f}^T\hat{K}_A\hat{f} = \hat{f}^T(\hat{K}+\hat{K}_A)\hat{f} \equiv \hat{f}^T\hat{R}\hat{f} \\ =& \hat{f}^T\hat{R}\hat{A}\hat{x} \equiv \hat{f}^T\hat{Q}\hat{x}, \\ \hat{Q} \equiv& \frac{1}{2}\hat{I} + \hat{K}_A\hat{A}. \end{aligned} \tag{38}$$

Note that because the arbitrary matrix $\hat{K}_A$ is of rank $n+1$, there are $\frac{n(n+1)}{2}$ free independent parameters, $a_1,\cdots,a_{(n(n+1))/2}$, in the matrix $\hat{Q}$ in (38).

Now, reinserting (38) into (34), we can solve $\hat{z}$ as

$$\hat{f}^T\hat{z} + B\hat{f}^T\hat{Q}\hat{x} = 0, \quad \Longrightarrow \hat{z} = -B\hat{Q}\hat{x}. \tag{39}$$

Note that because z is degree zero, we should ensure B is a homogenous function of degree -1. In this study, we choose $B = 1/x_{n+1}$. The choice of z given by (39) will guarantee the removal of dimension shift in the IBP relation. Furthermore, by choosing particular values of the free parameters of $\hat{Q}$, we may cancel several unwanted terms. Some examples are shown in later computations to illustrate this trick.

III. REDUCTION IN ONE-LOOP INTEGRALS

As mentioned in the introduction, one motivation of this study is to complete reduction in the scalar basis with general powers. Using the unitarity cut method in [3], we are able to find reduction coefficients of all bases, except the tadpole. In this section, we will use the improved IBP relation (32) to find the tadpole coefficients as well as other coefficients.

A. Bubble case

We begin with bubble topology. Although this was already done in (30), we redo it using the improved IBP relation (32). The parametric form of bubble is given by (18), (19), and (20). Using our label, we have

$$\hat{f} = \hat{A}\hat{x}, \quad \hat{A} = \begin{bmatrix} 2m_1^2 & \Delta & 1 \\ \Delta & 2m_2^2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \tag{40}$$

and

$$\begin{aligned} F =& \hat{f}^T\hat{K}\hat{f}, \\ \hat{K} =& \begin{bmatrix} \frac{1}{4p_1^2} & -\frac{1}{4p_1^2} & \frac{-m_1^2+m_2^2+p_1^2}{4p_1^2} \\ -\frac{1}{4p_1^2} & \frac{1}{4p_1^2} & \frac{m_1^2-m_2^2+p_1^2}{4p_1^2} \\ \frac{-m_1^2+m_2^2+p_1^2}{4p_1^2} & \frac{m_1^2-m_2^2+p_1^2}{4p_1^2} & \frac{\Delta^2-4m_1^2m_2^2}{4p_1^2} \end{bmatrix}. \end{aligned} \tag{41}$$

Adding the antisymmetric matrix K_A, we have

1) In general it is not necessary to make $\hat{A}$ be symmetry matrix, and this is just one choice. But for the simplification of the following calculation, since we will later set an antisymmetric matrix $\hat{K}_A$, it is convenient to make the convention to set $\hat{A}$ be symmetry matrix.

2) The antisymmetric matrix $\hat{K}_A$ will not contribute to the F, but it will change $\hat{Q}$ in (38), thus gives more free parameters in the solution of $\hat{z}$ in (39).

$$\hat{K}_A = \begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}, \ \hat{Q} = \begin{bmatrix} \frac{1+2a_2+2a_1m_1^2+2a_1m_2^2-2a_1p_1^2}{2} & a_2+2a_1m_2^2 & a_1 \\ a_3-2a_1m_1^2 & \frac{1+2a_3-2a_1m_1^2-2a_1m_2^2+2a_1p_1^2}{2} & -a_1 \\ -2a_2m_1^2-a_3(m_1^2+m_2^2-p_1^2) & -2a_3m_2^2-a_2(m_1^2+m_2^2-p_1^2) & \frac{1-2a_2-2a_3}{2} \end{bmatrix}. \tag{42}$$

1. *Deriving the recurrence relation*

Taking $B=-1/x_3$ in (34), solution (39) gives $z_i=(Q_{ij}x_j)/x_3$. Expanding (32), we obtain the IBP recurrence relation

$$\begin{aligned} &c_{m,n}i_{\lambda_0;m,n}+c_{m+1,n}i_{\lambda_0;m+1,n}+c_{m+1,n-1}i_{\lambda_0;m+1,n-1} \\ &+c_{m,n+1}i_{\lambda_0;m,n+1}+c_{m-1,n+1}i_{\lambda_0;m-1,n+1} \\ &+c_{m,n-1}i_{\lambda_0;m,n-1}+c_{m-1,n}i_{\lambda_0;m-1,n}+\delta_2=0, \end{aligned} \tag{43}$$

where δ_2 is the boundary term, which we will compute later. The other coefficients are

$$\begin{aligned} c_{m,n}&=Q_{11}(1+m)+Q_{22}(1+n)+Q_{33}(1+\lambda_3)+\lambda_0, \\ c_{m+1,n}&=Q_{31}\lambda_3=-\lambda_3(a_2A_{11}+a_3A_{21}), \\ c_{m+1,n-1}&=Q_{21}n=-n(a_1A_{11}-a_3A_{31}), \\ c_{m,n+1}&=Q_{32}\lambda_3=-\lambda_3(a_2A_{12}-a_3A_{22}), \\ c_{m-1,n+1}&=Q_{12}m=m(a_1A_{22}+a_2A_{32}), \\ c_{m,n-1}&=Q_{23}n=-n(a_1A_{13}-a_3A_{33}), \\ c_{m-1,n}&=Q_{13}m=m(a_1A_{32}+a_2A_{33}). \end{aligned} \tag{44}$$

Because we aim to obtain the reduction in $I_2(1,2)$, starting from $m=n=0$, we want to eliminate terms with the indices $(m+1,n)$ and $(m+1,n-1)$ while keeping the term with the index $(m,n+1)$. Thus, we impose $c_{m+1,n}=0$ and $c_{m+1,n-1}=0$, which can be satisfied by choosing the free parameters[1)]

$$\begin{aligned} a_2&=-\frac{a_1A_{21}}{A_{31}}=-a_1(m_1^2+m_2^2-p_1^2), \\ a_3&=\frac{a_1A_{11}}{A_{31}}=2a_1m_1^2. \end{aligned} \tag{45}$$

After this choice, the matrix $\hat{Q}$ becomes

$$\hat{Q}_{;r}=\begin{bmatrix} \frac{1}{2} & \frac{a_1}{A_{31}}(A_{22}A_{31}-A_{21}A_{32}) & \frac{a_1}{A_{31}}(A_{23}A_{31}-A_{21}A_{33}) \\ 0 & \frac{1}{2}-\frac{a_1}{A_{31}}(A_{12}A_{31}-A_{11}A_{32}) & \frac{a_1}{A_{31}}(A_{11}A_{33}-A_{13}A_{31}) \\ 0 & \frac{a_1}{A_{31}}(A_{12}A_{21}-A_{11}A_{22}) & \frac{1}{2}+\frac{a_1}{A_{31}}(A_{13}A_{21}-A_{11}A_{23}) \end{bmatrix},$$

leaving five terms with non-zero coefficients[2)].

$$\begin{aligned} c_{m,n+1}&=\frac{-a_1\lambda_3}{A_{31}}(A_{11}A_{22}-A_{12}A_{21})=\frac{-a_1\lambda_3}{A_{31}}|\tilde{A}_{33}|=a_1\lambda_3, \\ c_{m-1,n+1}&=-\frac{ma_1}{A_{31}}(A_{21}A_{32}-A_{22}A_{31})=-\frac{ma_1}{A_{31}}|\tilde{A}_{13}|=-a_1m(m_1^2-m_2^2-p_1^2), \\ c_{m,n-1}&=\frac{na_1}{A_{31}}(A_{11}A_{33}-A_{13}A_{31})=\frac{na_1}{A_{31}}|\tilde{A}_{22}|=-a_1n, \\ c_{m-1,n}&=-\frac{ma_1}{A_{31}}(A_{21}A_{33}-A_{23}A_{31})=\frac{-ma_1}{A_{31}}|\tilde{A}_{12}|=a_1m, \\ c_{m,n}&=\frac{a_1}{A_{31}}\Big((1+n)(A_{11}A_{32}-A_{12}A_{31})-(\lambda_3+1)(A_{11}A_{23}-A_{13}A_{21})\Big)=\frac{a_1}{A_{31}}(n-\lambda_3)|\tilde{A}_{23}| \\ &=\frac{a_1}{A_{31}}\Big((n-\lambda_3)(m_1^2-m_2^2+p_1^2)\Big). \end{aligned} \tag{46}$$

1) For this example, one can check that we can not add another constraint to fix a_1.

2) where we use the convention $|\tilde{A}_{ij}|$ means the cofactor of matrix element A_{ij}.

The boundary δ_2 term: The δ_2 term is given by

$$\delta_2 = \sum_{i=1}^{3} \delta_{\lambda_i,0} \int \mathrm{d}\Pi^{(2)} \left\{ z_i F^{\lambda_0} x_1^m x_2^n x_3^{\lambda_3+1} \right\}\Big|_{x_i=0}, \tag{47}$$

where λ_i represents the power of x_i. It is worth emphasizing that because z_i contains x_i, the total power λ_i of x_i is not equal to m, n, λ_3 in general. Expanding it, we get[1)]

$$\begin{aligned} \delta_2 =& \delta_{\lambda_1,0} \int \mathrm{d}\Pi^{(2)} \left(Q_{11} F^{\lambda_0} x_1^{m+1} x_2^n x_3^{\lambda_3} + Q_{12} F^{\lambda_0} x_1^m x_2^{n+1} x_3^{\lambda_3} + Q_{13} F^{\lambda_0} x_1^m x_2^n x_3^{\lambda_3+1} \right)\Big|_{x_1=0} \\ &+ \delta_{\lambda_2,0} \int \mathrm{d}\Pi^{(2)} \left(Q_{21} F^{\lambda_0} x_1^{m+1} x_2^n x_3^{\lambda_3} + Q_{22} F^{\lambda_0} x_1^m x_2^{n+1} x_3^{\lambda_3} + Q_{23} F^{\lambda_0} x_1^m x_2^n x_3^{\lambda_3+1} \right)\Big|_{x_2=0}. \end{aligned} \tag{48}$$

Remembering our extended notation explained under (22), we have

$$\int \mathrm{d}\Pi^{(2)} F|_{x_1=0}^{\lambda_0} x_2^n \equiv i_{\lambda_0;-1,n}, \quad \int \mathrm{d}\Pi^{(2)} F|_{x_2=0}^{\lambda_0} x_1^m \equiv i_{\lambda_0;m,-1}, \tag{49}$$

and the δ_2 term can be written as

$$\begin{aligned} \delta_{2;r} =& \delta_{\lambda_1,0} \left(Q_{11;r} i_{\lambda_0;m+1,n} + Q_{12;r} i_{\lambda_0;m,n+1} + Q_{13;r} i_{\lambda_0;m,n} \right) + \delta_{\lambda_2,0} \left(Q_{21;r} i_{\lambda_0;m+1,n} + Q_{22;r} i_{\lambda_0;m,n+1} + Q_{23;r} i_{\lambda_0;m,n} \right) \\ =& \delta_{m,-1} Q_{11;r} i_{\lambda_0;-1,n} + \delta_{m,0} Q_{12;r} i_{\lambda_0;,-1,n+1} + \delta_{m,0} Q_{13;r} i_{\lambda_0;-1,n} + \delta_{n,0} Q_{21;r} i_{\lambda_0;m+1,-1} + \delta_{n,-1} Q_{22;r} i_{\lambda_0;m,-1} + \delta_{n,0} Q_{23;r} i_{\lambda_0;m,-1}, \end{aligned} \tag{50}$$

where the subscript r in $\delta_{2;r}$ and $Q_{ij;r}$ indicates that a_2 and a_3 should be replaced by (45).

Because m and n cannot be -1, the first and fifth terms are actually zero.

Now, we can use (43) and (50) to get our result directly. Setting $m = 0$, $n = 0$, and all other terms in (43) equal to zero, and we are left with[2)]

$$c_{0,0} i_{\lambda_0;0,0} + c_{0,1} i_{\lambda_0;0,1} + \delta_{2;00} = 0, \tag{51}$$

with the coefficients

$$\begin{aligned} c_{0,0} =& -a_1(D-3)(m_1^2 - m_2^2 + p_1^2), \\ c_{0,1} =& a_1(D-3)\left(m_1^4 + m_2^4 p_1^4 - 2m_1^2 p_1^2 - 2m_2^2 p_1^2 - 2m_1^2 m_2^2 \right), \\ \delta_{2;00} =& Q_{12;r} i_{\lambda_0;-1,1} + Q_{13;r} i_{\lambda_0;-1,0} + Q_{21;r} i_{\lambda_0;1,-1} + Q_{23;r} i_{\lambda_0;0,-1}, \end{aligned} \tag{52}$$

where

$$\begin{aligned} Q_{21;r} =& \frac{-a_1}{A_{31}}(A_{21}A_{32} - A_{22}A_{31}) = \frac{-a_1}{A_{31}}|\tilde{A}_{13}|, \quad Q_{23;r} = \frac{-a_1}{A_{31}}(A_{11}A_{33} - A_{13}A_{31}) = \frac{-a_1}{A_{31}}|\tilde{A}_{22}|, \\ Q_{12;r} =& \frac{-a_1}{A_{31}}(A_{21}A_{32} - A_{22}A_{31}) = \frac{-a_1}{A_{31}}|\tilde{A}_{13}|, \quad Q_{13;r} = \frac{-a_1}{A_{31}}(A_{21}A_{33} - A_{23}A_{31}) = \frac{-a_1}{A_{31}}|\tilde{A}_{12}|. \end{aligned} \tag{53}$$

From this, we can directly write the solution as

$$i_{\lambda_0;0,1} = -\frac{c_{0,0}}{c_{0,1}} i_{\lambda_0;0,0} - \frac{Q_{21;r}}{c_{0,1}} i_{\lambda_0;1,-1} - \frac{Q_{23;r}}{c_{0,1}} i_{\lambda_0;0,-1} - \frac{Q_{12;r}}{c_{0,1}} i_{\lambda_0;-1,1} - \frac{Q_{13;r}}{c_{0,1}} i_{\lambda_0;-1,0}. \tag{54}$$

---

1) Since we have kept dimensional regularization ϵ, the λ_3 can not be zero, thus the corresponding boundary term does not exist.

2) When setting $m = n = 0$, except the boundary term δ_2, among other seven terms in (43), the coefficients of the second and the third terms have been chosen to be zero. For the other five terms, one can show that $c_{m-1,n+1}$, $c_{m,n-1}$, $c_{m-1,n}$ are zero by using the last line of (44). There is another technical point. When $m = n = 0$, the seventh term will contain $i_{\lambda_0;-1,0}$, which looks like the one defined in (49). But they are, in fact, different. The one appeared in (43) with the measure $\mathrm{d}\Pi^{(3)}$ while the one appeared in (49) with measure $\mathrm{d}\Pi^{(2)}$.

Translating back to scalar integrals, it is

$$
\begin{aligned}
I_2(1,2) =& c_{12\to 11} I_2(1,1) + c_{12\to 10} I_2(1,0) \\
&+ c_{12\to 20} I_2(2,0) + c_{12\to 01} I_2(0,1) \\
&+ c_{12\to 02} I_2(0,2),
\end{aligned} \tag{55}
$$

with $c_{12\to 20} = 0$ and

$$
\begin{aligned}
c_{12\to 11} =& \frac{-(-3+D)(m_1^2 - m_2^2 + p_1^2))}{(m_1^4 + (m_2^2 - p_1^2)^2 - 2m_1^2(m_2^2 + p_1^2)},\\
c_{12\to 10} =& \frac{D-2}{-2m_1^2\left(m_2^2+p_1^2\right)+m_1^4+\left(m_2^2-p_1^2\right)^2}\\
c_{12\to 01} =& \frac{2-D}{-2m_1^2\left(m_2^2+p_1^2\right)+m_1^4+\left(m_2^2-p_1^2\right)^2},\\
c_{12\to 02} =& \frac{-m_1^2+m_2^2+p_1^2}{-2m_1^2\left(m_2^2+p_1^2\right)+m_1^4+\left(m_2^2-p_1^2\right)^2}.
\end{aligned} \tag{56}
$$

Using $I_2(2,0) = \dfrac{D-2}{2m_1^2} I_2(1,0)$[1)] and $I_2(0,2) = \dfrac{D-2}{2m_2^2} I_2(0,1)$, we have our final results for the reduction in $I_2(1,2)$,

$$
I_2(1,2) = c_{2\to 2} I_2(1,1) + c_{2\to 1;\bar{2}} I_2(1,0) + c_{2\to 1;\bar{1}} I_2(0,1), \tag{57}
$$

with the coefficients

$$
\begin{aligned}
c_{2\to 2} =& -\frac{(D-3)\left(m_1^2-m_2^2+p_1^2\right)}{-2m_1^2\left(m_2^2+p_1^2\right)+m_1^4+\left(m_2^2-p_1^2\right)^2},\\
c_{2\to 1;\bar{2}} =& \frac{D-2}{-2m_1^2\left(m_2^2+p_1^2\right)+m_1^4+\left(m_2^2-p_1^2\right)^2},\\
c_{2\to 1;\bar{1}} =& -\frac{(D-2)\left(m_1^2+m_2^2-p_1^2\right)}{2m_2^2\left(-2m_1^2\left(m_2^2+p_1^2\right)+m_1^4+\left(m_2^2-p_1^2\right)^2\right)},
\end{aligned} \tag{58}
$$

which is given in (30).

B. General case for bubbles

Now, let us consider more complicated examples, that is, bubbles with general higher power propagators. With the choice of (45), we get an IBP recurrence relation (46) and use it to reduce the bubbles $i_{\lambda_0,m,n+1}$ to simpler bubbles, which have a lower total propagator power and no higher power in D_2. Similarly, by choosing different values of a_2 and a_3, we can obtain another IBP recurrence relation to reduce the integral to those with no higher power in D_1. The choice is

$$
a_2 = -\frac{a_1 A_{22}}{A_{32}}, \quad a_3 = \frac{a_1 A_{12}}{A_{32}}, \tag{59}
$$

and the corresponding IBP recurrence is

$$
\begin{aligned}
&c_{m+1,n} i_{\lambda_0,m+1,n} + c_{m+1,n-1} i_{\lambda_0,m+1,n-1} + c_{m,n-1} i_{\lambda_0,m,n-1}\\
&+ c_{m-1,n} i_{\lambda_0,m-1,n} + c_{m,n} i_{\lambda_0,m,n} + \delta_{2;r} = 0,
\end{aligned} \tag{60}
$$

with the coefficients

$$
\begin{aligned}
c_{m+1,n} =& (|\tilde{A}_{33}|)(D-3-m-n), \quad c_{m+1,n-1} = -n|\tilde{A}_{23}|,\\
c_{m,n-1} =& n|\tilde{A}_{21}|, \quad c_{m-1,n} = -m|\tilde{A}_{11}|,\\
c_{m,n} =& |\tilde{A}_{13}|(3+2m+n-D),
\end{aligned} \tag{61}
$$

and the boundary term

$$
\delta_{2;r'} = -\delta_{m,0}|\tilde{A}_{11}| i_{\lambda_0,m,n} + \delta_{n,0}\Big(-|\tilde{A}_{32}| i_{\lambda_0,m+1,n} + |\tilde{A}_{21}| i_{\lambda_0,m,n}\Big). \tag{62}
$$

Combining (46) and (60), we can reduce the general bubbles.

1. *Example:* $I_2(1,3)$

In the example $I_2(1,3)$, we simply need to reduce D_2 from power 3 to 1. The strategy is to use (46) twice. In the first step, by setting $m=0$ and $n=1$ in (46), we get

$$
\begin{aligned}
I_2(1,3) =& \frac{|\tilde{A}_{23}|(D-5)}{2|\tilde{A}_{33}|} I_2(1,2) + \frac{|\tilde{A}_{22}|(D-3)}{2|\tilde{A}_{33}|} I_2(1,1)\\
&+ \frac{-|\tilde{A}_{12}|(D-3)}{2|\tilde{A}_{33}|} I_2(0,2) + \frac{|\tilde{A}_{13}|}{|\tilde{A}_{33}|} I_2(0,3).
\end{aligned} \tag{63}
$$

For the first term in (63), setting $m=0$ and $n=0$ in (46) again, we have

$$
\begin{aligned}
I_2(1,2) =& \frac{|\tilde{A}_{23}|(D-3)}{|\tilde{A}_{33}|} I_2(1,1) + \frac{|\tilde{A}_{22}|(D-2)}{|\tilde{A}_{33}|} I_2(1,0)\\
&+ \frac{|\tilde{A}_{13}|}{|\tilde{A}_{33}|} I_2(0,2) + \frac{-|\tilde{A}_{12}|(D-2)}{|\tilde{A}_{33}|} I_2(0,1).
\end{aligned} \tag{64}
$$

1) The reduction of tadpole with higher power is simple. Noticing that $I_2(1,0) \propto (m_1^2)^{(D-2)/2}$ by dimensional analysis, one can take the derivative over m_1^2 to get the wanted reduction coefficients.

Inserting (64) into (63) and using the reduction in the tadpole[1)], we get

$$I_2(1,3) = c_{13\to11}I_2(1,1) + c_{13\to10}I_2(1,0) + c_{13\to01}I_2(0,1), \tag{65}$$

with the coefficients

$$\begin{aligned}
c_{13\to11} =& \frac{(|\tilde{A}_{23}||\tilde{A}_{33}| + |\tilde{A}_{23}|^2(D-5))(D-3)}{2|\tilde{A}_{33}|^2},\\
c_{13\to10} =& \frac{|\tilde{A}_{22}||\tilde{A}_{23}|(D-5)(D-2)}{2|\tilde{A}_{33}|^2},\\
c_{13\to01} =& \frac{(D-2)}{8|\tilde{A}_{33}|^2 m_2^4}A_{21}(2A_{32}|\tilde{A}_{23}|(D-5)m_2^2 + A_{32}A_{t33}(D-4)\\
&- 4A_{33}|\tilde{A}_{23}|(D-5)m_2^4 - 2A_{33}|\tilde{A}_{33}|(D-3)m_2^2)\\
&- A_{22}A_{31}(2|\tilde{A}_{23}|(D-5)m_2^2 + A_{t33}(D-4))\\
&+ 2A_{23}A_{31}m_2^2(2|\tilde{A}_{23}|(D-5)m_2^2 + |\tilde{A}_{33}|(D-3)).
\end{aligned} \tag{66}$$

The result is confirmed with FIRE6. In this example, we simply need to solve two equations to reduce the bubble topology.

2. *Example:* $I_2(3,5)$

For this example, we must use (60) to lower the power of D_1 and (46) to lower the power of D_2. Setting $m=1$ and $n=4$ in (60), we can reduce $I_2(3,5)$ to $I_2(2,4)$, $I_2(2,5)$, $I_2(1,5)$, and $I_2(3,4)$.

$$\begin{aligned}
I_2(3,5) =& \frac{|\tilde{A}_{11}|(D-7)}{2|\tilde{A}_{33}|}I_2(1,5) + \frac{-|\tilde{A}_{13}|(D-9)}{2|\tilde{A}_{33}|}I_2(2,5)\\
&+ \frac{-|\tilde{A}_{21}|(D-7)}{2|\tilde{A}_{33}|}I_2(2,4) + \frac{|\tilde{A}_{23}|}{|\tilde{A}_{33}|}I_2(3,4).
\end{aligned} \tag{67}$$

Then, setting $m=1$ and $n=3$ in (60), we reduce $I_2(3,4)$ to $I_2(1,4)$, $I_2(2,3)$, $I_2(2,4)$, and $I_2(3,3)$.

$$\begin{aligned}
I_2(3,4) =& \frac{-|\tilde{A}_{23}|}{|\tilde{A}_{33}|}I_2(3,3) + \frac{-|\tilde{A}_{13}|(D-8)}{2|\tilde{A}_{33}|}I_2(2,4)\\
&+ \frac{-|\tilde{A}_{21}|(D-6)}{2|\tilde{A}_{33}|}I_2(2,3) + \frac{|\tilde{A}_{11}|(D-6)}{2|\tilde{A}_{33}|}I_2(1,4).
\end{aligned} \tag{68}$$

Using the same idea, we must solve 14 equations to completely reduce $I_2(3,5)$. The analytic expressions for these 14 equations have also been confirmed by FIRE6.

C. Triangle case

The triangle $I_3(m+1,n+1,q+1)$ is given by

$$\begin{aligned}
&I_3(m+1,n+1,q+1)\\
&= \int \frac{\mathrm{d}^D l}{(l^2-m_1^2)^{m+1}((l-p_1)^2-m_2^2)^{n+1}((l+p_3)^2-m_3^2)^{q+1}}.
\end{aligned} \tag{69}$$

Its parametric form is

$$\begin{aligned}
&I_3(m+1,n+1,q+1)\\
&= \mathrm{i}(-1)^{3+m+n+q}\frac{\Gamma(-\lambda_0)}{\Gamma(m+1)\Gamma(n+1)\Gamma(q+1)\Gamma(\lambda_4+1)}i_{\lambda_0,m,n,q},
\end{aligned} \tag{70}$$

where

$$\begin{aligned}
&i_{\lambda_0;m,n,q} = \int \mathrm{d}\Pi^{(4)} F^{\lambda_0} x_1^m x_2^n x_3^q x_4^{\lambda_4}, \quad \lambda_0 = -\frac{D}{2},\\
&\lambda_4 = -4 - 2\lambda_0 - m - n - q = D - 4 - m - n - q.
\end{aligned} \tag{71}$$

Using expression (10), we have

$$\begin{aligned}
U(x) =& x_1 + x_2 + x_3, \quad V(x) = x_1x_2p_1^2 + x_1x_3p_3^2 + x_2x_3p_2^2,\\
f(x) =& -V + U\sum m_i^2 x_i = (x_1+x_2+x_3)(x_1m_1^2 + x_2m_2^2 + x_3m_3^2)\\
&- x_1x_2p_1^2 - x_2x_3p_2^2 - x_1x_3p_3^2,\\
F(x) =& U(x)x_4 + f(x) = \left(x_1+x_2+x_3\right)\\
&\times\left(m_1^2x_1 + m_2^2x_2 + m_3^2x_3 + x_4\right)\\
&- x_1x_2p_1^2 - x_2x_3p_2^2 - x_1x_3p_3^2.
\end{aligned} \tag{72}$$

Thus, we can express the matrices as

$$\begin{aligned}
\hat{A} =& \begin{bmatrix} 2m_1^2 & m_1^2+m_2^2-p_1^2 & m_1^2+m_3^2-p_3^2 & 1\\ m_1^2+m_2^2-p_1^2 & 2m_2^2 & m_2^2+m_3^2-p_2^2 & 1\\ m_1^2+m_3^2-p_3^2 & m_2^2+m_3^2-p_2^2 & 2m_3^2 & 1\\ 1 & 1 & 1 & 0 \end{bmatrix},\\
\hat{K}_A =& \begin{bmatrix} 0 & a_1 & a_2 & a_3\\ -a_1 & 0 & a_4 & a_5\\ -a_2 & -a_4 & 0 & a_6\\ -a_3 & -a_5 & -a_6 & 0 \end{bmatrix}, \quad \hat{Q} = \frac{1}{2}\hat{I} + \hat{K}_A\hat{A}.
\end{aligned} \tag{73}$$

1) In general, we could repeat the similar procedure to give the tadpoles' IBP recurrence relation, and calculate them step by step. Here, for simplicity, we could just use the trick, $I_2(1,0)\propto(m_1^2)^{(D-2)/2}$, and $I_2(0,1)\propto(m_2^2)^{(D-2)/2}$, to directly calculate the $I_2(2,0)=\partial/\partial m_1^2 I_2(1,0)=((D-2)/2m_1^2)I_2(1,0)$, $I_2(0,2)=\partial/\partial m_2^2 I_2(0,1)=((D-2)/2m_2^2)I_2(0,1)$, and $I_2(3,0)=\frac{1}{2}(\partial/\partial m_1^2)^2 I_2(1,0)=(((D-2)(D-4))/8m_1^4)I_2(1,0)$, $I_2(0,3)=\frac{1}{2}(\partial/\partial m_2^2)^2 I_2(0,3)=(((D-2)(D-4))/8m_2^4)I_2(0,1)$.

1. *Deriving the recurrence relation*

Taking $B=\frac{-1}{x_4}$ in (39), we get $z_i=\frac{Q_{ij}x_j}{x_4}$. Taking this relation into our IBP identities (32), we get

$$\sum_{i=1}^{4}\int \mathrm{d}\Pi^{(4)}\left\{z_i F^{\lambda_0}x_1^m x_2^n x_3^q x_4^{\lambda_4+1}\right\}+\delta_3=0, \tag{74}$$

for which we deal with the boundary δ_3 term later. After expanding the first term, we get

$$\begin{aligned}&c_{m,n,q}i_{\lambda_0;m,n,q}+c_{m+1,n,q}i_{\lambda_0;m+1,n,q}+c_{m+1,n,q-1}i_{\lambda_0;m+1,n,q-1}+c_{m+1,n-1,q}i_{\lambda_0;m+1,n-1,q}\\&+c_{m-1,n+1,q}i_{\lambda_0;m-1,q+1,q}+c_{m,n+1,q-1}i_{\lambda_0;m,n+1,q-1}+c_{m,n+1,q}i_{\lambda_0;m,n+1,q}+c_{m,n,q+1}i_{\lambda_0;m,n,q+1}\\&+c_{m,n-1,q+1}i_{\lambda_0;m,n-1,q+1}+c_{m-1,n,q+1}i_{\lambda_0;m-1,n,q+1}+c_{m-1,n,q}i_{\lambda_0;m-1,n,q}+c_{m,n-1,q}i_{m,n-1,q}\\&+c_{m,n,q-1}i_{\lambda_0;m,n,q-1}+\delta_3=0,\end{aligned} \tag{75}$$

with the coefficients

$$\begin{aligned}c_{m,n,q}=&\lambda_0+(m+1)Q_{11}+(n+1)Q_{22}+(q+1)Q_{33}+(\lambda_4+1)Q_{44},\\c_{m+1,n,q}=&\lambda_4Q_{41},\;\; c_{m+1,n,q-1}=qQ_{31},\;\; c_{m+1,n-1,q}=nQ_{21},\;\; c_{m,n,q-1}=qQ_{34},\\c_{m-1,n+1,q}=&mQ_{12},\;\; c_{m,n+1,q-1}=qQ_{32},\;\; c_{m,n+1,q}=\lambda_4Q_{42},\;\; c_{m,n,q+1}=\lambda_4Q_{43},\\c_{m,n-1,q+1}=&nQ_{23},\;\; c_{m-1,n,q+1}=mQ_{13},\;\; c_{m-1,n,q}=mQ_{14},\;\; c_{m,n-1,q}=nQ_{24}.\end{aligned} \tag{76}$$

Now, we can choose particular values for our six parameters a_1, a_2, a_3, a_4, a_5, and a_6 to let the coefficients $c_{m+1,n,q}$, $c_{m+1,n,q-1}$, $c_{m+1,n-1,q}$, $c_{m-1,n+1,q}$, $c_{m,n+1,q}$, and $c_{m,n+1,q}$ be zero. The solutions are

$$\begin{aligned}a_2=&-a_1\frac{A_{21}A_{42}-A_{22}A_{41}}{A_{31}A_{42}-A_{32}A_{41}}=-\frac{a_1\left(-m_1^2+m_2^2+p_1^2\right)}{-m_1^2+m_2^2+2(p_1\cdot p_2)+p_1^2},\\a_3=&\frac{a_1(A_{21}A_{32}-A_{22}A_{31})}{A_{31}A_{42}-A_{32}A_{41}}=-\frac{a_1\left(m_1^2-m_2^2-p_1^2\right)\left(m_2^2+m_3^2-p_2^2\right)}{-m_1^2+m_2^2+2(p_1\cdot p_2)+p_1^2}-2a_1m_2^2,\\a_4=&\frac{a_1(A_{11}A_{42}-A_{12}A_{41})}{A_{31}A_{42}-A_{32}A_{41}}=-\frac{a_1\left(m_1^2-m_2^2+p_1^2\right)}{-m_1^2+m_2^2+2(p_1\cdot p_2)+p_1^2},\\a_5=&\frac{-a_1(A_{11}A_{32}-A_{12}A_{31})}{A_{31}A_{42}-A_{32}A_{41}}=\frac{a_1\left(m_1^2-m_2^2+p_1^2\right)\left(m_1^2+m_3^2-2(p_1\cdot p_2)-p_1^2-p_2^2\right)}{-m_1^2+m_2^2+2(p_1\cdot p_2)+p_1^2}+2a_1m_1^2,\\a_6=&\frac{a_1(A_{11}A_{22}-A_{12}A_{21})}{A_{31}A_{42}-A_{32}A_{41}}=\frac{a_1\left(m_1^4-2m_1^2\left(m_2^2+p_1^2\right)+\left(m_2^2-p_1^2\right)^2\right)}{-m_1^2+m_2^2+2(p_1\cdot p_2)+p_1^2}.\end{aligned} \tag{77}$$

Then, the matrix $\hat{Q}$ becomes

$$\hat{Q}_r=\frac{1}{\Delta_A}\begin{bmatrix}\frac{1}{2}\Delta_A & 0 & a_1|\tilde{A}_{14}| & a_1|\tilde{A}_{13}|\\ 0 & \frac{1}{2}\Delta_A & -a_1|\tilde{A}_{24}| & a_1|\tilde{A}_{23}|\\ 0 & 0 & \frac{1}{2}\Delta_A+a_1|\tilde{A}_{34}| & a_1|\tilde{A}_{33}|\\ 0 & 0 & -a_1|\tilde{A}_{44}| & \frac{1}{2}\Delta_A-a_1|\tilde{A}_{43}|\end{bmatrix},\quad \Delta_A=\mathrm{Det}\begin{bmatrix}A_{31} & A_{32}\\ A_{41} & A_{42}\end{bmatrix}=A_{31}A_{42}-A_{32}A_{41}.$$

After this, we obtain the reduced IBP relation, where only the propagator $D_3=(l+p_3)^2-m_3^2$ has one increasing power,

$$
\begin{aligned}
&c_{m,n,q} i_{\lambda_0;m,n,q} + c_{m,n,q+1} i_{\lambda_0;m,n,q+1} + c_{m,n-1,q+1} i_{\lambda_0;m,n-1,q+1} + c_{m-1,n,q+1} i_{\lambda_0;m-1,n,q+1} \\
&+ c_{m-1,n,q} i_{\lambda_0;m-1,n,q} + c_{m,n-1,q} i_{\lambda_0;m,n-1,q} + c_{m,n,q-1} i_{\lambda_0;m,n,q-1} + \delta_{3;r} = 0,
\end{aligned} \tag{78}
$$

with the coefficients

$$
\begin{aligned}
c_{m,n,q} =& \lambda_0 + mQ_{11;r} + nQ_{22;r} + qQ_{33;r} + Q_{11;r} + Q_{22;r} + Q_{33;r} + \lambda_4 Q_{44;r} + Q_{44;r}, \\
c_{m,n,q+1} =& \lambda_4 Q_{43;r} = \frac{-a_1\lambda_4}{A_{31}A_{42} - A_{32}A_{41}} |\tilde{A}_{44}|, \;\; c_{m,n-1;q+1} = nQ_{23;r} = \frac{-a_1 n}{A_{31}A_{42} - A_{32}A_{41}} |\tilde{A}_{24}|, \\
c_{m-1,n,q+1} =& mQ_{13;r} = \frac{a_1 m}{A_{31}A_{42} - A_{32}A_{41}} |\tilde{A}_{14}|, \;\; c_{m-1,n,q} = mQ_{14;r} \frac{a_1 m}{A_{31}A_{42} - A_{32}A_{41}} |\tilde{A}_{13}|, \\
c_{m,n-1,q} =& nQ_{24;r} = \frac{-a_1 n}{A_{31}A_{42} - A_{32}A_{41}} |\tilde{A}_{23}|, \;\; c_{m,n,q-1} = qQ_{34;r} = \frac{a_1 q}{A_{31}A_{42} - A_{23}A_{41}} |\tilde{A}_{33}|,
\end{aligned} \tag{79}
$$

where the subscript r in $\delta_{3;r}$ and $Q_{ij;r}$ indicates that the parameters a_2 to a_6 should be replaced by (77).

The reduction in the boundary δ_3 part: Similar to the bubble situation, inserting the value of z_i into the δ_3 part, we obtain

$$
\begin{aligned}
\delta_{3;r} =& \left(\delta_{m+1,0} Q_{11;r} + \delta_{m,0} Q_{14;r}\right) i_{\lambda_0,-1,n,q} + \delta_{m,0} Q_{12;r} i_{\lambda_0,-1,n+1,q} + \delta_{m,0} Q_{13;r} i_{\lambda_0,-1,n,q+1} \\
&+ \delta_{n,0} Q_{21;r} i_{\lambda_0,m+1,-1,q} + \left(\delta_{n+1,0} Q_{22;r} + \delta_{n,0} Q_{24;r}\right) i_{\lambda_0,m,-1,q} + \delta_{n,0} Q_{23;r} i_{\lambda_0,m,-1,q+1} \\
&+ \delta_{q,0} Q_{31;r} i_{\lambda_0,m+1,n,-1} + \delta_{q,0} Q_{32;r} i_{\lambda_0,m,n+1,-1} + \left(\delta_{q+1,0} Q_{33;r} + \delta_{q,0} Q_{34;r}\right) i_{\lambda_0,m,n,-1},
\end{aligned} \tag{80}
$$

where $i_{\lambda_0,m,n,-1}$, $i_{\lambda_0,m,-1,q}$, and $i_{\lambda_0,-1,n,q}$ contribute to the sub-topology of the triangle, that is, the bubble[1)].

2. *Triangle example:* $I_3(1,1,2)$

Now, we apply the complete recurrence relation to the example $I_3(1,1,2)$. Setting $m = n = q = 0$ in (78), we obtain

$$
c_{0,0,0} i_{\lambda_0,0,0,0} + c_{0,0,1} i_{\lambda_0,0,0,1} + \delta_{3;000} = 0, \tag{81}
$$

with the coefficients

$$
\begin{aligned}
c_{0,0,1} =& \lambda_4 Q_{43;r} = -\frac{1}{-m_1^2 + m_2^2 + 2(p_1\cdot p_2) + p_1^2} \times \Big\{ 2a_1(D-4)\Big(m_1^4 p_2^2 - 2m_1^2 (m_2^2((p_1\cdot p_2) + p_2^2) - m_3^2 (p_1\cdot p_2) \\
&+ p_2^2((p_1\cdot p_2) + p_1^2)) + m_2^4 (2(p_1\cdot p_2) + p_1^2 + p_2^2) + m_2^2 (2(p_1\cdot p_2)(2(p_1\cdot p_2) + p_1^2 + p_2^2) \\
&- 2m_3^2((p_1\cdot p_2) + p_1^2)) + p_1^2 (m_3^4 - 2m_3^2((p_1\cdot p_2) + p_2^2) + p_2^2(2(p_1\cdot p_2) + p_1^2 + p_2^2))\Big)\Big\}, \\
c_{0,0,0} =& -\frac{D}{2} + Q_{11;r} + Q_{22;r} + Q_{33;r} + (D-3) Q_{44;r} \\
=& -\frac{2a_1(D-4)\Big(m_1^2(p_1\cdot p_2) - m_2^2((p_1\cdot p_2) + p_1^2) + p_1^2\Big(m_3^2 - (p_1\cdot p_2) - p_2^2\Big)\Big)}{-m_1^2 + m_2^2 + 2(p_1\cdot p_2) + p_1^2}.
\end{aligned} \tag{82}
$$

In (81), only two terms of triangle topology remain: one is the scalar basis, and the other is the target we want to reduce. The other five terms in (78) disappear owing to the expression in (79). Thus, there is no need to solve mixed IBP relations. The δ_3 term becomes

$$
\begin{aligned}
\delta_{p;000} \equiv \delta_{p;r}|_{m=0,n=0,q=0} =& Q_{14;r} i_{\lambda_0,-1,0,0} + Q_{12;r} i_{\lambda_0,-1,1,0} + Q_{13;r} i_{\lambda_0,-1,0,1} + Q_{21;r} i_{\lambda_0,1,-1,0} + Q_{24;r} i_{\lambda_0,0,-1,0} + Q_{23;r} i_{\lambda_0,0,-1,1} \\
&+ Q_{31;r} i_{\lambda_0,1,0,-1} + Q_{32;r} i_{\lambda_0,0,1,-1} + Q_{34;r} i_{\lambda_0,0,0,-1}.
\end{aligned} \tag{83}
$$

1) Since the boundary term having only one $x_i = 0$, it reduces to the sub-topologies with only one propagator pinched.

Translating back to the I form, we obtain the result

$$\begin{aligned} I_3(1,1,2) =& c_{3\to 111}I_3(1,1,1)+c_{3\to 110}I_3(1,1,0)+c_{3\to 101}I_3(1,0,1)+c_{3\to 011}I_3(0,1,1)\\ & c_{3\to 210}I_3(2,1,0)+c_{3\to 201}I_3(2,0,1)+c_{3\to 120}I_3(1,2,0)+c_{3\to 021}I_3(0,2,1)\\ & c_{3\to 102}I_3(1,0,2)+c_{3\to 012}I_3(0,1,2), \end{aligned} \tag{84}$$

with the coefficients

$$\begin{aligned} & c_{3\to 111}=\frac{c_{0,0,0}\Gamma(D-3)}{c_{0,0,1}\Gamma(D-4)},\quad c_{3\to 110}=-\frac{Q_{34;r}\Gamma(D-2)}{c_{0,0,1}\Gamma(D-4)},\quad c_{3\to 101}=-\frac{Q_{24;r}\Gamma(D-2)}{c_{0,0,1}\Gamma(D-4)},\quad c_{3\to 011}=-\frac{Q_{14;r}\Gamma(D-2)}{c_{0,0,1}\Gamma(D-4)},\\ & c_{3\to 210}=\frac{Q_{31;r}\Gamma(D-3)}{c_{0,0,1}\Gamma(D-4)},\quad c_{3\to 201}=\frac{Q_{21;r}\Gamma(D-3)}{c_{0,0,1}\Gamma(D-4)},\quad c_{3\to 021}=\frac{Q_{12;r}\Gamma(D-3)}{c_{0,0,1}\Gamma(D-4)},\quad c_{3\to 120}=\frac{Q_{32;r}\Gamma(D-3)}{c_{0,0,1}\Gamma(D-4)},\\ & c_{3\to 102}=\frac{Q_{23;r}\Gamma(D-3)}{c_{0,0,1}\Gamma(D-4)},\quad c_{3\to 012}=\frac{Q_{13;r}\Gamma(D-3)}{c_{0,0,1}\Gamma(D-4)}. \end{aligned} \tag{85}$$

The final step is to reduce bubbles that have one propagator with the power two. This problem has been solved in the previous subsection (see (57)). With proper relabeling of the external variables of the last six terms in (84) and collecting all coefficients together, we get

$$\begin{aligned} I_3(1,1,2) =& c_{3\to 3}I_3(1,1,1)+c_{3\to 2;\bar{3}}I_3(1,1,0)+c_{3\to 2;\bar{2}}I_3(1,0,1)+c_{3\to 2;\bar{1}}I_3(0,1,1)\\ & +c_{3\to 1;\bar{2}\bar{3}}I_3(1,0,0)+c_{3\to 1;\bar{1}\bar{3}}I_3(0,1,0)+c_{3\to 1;\bar{1}\bar{2}}I_3(0,0,1). \end{aligned} \tag{86}$$

Because the explicit expressions of these coefficients are long, they are provided in the companion Mathematica notebook. The result is confirmed by FIRE6.

3. *General case in triangles*

Similar to the bubble case, with different choices, we can obtain three IBP recurrence relations. In each of these relations, only one term has a propagator with a higher power. For simplicity, we label the IBP recurrence relation eq_i, which shifts the propagator D_i. Now, we can use eq_i with $i=1,2,3$ to calculate the general case for triangles. Let us denote

$$\begin{aligned} & eq_1:\left(a_{1^+}1^+ + a_{1^+3^-}1^+3^- + a_{1^+2^-}1^+2^- + a_{3^-}3^- + a_{2^-}2^- + a_{1^-}1^- + a_0\right)i_{\lambda_0,m,n,q}+\delta_{3;r,eq1}=0,\\ & eq_2:\left(b_{2^+}2^+ + b_{2^+3^-}2^+3^- + b_{1^-2^+}1^-2^+ + b_{3^-}3^- + b_{2^-}2^- + b_{1^-}1^- + b_0\right)i_{\lambda_0,m,n,q}+\delta_{3;r,eq2}=0,\\ & eq_3:\left(c_{3^+}3^+ + c_{2^-3^+}2^-3^+ + c_{1^-3^+}1^-3^+ + c_{3^-}3^- + c_{2^-}2^- + c_{1^-}1^- + c_0\right)i_{\lambda_0,m,n,q}+\delta_{3;r,eq3}=0, \end{aligned} \tag{87}$$

where all coefficients have the same form as in (78). Combining these, we can reduce the general triangles. For example, for $I_3(2,2,3)$, after setting $m=0$, $n=1$, and $q=2$ in eq_1, we can reduce $I_3(2,2,3)$ to $I_3(1,1,3)$, $I_3(1,2,2)$, $I_3(1,2,3)$, $I_3(2,1,3)$, $I_3(2,2,2)$ and boundary terms, the general bubbles. Then, setting $m=0$, $n=0$, and $q=2$ in eq_1, we can reduce $I_3(2,1,3)$ to $I_3(1,1,2)$, $I_3(1,1,3)$, and $I_3(2,1,2)$. After **12** steps, we get the result for the reduction in the triangle topology. The boundary terms involve bubbles and tadpoles, which have been dealt with in previous subsections. Finally, we can obtain all coefficients from $I_3(2,2,3)$ to all scalar bases.

D. Box case

The general form of a box is given by

$$I_4(n_1+1,n_2+1,n_3+1,n_4+1)=\int\frac{\mathrm{d}^D l}{D_1^{n_1+1}D_2^{n_2+1}D_3^{n_3+1}D_4^{n_4+1}}, \tag{88}$$

with

$$D_1 = \quad l^2 - m_1^2, \quad D_2 = (l-p_1)^2 - m_2^2, \quad D_3 = (l-p_1-p_2)^2 - m_3^2, \quad D_4 = (l+p_4)^2 - m_4^2. \tag{89}$$

The parametric form of $I_4(n_1+1, n_2+1, n_3+1, n_4+1)$ can be written as

$$I_4(n_1+1, n_2+1, n_3+1, n_4+1) = \frac{i(-1)^{4+n_1+n_2+n_3+n_4}\Gamma(-\lambda_0)}{\Gamma(n_1+1)\Gamma(n_2+1)\Gamma(n_3+1)\Gamma(n_4+1)\Gamma(\lambda_5+1)} i_{\lambda_0;n_1,n_2,n_3,n_4}, \tag{90}$$

where

$$\begin{aligned} i_{\lambda_0;n_1,n_2,n_3,n_4} &= \int \mathrm{d}\Pi^{(5)} F^{\lambda_0} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{\lambda_5} = \int \mathrm{d}\Pi^{(5)} (Ux_5 + f)^{\lambda_0} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{\lambda_5} \\ \mathrm{d}\Pi^{(5)} &= \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5 \delta(\sum x_j - 1), \qquad \lambda_0 = -\frac{D}{2} \\ \lambda_5 &= -5 - n_1 - n_2 - n_3 - n_4 - 2\lambda_0 = D - 5 - n_1 - n_2 - n_3 - n_4, \end{aligned} \tag{91}$$

and the functions are

$$\begin{aligned} U(x) =& x_1 + x_2 + x_3 + x_4, \\ V(x) =& x_1 x_2 p_1^2 + x_1 x_3 (p_1+p_2)^2 + x_1 x_4 (p_1+p_2+p_3)^2 + x_2 x_3 p_2^2 + x_2 x_4 (p_2+p_3)^2 + x_3 x_4 p_3^2 \\ f(x) =& -V(x) + U(x) \sum m_i^2 x_i = m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3 + m_4^2 x_4 + (m_1^2 + m_2^2 - p_1^2) x_1 x_2 \\ &+ [m_1^2 + m_3^2 - (p_1+p_2)^2] x_1 x_3 + [m_1^2 + m_4^2 - (p_1+p_2+p_3)^2] x_1 x_4, \\ &+ (m_2^2 + m_3^2 - p_2^2) x_2 x_3 + [m_2^2 + m_4^2 - (p_2+p_3)^2] x_2 x_4 + (m_3^2 + m_4^2 - p_3^2) x_3 x_4, \\ F(x) =& U(x) x_5 + f(x) = m_1^2 x_1^2 + m_2^2 x_2^2 + m_3^2 x_3^2 + m_4^2 x_4^2 \\ &+ (m_1^2 + m_2^2 - p_1^2) x_1 x_2 + [m_1^2 + m_3^2 - (p_1+p_2)^2] x_1 x_3 + [m_1^2 + m_4^2 - (p_1+p_2+p_3)^2] x_1 x_4 \\ &+ [m_2^2 + m_3^2 - p_2^2] x_2 x_3 + [m_2^2 + m_4^2 - (p_2+p_3)^2] x_2 x_4 + [m_3^2 + m_4^2 - p_3^2] x_3 x_4 \\ &+ x_1 x_5 + x_2 x_5 + x_3 x_5 + x_4 x_5 \\ =& (x_1 + x_2 + x_3 + x_4)(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3 + m_4^2 x_4 + x_5) \\ &- x_1 x_2 p_1^2 - x_1 x_3 (p_1+p_2)^2 - x_1 x_4 (p_1+p_2+p_3)^2 - x_2 x_3 p_2^2 - x_2 x_4 (p_2+p_3)^2 - x_3 x_4 p_3^2. \end{aligned} \tag{92}$$

Now, the matrices are given by

$$\hat{A} = \begin{bmatrix} 2m_1^2 & m_1^2 + m_2^2 - p_1^2 & m_1^2 + m_3^2 - p_{12}^2 & m_1^2 + m_4^2 - p_{13}^2 & 1 \\ m_1^2 + m_2^2 - p_1^2 & 2m_2^2 & m_2^2 + m_3^2 - p_2^2 & m_2^2 + m_4^2 - p_{23}^2 & 1 \\ m_1^2 + m_3^2 - p_{12}^2 & m_2^2 + m_3^2 - p_2^2 & 2m_3^2 & m_3^2 + m_4^2 - p_3^2 & 1 \\ m_1^2 + m_4^2 - p_{13}^2 & m_2^2 + m_4^2 - p_{23}^2 & m_3^2 + m_4^2 - p_3^2 & 2m_4^2 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad K_A = \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ -a_1 & 0 & a_5 & a_6 & a_7 \\ -a_2 & -a_5 & 0 & a_8 & a_9 \\ -a_3 & -a_6 & -a_8 & 0 & a_{10} \\ -a_4 & -a_7 & -a_9 & -a_{10} & 0 \end{bmatrix}, \tag{93}$$

where $p_{ij} \equiv p_i + p_{i+1} \cdots p_j$.

1. *Deriving the recurrence relation*

Taking $B = \dfrac{-1}{x_5}$ in (39), we get

$$\begin{aligned}&\Big\{c_{n_1+1,n_2,n_2,n_4}1^+ + c_{n_1+1,n_2,n_3,n_4-1}{}^{\text{‘}+}4^- + c_{n_1+1,n_2,n_3-1,n_4}1^+3^- + c_{n_1+1,n_2-1,n_3,n_4}1^+2^- \\ &+ c_{n_1,n_2+1,n_3,n_4}2^+ + c_{n_1,n_2+1,n_3,n_4-1}2^+4^- + c_{n_1,n_2+1,n_3-1,n_4}2^+3^- + c_{n_1-1,n_2+1,n_3,n_4}2^+1^- \\ &+ c_{n_1,n_2,n_3+1,n_4}3^+ + c_{n_1,n_2,n_3+1,n_4-1}3^+4^- + c_{n_1,n_2-1,n_3+1,n_4}3^+2^- + c_{n_1-1,n_2,n_3+1,n_4}3^+1^- \\ &+ c_{n_1,n_2,n_3,n_4+1}4^+ + c_{n_1,n_2,n_3-1,n_4+1}4^+3^- + c_{n_1,n_2-1,n_3,n_4+1}4^+2^- + c_{n_1-1,n_2,n_3,n_4+1}4^+1^- \\ &+ c_{n_1,n_2,n_3,n_4-1}4^- + c_{n_1,n_2,n_3-1,n_4}3^- + c_{n_1,n_2-1,n_3,n_4}2^- + c_{n_1-1,n_2,n_3,n_4}1^- + c_{n_1,n_2,n_3,n_4}\Big\}i_{n_1,n_2,n_3,n_4} + \delta_4 = 0,\end{aligned} \tag{94}$$

where

$$j^+ i_{n_1\cdots n_j\cdots n_k} = i_{n_1\cdots n_j+1\cdots n_k}, \quad j^- i_{n_1\cdots n_j\cdots n_k} = i_{n_1\cdots n_j-1\cdots n_k}. \tag{95}$$

Similarly, we can choose particular values of the parameters a_2 to a_{10} with a free a_1 to ensure the coefficients of the terms in the first three lines of (94) equal zero. The analytic solution is provided in the companion Mathematica notebook. Here, we can express the solution for the parameters using the matrix elements of $\hat{A}$.

$$\begin{aligned}&a_2 = \frac{-a_1}{\Delta_{\rm Box}}|\tilde{A}_{13,45}|,\ \ a_3 = \frac{a_1}{\Delta_{\rm Box}}|\tilde{A}_{14,45}|,\ \ a_4 = \frac{-a_1}{\Delta_{\rm Box}}|\tilde{A}_{15,45}|,\ \ a_5 = \frac{a_1}{\Delta_{\rm Box}}|\tilde{A}_{23,45}|,\ \ a_6 = \frac{-a_1}{\Delta_{\rm Box}}|\tilde{A}_{24,45}|,\ \ a_7 = \frac{a_1}{\Delta_{\rm Box}}|\tilde{A}_{25,45}|,\\ &a_8 = \frac{a_1}{\Delta_{\rm Box}}|\tilde{A}_{34,45}|,\ \ a_9 = \frac{-a_1}{\Delta_{\rm Box}}|\tilde{A}_{35,45}|,\ \ a_{10} = \frac{a_1}{\Delta_{\rm Box}}|\tilde{A}_{45,45}|,\ \ \Delta_{\rm Box} = \begin{vmatrix} A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \\ A_{51} & A_{52} & A_{53}\end{vmatrix},\end{aligned}$$

where $|\tilde{A}_{ij,kl}|$ represents the determinant of the matrix A after we removed the i,jth rows and k,lth columns. Then, the matrix $\hat{Q}$ becomes

$$\hat{Q}_r = \frac{1}{\Delta_{\rm Box}}\begin{bmatrix} \frac{1}{2}\Delta_{\rm Box} & 0 & 0 & -a_1|\tilde{A}_{15}| & -a_1|\tilde{A}_{14}| \\ 0 & \frac{1}{2}\Delta_{\rm Box} & 0 & a_1|\tilde{A}_{25}| & a_1|\tilde{A}_{24}| \\ 0 & 0 & \frac{1}{2}\Delta_{\rm Box} & -a_1|\tilde{A}_{35}| & -a_1|\tilde{A}_{34}| \\ 0 & 0 & 0 & \frac{1}{2}\Delta_{\rm Box} + a_1|\tilde{A}_{45}| & a_1|\tilde{A}_{44}| \\ 0 & 0 & 0 & -a_1|\tilde{A}_{55}| & \frac{1}{2}\Delta_{\rm Box} - a_1|\tilde{A}_{54}| \end{bmatrix}.$$

We then obtain the simplified recurrence relation

$$\begin{aligned}&c_{n_1,n_2,n_3,n_4+1}i_{n_1,n_2,n_3,n_4+1} + c_{n_1,n_2,n_3-1,n_4+1}i_{n_1,n_2,n_3-1,n_4+1} + c_{n_1,n_2-1,n_3,n_4+1}i_{n_1,n_2-1,n_3,n_4+1} + c_{n_1-1,n_2,n_3,n_4}i_{n_1-1,n_2,n_3,n_4} \\ &+ c_{n_1,n_2,n_3,n_4-1}i_{n_1,n_2,n_3,n_4-1} + c_{n_1,n_2,n_3-1,n_4}i_{n_1,n_2,n_3-1,n_4} + c_{n_1,n_2-1,n_3,n_4}i_{n_1,n_2-1,n_3,n_4} + c_{n_1-1,n_2,n_3,n_4}i_{n_1-1,n_2,n_3,n_4} \\ &+ c_{n_1,n_2,n_3,n_4}i_{n_1,n_2,n_3,n_4} + \delta_{4;r} = 0.\end{aligned} \tag{96}$$

Now, we must calculate the δ_4 term.

The reduction in the boundary δ_4 term: Similar to the former case, we can expand the δ_4 term and take the values of the parameters a_2 to a_{10} into the δ_4 part. Subsequently, we get

$$\begin{aligned}\delta_{4;r} =& \delta_{n_1+1,0}Q_{11;r}i_{-1,n_2,n_3,n_4} + \delta_{n_1,0}Q_{12;r}i_{-1,n_2+1,n_3,n_4} + \delta_{n_1,0}Q_{13;r}i_{-1,n_2,n_3+1,n_4} + \delta_{n_1,0}Q_{14;r}i_{-1,n_2,n_3,n_4+1} \\ &+ \delta_{n_1,0}Q_{15;r}i_{-1,n_2,n_3,n_4} + \delta_{n_2,0}Q_{21;r}i_{n_1+1,-1,n_3,n_4} + \delta_{n_2+1,0}Q_{22;r}i_{n_1,-1,n_3,n_4} + \delta_{n_2,0}Q_{23;r}i_{n_1,-1,n_3+1,n_4} \\ &+ \delta_{n_2,0}Q_{24;r}i_{n_1,-1,n_3,n_4+1} + \delta_{n_2,0}Q_{25;r}i_{n_1,-1,n_3,n_4} + \delta_{n_3,0}Q_{31;r}i_{n_1+1,n_2,-1,n_4} + \delta_{n_3,0}Q_{32;r}i_{n_1,n_2+1,-1,n_4} \\ &+ \delta_{n_3+1,0}Q_{33;r}i_{n_1,n_2,-1,n_4} + \delta_{n_3,0}Q_{34;r}i_{n_1,n_2,-1,n_4+1} + \delta_{n_3,0}Q_{35;r}i_{n_1,n_2,-1,n_4} + \delta_{n_4,0}Q_{41;r}i_{n_1+1,n_2,n_3,-1} \\ &+ \delta_{n_4,0}Q_{42;r}i_{n_1,n_2+1,n_3,-1} + \delta_{n_4,0}Q_{43;r}i_{n_1,n_2,n_3+1,-1} + \delta_{n_4+1,0}Q_{44;r}i_{n_1,n_2,n_3,-1} + \delta_{n_4,0}Q_{45;r}i_{n_1,n_2,n_3,-1},\end{aligned} \tag{97}$$

where the subscript "r" represents the value of the parameter Q after we set a_2 to a_{10}.

2. *Example:* $I_4(1,1,1,2)$

Now, we can use recurrence relation (96) to calculate the example $I_4(1,1,1,2)$. Letting $n_1=n_2=n_3=n_4=0$, we get (the coefficients of the other terms are all zero)

$$c_{0,0,0,0}i_{0,0,0,0}+c_{0,0,0,1}i_{0,0,0,1}+\delta_{4;0000}=0, \tag{98}$$

where $\delta_{4;0000}\equiv\delta_{4;r}|_{n_1=n_2=n_3=n_4=0}$. Translating to I, we obtain the following result:

$$\begin{aligned}I_4(1,1,1,2)=&c_{4\to1111}I_4(1,1,1,1)+c_{4\to1110}I_4(1,1,1,0)+c_{4\to1101}I_4(1,1,0,1)+c_{4\to1011}I_4(1,0,1,1)+c_{4\to0111}I_4(0,1,1,1)\\&+c_{4\to2110}I_4(2,1,1,0)+c_{4\to2101}I_4(2,1,0,1)+c_{4\to2011}I_4(2,0,1,1)+c_{4\to1210}I_4(1,2,1,0)+c_{4\to1201}I_4(1,2,0,1)\\&+c_{4\to0211}I_4(0,2,1,1)+c_{4\to1120}I_4(1,1,2,0)+c_{4\to1021}I_4(1,0,2,1)+c_{4\to0121}I_4(0,1,2,1)\\&+c_{4\to1102}I_4(1,1,0,2)+c_{4\to1012}I_4(1,0,1,2)+c_{4\to0112}I_4(0,1,1,2),\end{aligned} \tag{99}$$

with the coefficients

$$\begin{aligned}&c_{4\to1111}=\frac{c_{0,0,0,0}}{c_{0,0,0,1}}(D-5)=\frac{\mathrm{Tr}\hat{Q}_{ij;r}+(D-5)Q_{55;r}-\frac{D}{2}}{Q_{54;r}},\quad c_{4\to0111}=-\frac{Q_{15;r}\Gamma(D-3)}{c_{0,0,0,1}\Gamma(D-5)},\\&c_{4\to1011}=-\frac{Q_{25;r}\Gamma(D-3)}{c_{0,0,0,1}\Gamma(D-5)},\quad c_{4\to1101}=-\frac{Q_{35;r}\Gamma(D-3)}{c_{0,0,0,1}\Gamma(D-5)},\quad c_{4\to1110}=-\frac{Q_{45;r}\Gamma(D-3)}{c_{0,0,0,1}\Gamma(D-5)},\\&c_{4\to0211}=\frac{Q_{12;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)},\quad c_{4\to0121}=\frac{Q_{13;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)},\quad c_{4\to0112}=\frac{Q_{14;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)},\\&c_{4\to2011}=\frac{Q_{21;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)},\quad c_{4\to1021}=\frac{Q_{23;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)},\quad c_{4\to1012}=\frac{Q_{24;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)},\\&c_{4\to2101}=\frac{Q_{31;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)},\quad c_{4\to1201}=\frac{Q_{32;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)},\quad c_{4\to1102}=\frac{Q_{34;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)},\\&c_{4\to2110}=\frac{Q_{41;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)},\quad c_{4\to1210}=\frac{Q_{42;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)},\quad c_{4\to1120}=\frac{Q_{43;r}\Gamma(D-4)}{c_{0,0,0,1}\Gamma(D-5)}.\end{aligned} \tag{100}$$

Next, we must use the reduction in triangles with one double propagator given in (86). Inserting them into (99), we obtain the complete reduction in the box $I_4(1,1,1,2)$.

$$\begin{aligned}I_4(1,1,1,2)=&c_{4\to4}I_4(1,1,1,1)+c_{4\to3;\bar{1}}I_4(0,1,1,1)+c_{4\to3;\bar{2}}I_4(1,0,1,1)+c_{4\to3;\bar{3}}I_4(1,1,0,1)\\&+c_{4\to3;\bar{4}}I_4(1,1,1,0)+c_{4\to2;\bar{1}\bar{2}}I_4(0,0,1,1)+c_{4\to2;\bar{1}\bar{3}}I_4(0,1,0,1)+c_{4\to2;\bar{1}\bar{4}}I_4(0,1,1,0)\\&+c_{4\to2;\bar{2}\bar{3}}I_4(1,0,0,1)+c_{4\to2;\bar{2}\bar{4}}I_4(1,0,1,0)+c_{4\to2;\bar{3}\bar{4}}I_4(1,1,0,0)\\&+c_{4\to1;D_1}I_4(1,0,0,0)+c_{4\to1;D_2}I_4(0,1,0,0)+c_{4\to1;D_3}I_4(0,0,1,0)+c_{4\to1;D_4}I_4(0,0,0,1),\end{aligned} \tag{101}$$

the long coefficient expressions of which are given in the companion Mathematica notebook. The result is confirmed by FIRE6.

E. Pentagon case

The general form of a pentagon is given by

$$I_5(n_1+1,n_2+1,n_3+1,n_4+1,n_5+1)=\int\frac{\mathrm{d}^Dl}{D_1^{n_1+1}D_2^{n_2+1}D_3^{n_3+1}D_4^{n_4+1}D_5^{n_5+1}} \tag{102}$$

with

$$D_1 = l^2 - m_1^2, \quad D_2 = (l-p_1)^2 - m_2^2, \quad D_3 = (l-p_1-p_2)^2 - m_3^2, \quad D_4 = (l-p_1-p_2-p_3)^2 - m_4^2, \;\; D_5 = (l+p_5)^2 - m_5^2. \tag{103}$$

The parametric form of $I_5(n_1+1, n_2+1, n_3+1, n_4+1, n_5+1)$ can be written as

$$I_5(n_1+1,n_2+1,n_3+1,n_4+1,n_5+1) = \frac{i(-1)^{5+n_1+n_2+n_3+n_4+n_5}\Gamma(-\lambda_0)}{\sum_{i=1}^{5}\Gamma(n_i+1)\Gamma(\lambda_6+1)} i_{\lambda_0;n_1,n_2,n_3,n_4,n_5}, \tag{104}$$

where

$$\begin{aligned} i_{\lambda_0;n_1,n_2,n_3,n_4,n_5} &= \int \mathrm{d}\Pi^{(6)} F^{\lambda_0} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5} x_6^{\lambda_6+1}, \\ \mathrm{d}\Pi^{(5)} =& \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5 \mathrm{d}x_6 \delta(\sum x_j - 1), \\ \lambda_0 =& -\frac{D}{2}, \quad \lambda_6 = (D-6) - n_1 - n_2 - n_3 - n_4 - n_5, \end{aligned} \tag{105}$$

and the function

$$\begin{aligned} U(x) =& x_1 + x_2 + x_3 + x_4 + x_5, \\ V(x) =& x_1x_2p_1^2 + x_1x_3p_{12}^2 + x_1x_4p_{13}^2 + x_1x_5p_{14}^2 + x_2x_3p_2^2 + x_2x_4p_{23}^2 + x_2x_5p_{24}^2 + x_3x_4p_3^2 + x_3x_5p_{34}^2 + x_4x_5p_4^2, \\ f(x) =& (x_1+x_2+x_3+x_4+x_5)(m_1^2x_1 + m_2^2x_2 + m_3^2x_3 + m_4^2x_4 + m_5^2x_5) - x_1x_2p_1^2 - x_1x_3p_{12}^2 - x_1x_4p_{13}^2 - x_1x_5p_{14}^2 \\ & - x_2x_3p_2^2 - x_2x_4p_{23}^2 - x_2x_5p_{24}^2 - x_3x_4p_3^2 - x_3x_5p_{34}^2 - x_4x_5p_4^2, \\ F(x) =& (x_1+x_2+x_3+x_4+x_5)(m_1^2x_1 + m_2^2x_2 + m_3^2x_3 + m_4^2x_4 + m_5^2x_5 + x_6) - x_1x_2p_1^2 - x_1x_3p_{12}^2 - x_1x_4p_{13}^2 \\ & - x_1x_5p_{14}^2 - x_2x_3p_2^2 - x_2x_4p_{23}^2 - x_2x_5p_{24}^2 - x_3x_4p_3^2 - x_3x_5p_{34}^2 - x_4x_5p_4^2, \end{aligned} \tag{106}$$

where $p_{ij} \equiv p_i + p_{i+1} + \cdots p_{j-1} + p_j$. Now the matrix are given by

$$\hat{A} = \begin{bmatrix} 2m_1^2 & m_1^2+m_2^2-p_1^2 & m_1^2+m_3^2-p_{12}^2 & m_1^2+m_4^2-p_{13}^2 & m_1^2+m_5^2-p_1^2 & 1 \\ m_1^2+m_2^2-p_1^2 & 2m_2^2 & m_2^2+m_3^2-p_2^2 & m_2^2+m_4^2-p_{23}^2 & m_2^2+m_5^2-p_{24}^2 & 1 \\ m_1^2+m_3^2-p_{12}^2 & m_2^2+m_3^2-p_{23}^2 & 2m_3^2 & m_3^2+m_4^2-p_3^2 & m_3^2+m_5^2-p_{34}^2 & 1 \\ m_1^2+m_4^2-p_{13}^2 & m_2^2+m_4^2-p_{23}^2 & m_3^2+m_4^2-p_3^2 & 2m_4^2 & m_4^2+m_5^2-p_4^2 & 1 \\ m_1^2+m_5^2-p_1^2 & m_2^2+m_5^2-p_{24}^2 & m_3^2+m_5^2-p_{34}^2 & m_4^2+m_5^2-p_4^2 & 2m_5^2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

$$\hat{K}_A = \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ -a_1 & 0 & a_6 & a_7 & a_8 & a_9 \\ -a_2 & -a_6 & 0 & a_{10} & a_{11} & a_{12} \\ -a_3 & -a_7 & -a_{10} & 0 & a_{13} & a_{14} \\ -a_4 & -a_8 & -a_{11} & -a_{13} & 0 & a_{15} \\ -a_5 & -a_9 & -a_{12} & -a_{14} & -a_{15} & 0 \end{bmatrix}. \tag{107}$$

Taking $B=-1/x_6$, and inserting z_i into the IBP identities,

$$\sum_{i=1}^{6}\int\frac{\partial}{\partial x_i}\left\{z_i F^{\lambda_0}x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}x_5^{n_5}x_6^{\lambda_6+1}\right\}+\delta_5=\quad 0,\tag{108}$$

where δ_5 is given by

$$\delta_5=\sum_{i=1}^{5}\delta_{\lambda_i,0}\int \mathrm{d}\Pi^{(5)}\left\{z_i F^{\lambda_0}x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}x_5^{n_5}x_6^{\lambda_6+1}\right\}|_{x_i=0}.\tag{109}$$

1. *Deriving the recurrence relation*

Similar to the previous subsections, by expanding the IBP relation, we get

$$\begin{aligned}&\Big\{c_{n_1+1,n_2,n_3,n_4,n_5}1^+ + c_{n_1+1,n_2-1,n_3,n_4,n_5}1^+2^- + c_{n_1+1,n_2,n_3-1,n_4,n_5}1^+3^- + c_{n_1+1,n_2,n_3,n_4-1,n_5}1^+4^-\\&+c_{n_1+1,n_2,n_3,n_4,n_5-1}1^+5^- + c_{n_1,n_2+1,n_3,n_4,n_5}2^+ + c_{n_1-1,n_2+1,n_3,n_4,n_5}1^-2^+ + c_{n_1,n_2+1,n_3-1,n_4,n_5}2^+3^-\\&+c_{n_1,n_2+1,n_3,n_4-1,n_5}2^+4^- + c_{n_1,n_2+1,n_3,n_4,n_5-1}2^+5^- + c_{n_1,n_2,n_3+1,n_4,n_5}3^+ + c_{n_1-1,n_2,n_3+1,n_4,n_5}1^-3^+\\&+c_{n_1,n_2-1,n_3+1,n_4,n_5}2^-3^+ + c_{n_1,n_2,n_3+1,n_4-1,n_5}3^+4^- + c_{n_1,n_2,n_3+1,n_4,n_5-1}3^+5^- + c_{n_1,n_2,n_3,n_4+1,n_5}4^+\\&+c_{n_1-1,n_2,n_3,n_4+1,n_5}1^-4^+ + c_{n_1,n_2-1,n_3,n_4+1,n_5}2^-4^+ + c_{n_1,n_2,n_3-1,n_4+1,n_5}3^-4^+ + c_{n_1,n_2,n_3,n_4+1,n_5-1}4^+5^-\\&+c_{n_1,n_2,n_3,n_4,n_5+1}5^+ + c_{n_1-1,n_2,n_3,n_4,n_5+1}1^-5^+ + c_{n_1,n_2-1,n_3,n_4,n_5+1}2^-5^+ + c_{n_1,n_2,n_3-1,n_4,n_5+1}3^-5^+\\&+c_{n_1,n_2,n_3,n_4-1,n_5+1}4^-5^+ + c_{n_1-1,n_2,n_3,n_4,n_5}1^- + c_{n_1,n_2-1,n_3,n_4,n_5}2^- + c_{n_1,n_2,n_3-1,n_4,n_5}3^-\\&+c_{n_1,n_2,n_3,n_4-1,n_5}4^- + c_{n_1,n_2,n_3,n_4,n_5-1}5^- + c_{n_1,n_2,n_3,n_4,n_5}\Big\}i_{n_1,n_2,n_3,n_4,n_5}+\delta_5=0.\end{aligned}\tag{110}$$

We can choose particular values for parameters a_2 to a_{15} to ensure the coefficients of the first three line of (110) equal zero. The solution is

$$\begin{aligned}&a_2=\frac{-a_1}{\Delta_{\rm pen}}|\tilde{A}_{13,56}|,\;\; a_3=\frac{a_1}{\Delta_{\rm pen}}|\tilde{A}_{14,56}|,\;\; a_4=\frac{-a_1}{\Delta_{\rm pen}}|\tilde{A}_{15,56}|,\;\; a_5=\frac{a_1}{\Delta_{\rm pen}}|\tilde{A}_{16,56}|,\;\; a_6=\frac{a_1}{\Delta_{\rm pen}}|\tilde{A}_{23,56}|,\\&a_7=\frac{-a_1}{\Delta_{\rm pen}}|\tilde{A}_{24,56}|,\;\; a_8=\frac{a_1}{\Delta_{\rm pen}}|\tilde{A}_{25,56}|,\;\; a_9=\frac{-a_1}{\Delta_{\rm pen}}|\tilde{A}_{26,56}|,\;\; a_{10}=\frac{a_1}{\Delta_{\rm pen}}|\tilde{A}_{34,56}|,\;\; a_{11}=\frac{-a_1}{\Delta_{\rm pen}}|\tilde{A}_{35,56}|,\\&a_{12}=\frac{a_1}{\Delta_{\rm pen}}|\tilde{A}_{36,56}|,\;\; a_{13}=\frac{a_1}{\Delta_{\rm pen}}|\tilde{A}_{45,56}|,\;\; a_{14}=\frac{-a_1}{\Delta_{\rm pen}}|\tilde{A}_{46,56}|,\;\; a_{15}=\frac{a_1}{\Delta_{\rm pen}}|\tilde{A}_{56,56}|,\end{aligned}\tag{111}$$

where

$$\Delta_{\rm pen}=\begin{vmatrix}A_{31}&A_{32}&A_{33}&A_{34}\\A_{41}&A_{42}&A_{43}&A_{44}\\A_{51}&A_{52}&A_{53}&A_{54}\\A_{61}&A_{62}&A_{63}&A_{64}\end{vmatrix}.$$

Subsequently, we get

$$\begin{aligned}&\Big\{c_{n_1,n_2,n_3,n_4,n_5+1;r}5^+ + c_{n_1-1,n_2,n_3,n_4,n_5+1;r}1^-5^+ + c_{n_1,n_2-1,n_3,n_4,n_5+1;r}2^-5^+ + c_{n_1,n_2,n_3-1,n_4,n_5+1;r}3^-5^+\\&+c_{n_1,n_2,n_3,n_4-1,n_5+1;r}4^-5^+c_{n_1-1,n_2,n_3,n_4,n_5;r}1^- + c_{n_1,n_2-1,n_3,n_4,n_5;r}2^- + c_{n_1,n_2,n_3-1,n_4,n_5;r}3^-\\&+c_{n_1,n_2,n_3,n_4-1,n_5;r}4^- + c_{n_1,n_2,n_3,n_4,n_5-1;r}5^- + c_{n_1,n_2,n_3,n_4,n_5;r}\Big\}i_{\lambda_0;n_1,n_2,n_3,n_4,n_5}+\delta_{5;r}=0,\end{aligned}\tag{112}$$

where we define

$$i^{+}i_{\lambda_0;n_1,n_2,n_3,n_4,n_5} \equiv i_{\lambda_0;n_1,\cdots n_i+1,\cdots n_5}, \qquad i^{-}i_{\lambda_0;n_1,n_2,n_3,n_4,n_5} \equiv i_{\lambda_0;n_1,\cdots n_i-1,\cdots n_5}, \tag{113}$$

with the coefficients

$$\begin{aligned} c_{0,0,0,0,1} =& Q_{65;r}\lambda_6,\ c_{-1,0,0,0,1}=n_1Q_{15;r},\ c_{0,-1,0,0,1}=n_2Q_{25;r},\ c_{0,0,-1,0,1}=n_3Q_{35;r},\ c_{0,0,0,-1,1}=n_4Q_{45;r},\\ c_{-1,0,0,0,0} =& n_1Q_{16;r},\ c_{0,-1,0,0,0}=n_2Q_{26;r},\ c_{0,0,-1,0,0}=n_3Q_{36;r},\ c_{0,0,0,-1,0}=n_4Q_{46;r},\ c_{0,0,0,0,-1}=n_5Q_{56;r},\\ c_{00000;r} =& \mathrm{Tr}\hat{Q}_{ij;r}+((D-6))Q_{66;rr}-\frac{D}{2}+n_1Q_{11;r}+n_2Q_{22;r}+n_3Q_{33;r}+n_4Q_{44;r}+n_5Q_{55;r}, \end{aligned} \tag{114}$$

while the matrix $\hat{Q}$ becomes

$$\hat{Q}_r = \frac{1}{\Delta_{\rm pen}}\begin{bmatrix} \frac{1}{2}\Delta_{\rm pen} & 0 & 0 & 0 & a_1|\tilde{A}_{1,6}| & a_1|\tilde{A}_{1,5}| \\ 0 & \frac{1}{2}\Delta_{\rm pen} & 0 & 0 & -a_1|\tilde{A}_{2,6}| & -a_1|\tilde{A}_{2,5}| \\ 0 & 0 & \frac{1}{2}\Delta_{\rm pen} & 0 & a_1|\tilde{A}_{3,6}| & a_1|\tilde{A}_{3,5}| \\ 0 & 0 & 0 & \frac{1}{2}\Delta_{\rm pen} & -a_1|\tilde{A}_{4,6}| & -a_1|\tilde{A}_{4,5}| \\ 0 & 0 & 0 & 0 & \frac{1}{2}\Delta_{\rm pen}+a_1|\tilde{A}_{5,6}| & a_1|\tilde{A}_{5,5}| \\ 0 & 0 & 0 & 0 & -a_1|\tilde{A}_{6,6}| & \frac{1}{2}\Delta_{\rm pen}-a_1|\tilde{A}_{6,5}| \end{bmatrix}.$$

F. Reducing the δ_5 term

Similar to the former situation, the $\delta_{6;r}$ term is given by

$$\begin{aligned} \delta_{5;r} =& Q_{11;r}\delta_{n_1,-1}i_{-1,n_2,n_3,n_4,n_5}+Q_{12;r}\delta_{n_1,0}i_{-1,n_2+1,n_3,n_4,n_5}+Q_{13;r}\delta_{n_1,0}i_{-1,n_2,n_3+1,n_4,n_5}+Q_{14;r}\delta_{n_1,0}i_{-1,n_2,n_3,n_4+1,n_5}+Q_{15;r}\delta_{n_1,0}i_{-1,n_2,n_3,n_4,n_5+1}\\ &+Q_{16;r}\delta_{n_1,0}i_{-1,n_2,n_3,n_4,n_5}+Q_{21;r}\delta_{n_2,0}i_{n_1+1,-1,n_3,n_4,n_5}+Q_{22;r}\delta_{n_2,-1}i_{n_1,-1,n_2,n_3,n_4,n_5}+Q_{23;r}\delta_{n_2,0}i_{n_1,-1,n_3+1,n_4,n_5}+Q_{24;r}\delta_{n_2,0}i_{n_1,-1,n_3,n_4+1,n_5}\\ &+Q_{25;r}\delta_{n_2,0}i_{n_1,-1,n_3,n_4,n_5+1}+Q_{26;r}\delta_{n_2,0}i_{n_1,-1,n_3,n_4,n_5}+Q_{31;r}\delta_{n_3,0}i_{n_1+1,n_2,-1,n_4,n_5}Q_{32;r}\delta_{n_3,0}i_{n_1,n_2+1,-1,n_4,n_5}+Q_{33;r}\delta_{n_3,-1}i_{n_1,n_2,-1,n_4,n_5}\\ &+Q_{34;r}\delta_{n_3,0}i_{n_1,n_2,-1,n_4+1,n_5}+Q_{35;r}\delta_{n_3,0}i_{n_1,n_2,-1,n_4,n_5+1}+Q_{36;r}\delta_{n_3,0}i_{n_1,n_2,-1,n_4,n_5}Q_{41;r}\delta_{n_4,0}i_{n_1+1,n_2,n_3,-1,n_5}+Q_{42;r}\delta_{n_4,0}i_{n_1,n_2+1,n_3,-1,n_5}\\ &+Q_{43;r}\delta_{n_4,0}i_{n_1,n_2,n_3+1,-1,n_5}+Q_{44;r}\delta_{n_4,-1}i_{n_1,n_2,n_3,-1,n_5}+Q_{45;r}\delta_{n_4,0}i_{n_1,n_2,n_3,-1,n_5+1}+Q_{46;r}\delta_{n_4,0}i_{n_1,n_2,n_3,-1,n_5}+Q_{51;r}\delta_{n_5,0}i_{n_1+1,n_2,n_3,n_4,-1}\\ &+Q_{52;r}\delta_{n_5,0}i_{n_1,n_2+1,n_3,n_4,-1}+Q_{53;r}\delta_{n_5,0}i_{n_1,n_2,n_3+1,n_4,-1}+Q_{54;r}\delta_{n_5,0}i_{n_1,n_2,n_3,n_4+1,-1}+Q_{55;r}\delta_{n_5,-1}i_{n_1,n_2,n_3,n_4,-1}+Q_{56;r}\delta_{n_5,0}i_{n_1,n_2,n_3,n_4,n_5}. \end{aligned} \tag{115}$$

G. Example: $I_5(1,1,1,1,2)$

Setting $n_1=n_2=n_3=n_4=n_5=0$, we get the IBP recurrence relation (other coefficients are all zero)

$$c_{0,0,0,0,1}i_{\lambda_0;0,0,0,0,1}+c_{0,0,0,0,0}i_{\lambda_0;0,0,0,0,0}+\delta_{5;00000}=0, \tag{116}$$

where $\delta_{5;00000}\equiv\delta_{5;r}|_{n_1=n_2=n_3=n_4=n_5=0}$.

Comparing them with our scalar basis, we have the result

$$\begin{aligned} I_5(1,1,1,1,2) =& c_{5\to5}I_5(1,1,1,1,1)+c_{5\to01111}I_4(0,1,1,1,1)+c_{5\to10111}I_5(1,0,1,1,1)\\ &+c_{5\to11011}I_5(1,1,0,1,1)+c_{5\to11101}I_5(1,1,1,0,1)+c_{5\to11110}I_5(1,1,1,1,0)\\ &+c_{5\to20111}I_5(2,0,1,1,1)+c_{5\to21011}I_5(2,1,0,1,1)+c_{5\to21101}I_5(2,1,1,0,1) \end{aligned}$$

$$
\begin{aligned}
&+c_{5\to 21110}I_5(2,1,1,1,0)+c_{5\to 02111}I_5(0,2,1,1,1)+c_{5\to 12011}I_5(1,2,0,1,1)\\
&+c_{5\to 12101}I_5(1,2,1,0,1)+c_{5\to 12110}I_5(1,2,1,1,0)+c_{5\to 01211}I_5(0,1,2,1,1)\\
&+c_{5\to 10211}I_5(1,0,2,1,1)+c_{5\to 11201}I_5(1,1,2,0,1)+c_{5\to 11210}I_5(1,1,2,1,0)\\
&+c_{5\to 01121}I_5(0,1,1,2,1)+c_{5\to 10121}I_5(1,0,1,2,1)+c_{5\to 11021}I_5(1,1,0,2,1)\\
&+c_{5\to 11120}I_5(1,1,1,2,0)+c_{5\to 01112}I_5(0,1,1,1,2)+c_{5\to 10112}I_5(1,0,1,1,2)\\
&+c_{5\to 11012}I_5(1,1,0,1,2)+c_{5\to 11102}I_5(1,1,1,0,2),
\end{aligned}
\tag{117}
$$

with the coefficients

$$
\begin{aligned}
&c_{5\to 5}=\frac{(D-6)c_{0,0,0,0,0}}{c_{0,0,0,0,1}},\quad c_{5\to 01111}=\frac{(D-6)(5-D)Q_{16;r}}{c_{0,0,0,0,1}},\quad c_{5\to 4;10111}=\frac{(D-6)(5-D)Q_{26;r}}{c_{0,0,0,0,1}},\\
&c_{5\to 4;11011}=\frac{(D-6)(5-D)Q_{36;r}}{c_{0,0,0,0,1}},\quad c_{5\to 4;11101}=\frac{(D-6)(5-D)Q_{46;r}}{c_{0,0,0,0,1}},\quad c_{5\to 4;11110}=\frac{(D-6)(5-D)Q_{56;r}}{c_{0,0,0,0,1}},\\
&c_{5\to 20111}=\frac{(D-6)Q_{21;r}}{c_{0,0,0,0,1}},\quad c_{5\to 21011}=\frac{(D-6)Q_{31;r}}{c_{0,0,0,0,1}},\quad c_{5\to 21101}=\frac{(D-6)Q_{41;r}}{c_{0,0,0,0,1}},\quad c_{5\to 21110}=\frac{(D-6)Q_{51;r}}{c_{0,0,0,0,1}},\\
&c_{5\to 02111}=\frac{(D-6)Q_{12;r}}{c_{0,0,0,0,1}},\quad c_{5\to 12011}=\frac{(D-6)Q_{32;r}}{c_{0,0,0,0,1}},\quad c_{5\to 12101}=\frac{(D-6)Q_{42;r}}{c_{0,0,0,0,1}},\quad c_{5\to 12110}=\frac{(D-6)Q_{52;r}}{c_{0,0,0,0,1}},\\
&c_{5\to 01211}=\frac{(D-6)Q_{13;r}}{c_{0,0,0,0,1}},\quad c_{5\to 10211}=\frac{(D-6)Q_{23;r}}{c_{0,0,0,0,1}},\quad c_{5\to 11201}=\frac{(D-6)Q_{43;r}}{c_{0,0,0,0,1}},\quad c_{5\to 11210}=\frac{(D-6)Q_{53;r}}{c_{0,0,0,0,1}},\\
&c_{5\to 01121}=\frac{(D-6)Q_{14;r}}{c_{0,0,0,0,1}},\quad c_{5\to 10121}=\frac{(D-6)Q_{24;r}}{c_{0,0,0,0,1}},\quad c_{5\to 11021}=\frac{(D-6)Q_{34;r}}{c_{0,0,0,0,1}},\quad c_{5\to 11120}=\frac{(D-6)Q_{54;r}}{c_{0,0,0,0,1}},\\
&c_{5\to 01112}=\frac{(D-6)Q_{15;r}}{c_{0,0,0,0,1}},\quad c_{5\to 10112}=\frac{(D-6)Q_{25;r}}{c_{0,0,0,0,1}},\quad c_{5\to 11012}=\frac{(D-6)Q_{35;r}}{c_{0,0,0,0,1}},\quad c_{5\to 11102}=\frac{(D-6)Q_{45;r}}{c_{0,0,0,0,1}}.
\end{aligned}
\tag{118}
$$

The final step is to reduce the coefficients of the general boxes to the scalar basis.

After this reduction, we obtain the final solution.

$$
\begin{aligned}
I_5(1,1,1,1,2)=&c_{5\to 5}I_5(1,1,1,1,1)+c_{5\to 4;\bar{1}}I_5(0,1,1,1,1)+c_{5\to 4;\bar{2}}I_5(1,0,1,1,1)+c_{5\to 4;\bar{3}}I_5(1,1,0,1,1)\\
&+c_{5\to 4;\bar{4}}I_5(1,1,1,0,1)+c_{5\to 4;\bar{5}}I_5(1,1,1,1,0)+c_{5\to 3;\bar{1}\bar{2}}I_5(0,0,1,1,1)+c_{5\to 3;\bar{1}\bar{3}}I_5(0,1,0,1,1)\\
&+c_{5\to 3;\bar{1}\bar{4}}I_5(0,1,1,0,1)+c_{5\to 3;\bar{1}\bar{5}}I_5(0,1,1,1,0)+c_{5\to 3;\bar{2}\bar{3}}I_5(1,0,0,1,1)+c_{5\to 3;\bar{2}\bar{4}}I_5(1,0,1,0,1)\\
&+c_{5\to 3;\bar{2}\bar{5}}I_5(1,0,1,1,0)c_{5\to 3;\bar{3}\bar{4}}I_5(1,1,0,0,1)+c_{5\to 3;\bar{3}\bar{5}}I_5(1,1,0,1,0)+c_{5\to 3;\bar{4}\bar{5}}I_5(1,1,1,0,0)\\
&+c_{5\to 2;D_1D_2}I_5(1,1,0,0,0)+c_{5\to 2;D_1D_3}I_5(1,0,1,0,0)+c_{5\to 2;D_1D_4}I_5(1,0,0,1,0)+c_{5\to 2;D_1D_5}I_5(1,0,0,0,1)\\
&+c_{5\to 2;D_2D_3}I_5(0,1,1,0,0)+c_{5\to 2;D_2D_4}I_5(0,1,0,1,0)+c_{5\to 2;D_2D_5}I_5(0,1,0,0,1)+c_{5\to 2;D_3D_4}I_5(0,0,1,1,0)\\
&+c_{5\to 2;D_3D_5}I_5(0,0,1,0,1)+c_{5\to 2;D_4D_5}I_5(0,0,0,1,1)+c_{5\to 1;D_1}I_5(1,0,0,0,0)+c_{5\to 1;D_2}I_5(0,1,0,0,0)\\
&+c_{5\to 1;D_3}I_5(0,0,1,0,0)+c_{5\to 1;D_4}I_5(0,0,0,1,0)+c_{5\to 1;D_5}I_5(0,0,0,0,1),
\end{aligned}
\tag{119}
$$

with the coefficients given in the attached Mathematica notebook. Now, all coefficients are complete.

IV. ANALYTIC RESULTS OF THE COEFFICIENTS

The analytic results are provided in the Mathematica notebooks, which are publicly available at https://github.com/Wanghongbin123/oneloop_parametric.

V. SUMMARY AND FURTHER DISCUSSION

In this paper, we consider one-loop scalar integrals in the parametric representation given by Chen. However, in the recurrence relation, there are typically several terms that we do not want as well as terms with dimensional shifting in general, which makes calculations difficult and inefficient. In Chen's later paper [2], he used a method based on non-commutative algebra to cancel the dimension shift. Unlike other methods, the one-loop case involves a straightforward method in which linear equation systems are solved to simplify the IBP recurrence relation in the parametric representation. Benefiting from the fact that F is a homogeneous function of x_i with a degree of two in the one-loop situation, we can solve x_i using $\partial F/\partial x_i$ with several free parameters. Then, combining all the IBP identities with a particular factor z_i and choosing particular values for the free parameters, we succeed in canceling the dimension shift and terms with higher total power. As a complement to the tadpole coefficients of the reduction explored within our previous paper, we calculate several examples and provide an analytic result of the reduction.

For further research, there are several factors to be considered. In our calculations, the constructed coefficients z_i are not polynomial since they have a denominator with the form x^{γ}_{n+1}; therefore, we cannot directly use the technique of syzygy. Moreover, the application of Chen's method to a higher loop is definitely another future research direction. For this case, the homogeneous function $F(x)$ is of degree $L+1$, where L is the number of loops. For the high loop case, we should consider how to construct the coefficients z_i efficiently and find a relation similar to (37) to cancel the terms we do not need. Finally, the sub-topologies are entirely decided by the boundary term in the parametric representation, which may lead to simplification of calculation.

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References

[1] W. Chen, JHEP **02**, 115 (2020), arXiv:1902.10387
[2] W. Chen, *Reduction of Feynman Integrals in the Parametric Representation II: Reduction of Tensor Integrals*, arXiv: 1912.08606
[3] B. Feng and H. Wang, *Reduction of one-loop integrals with higher poles by unitarity cut method*, arXiv: 2104.00922
[4] G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160**, 151 (1979)
[5] V. A. Smirnov, Springer Tracts in Modern Physics **211**, (2005)
[6] F. V. Tkachov, Phys. Lett. B **100**, 65 (1981)
[7] K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192**, 159 (1981)
[8] Z. Bern, L. J. Dixon, D. C. Dunbar *et al.*, Nucl. Phys. B **425**, 217 (1994), arXiv:hep-ph/9403226
[9] Z. Bern, L. J. Dixon, D. C. Dunbar *et al.*, Nucl. Phys. B **435**, 59 (1995), arXiv:hep-ph/9409265
[10] R. Britto, F. Cachazo, and B. Feng, Nucl. Phys. B **725**, 275 (2005), arXiv:hep-th/0412103
[11] F. Cachazo, P. Svrcek, and E. Witten, JHEP **10**, 074 (2004), arXiv:hep-th/0406177
[12] R. Britto, E. Buchbinder, F. Cachazo *et al.*, Phys. Rev. D **72**, 065012 (2005), arXiv:hep-ph/0503132
[13] C. Anastasiou, R. Britto, B. Feng *et al.*, Phys. Lett. B **645**, 213 (2007), arXiv:hep-ph/0609191
[14] R. Britto and B. Feng, Phys. Rev. D **75**, 105006 (2007), arXiv:hep-ph/0612089
[15] C. Anastasiou, R. Britto, B. Feng *et al.*, JHEP **03**, 111 (2007), arXiv:hep-ph/0612277
[16] R. Britto, B. Feng, and P. Mastrolia, Phys. Rev. D **73**, 105004 (2006), arXiv:hep-ph/0602178
[17] R. Britto and B. Feng, Phys. Lett. B **681**, 376 (2009), arXiv:0904.2766
[18] Y. Zhang, *An Introduction to Integrals with Uniformally Transcendental Weights, Canonical Differential Equation, Symbols and Polylogarithms*, http://staff.ustc.edu.cn/~yzhphy/teaching/summer2021/UT_2021.pdf, 2021
[19] J. M. Henn, J. Phys. A **48**, 153001 (2015), arXiv:1412.2296
[20] P. A. Baikov, Phys. Lett. B **385**, 404 (1996), arXiv:hep-ph/9603267
[21] P. A. Baikov, *Explicit solutions of n loop vacuum integral recurrence relations*, hep-ph/9604254
[22] Z. Bern, L. J. Dixon, and D. A. Kosower, Phys. Lett. B **302**, 299 (1993), arXiv:hep-ph/9212308
[23] Z. Bern, L. J. Dixon, and D. A. Kosower, Nucl. Phys. B **412**, 751 (1994), arXiv:hep-ph/9306240
[24] J. Gluza, K. Kajda, and D.A. Kosower, *Towards a basis for planar two-loop integrals*, arXiv: 1009.0472
[25] K. J. Larsen and Y. Zhang, *Integration-by-parts reductions from unitarity cuts and algebraic geometry*, arXiv: 1511.01071
[26] K. J. Larsen and Y. Zhang, *Integration-by-parts reductions from the viewpoint of computational algebraic geometry*, arXiv: 1606.09447
[27] K. J. Larsen and Y. Zhang, Phys. Rev. D **93**, 041701 (2016), arXiv:1511.01071
[28] K. J. Larsen and Y. Zhang, PoS **LL2016**, 029 (2016), arXiv:1606.09447
[29] Y. Zhang, *Lecture Notes on Multi-loop Integral Reduction and Applied Algebraic Geometry*, 12, 2016 [1612.02249]
[30] Y. Jiang and Y. Zhang, JHEP **03**, 087 (2018), arXiv:1710.04693
[31] A.V. Smirnov and F. S. Chuharev, Comput. Phys. Commun. **247**, 106877 (2020), arXiv:1901.07808
[32] R. N. Lee, J. Phys. Conf. Ser. **523**, 012059 (2014), arXiv:1310.1145