Validity of thermodynamic laws and weak cosmic censorship for AdS black holes and black holes in a cavity*

Peng Wang(王鹏)[1†] Houwen Wu(伍厚文)[1‡] Shuxuan Ying(应蜀翾)[1,2§]

[1]Center for Theoretical Physics, College of Physics, Sichuan University, Chengdu 610064, China
[2]Department of Physics, Chongqing University, Chongqing 401331, China

Abstract: By throwing a test charged particle into a Reissner-Nordstrom (RN) black hole, we test the validity of the first and second laws of thermodynamics and the weak cosmic censorship conjecture (WCCC) with two types of boundary conditions: the asymptotically anti-de Sitter (AdS) space and a Dirichlet cavity wall placed in an asymptotically flat space. For the RN-AdS black hole, the second law of thermodynamics is satisfied, and the WCCC is violated for both extremal and near-extremal black holes. For the RN black hole in a cavity, the entropy can either increase or decrease depending on the change in the charge, and the WCCC is satisfied/violated for the extremal/near-extremal black hole. Our results indicate that there may be a connection between the black hole thermodynamics and the boundary condition imposed on the black hole.

Keywords: weak cosmic censorship, Reissner-Nordstrom black hole, black hole thermodynamics

DOI: 10.1088/1674-1137/abeb5e

I. INTRODUCTION

Studying the thermodynamic properties of black holes can have a deep impact upon the understanding of quantum gravity. Penrose first noticed that a particle can extract energy from a black hole with an ergosphere [1], which led to the discovery of the irreducible mass [2-4]. The square of the irreducible mass of a black hole can be interpreted as the black hole entropy [5, 6]. Later, analogous to the laws of thermodynamics, the four laws of black hole mechanics were proposed [7]. With the advent of the AdS/CFT correspondence [8], there has been substantial interest in studying the thermodynamics and phase structure of AdS black holes [9-22], where some intriguing phase behavior, e.g., reentrant phase transitions and tricritical points, has been found. It is also worth noting that black holes can become thermally stable in AdS space because the AdS boundary acts as a reflecting wall.

Along with the development of black hole thermodynamics, the weak cosmic censorship conjecture (WCCC) was proposed to hide singularities by event horizons [1]. If the WCCC is valid, the singularities cannot be seen by observers at the future null infinity. To test the validity of the WCCC, Wald constructed a gedanken experiment to destroy an extremal Kerr-Newman black hole by overcharging or overspinning the black hole via throwing a test particle into it [23]. Nevertheless, the extremal Kerr-Newman black hole was shown to be incapable of capturing particles with sufficient charge or angular momentum to overcharge or overspin the black hole. Later, near-extremal charged/rotating black holes were found to be overcharged/overspun by absorbing a particle [24-26], and hence, the WCCC was violated. However, subsequent studies showed that the WCCC might be still valid if the backreaction and self-force effects were considered [27-32]. As a general proof of the WCCC is still lacking, its validity has been tested in various black holes [33-72]. In particular, the thermodynamics and WCCC have been considered for a Reissner-Nordstrom (RN)-AdS black hole via charged particle absorption in the normal and extended phase spaces [46, 73]. In the normal phase space, in which the cosmological constant is fixed, the first and second laws of thermodynamics were shown to be satisfied while the WCCC was violated even for an extremal RN-AdS black hole.

Instead of the AdS boundary, York showed that placing Schwarzschild black holes inside a cavity, on the wall of which the metric is fixed, can make them thermally stable [74]. The thermodynamics and phase structure of RN black holes in a cavity were studied in a grand canonical ensemble [75] and a canonical ensemble [76, 77]. It was found that the Schwarzschild and RN

Received 16 December 2020; Accepted 2 March 2021; Published online 22 March 2021

* Supported in part by National Natural Science Foundation of China (11875196, 11375121, 11005016)

† E-mail: pengw@scu.edu.cn
‡ E-mail: iverwu@scu.edu.cn
§ E-mail: ysxuan@cqu.edu.cn

black holes in a cavity have quite similar phase structure and transition to these of the AdS counterparts. Afterwards, various black brane systems [78-83], a Gauss-Bonnet black hole [84], hairy black holes [85-88], and boson stars [89-92] in a cavity were extensively investigated, and it was also shown that the behavior of the gravity systems in a cavity is strikingly similar to that of their counterparts in AdS gravity. However, we have recently studied the phase structure of Born-Infeld black holes enclosed in a cavity [93, 94] and thermodynamic geometry of RN black holes in a cavity [95] and found their behavior is dissimilar from that of the corresponding AdS black holes. Note that the thermodynamics and critical behavior of de Sitter black holes in a cavity were investigated in [96, 97].

Although a considerable amount of work is in progress on the thermodynamic laws and WCCC for various black holes of different theories of gravity in spacetimes with differing asymptotics, little is known about the second law of thermodynamics and WCCC for a black hole enclosed in a cavity. As RN-AdS black holes and RN black holes in a cavity are thermally stable, they provide appropriate scenarios to explore whether or not the thermodynamic laws and WCCC are sensitive to the boundary condition of black holes. To this end, we study the thermodynamic laws and WCCC for an RN black hole in a cavity in this paper.

The rest of this paper is organized as follows. In Section II, we review the discussion of the thermodynamic laws and WCCC for an RN-AdS black hole to be self-contained and introduce the method used in this paper. The thermodynamic laws and WCCC of an RN black hole in a cavity are then tested via the absorption of a charged particle in Section III. We summarize our results with a brief discussion in Section IV. For simplicity, we set $G=\hbar=c=k_B=1$ in this paper.

II. RN-ADS BLACK HOLE

In this section, we discuss the first and second laws of thermodynamics and WCCC for an RN-AdS black hole by throwing a test particle into the black hole. First, we consider the motion of a test particle of energy E, charge q, and mass m in a four-dimensional charged static black hole with the line element

$$\mathrm{d}s^2=-f(r)\mathrm{d}t^2+\frac{1}{f(r)}\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right), \tag{1}$$

and the electromagnetic potential A_μ,

$$A_\mu=A_t(r)\delta_{\mu t}. \tag{2}$$

We also suppose that the outermost horizon of the black hole is at $r=r_+$, where $f(r_+)=0$. In [58], the Hamilton-Jacobi equation of the test particle was given by

$$-\frac{[E+qA_t(r)]^2}{f(r)}+\frac{[P^r(r)]^2}{f(r)}+\frac{L^2}{r^2}=m^2, \tag{3}$$

where L and $P^r(r)$ are the particle's angular momentum and radial momentum, respectively. It was shown in [98, 99] that $P^r(r_+)$ is finite and proportional to the Hawking temperature of the black hole. Eqn. (3) gives

$$E=-qA_t(r)+\sqrt{f(r)\left(m^2+\frac{L^2}{r^2}\right)+[P^r(r)]^2}, \tag{4}$$

where we choose the positive sign in front of the square root as the energy of the particle is required to be a positive value [2, 4]. At the horizon $r=r_+$, the above equation reduces to

$$E=q\Phi+\left|P^r(r_+)\right|, \tag{5}$$

where $\Phi\equiv-A_t(r_+)$ is the electric potential of the black hole. Eqn. (5) relates the energy of the particle to its radial momentum and potential energy just before the particle enters the horizon.

To check whether a particle can reach or exist near the black hole horizon, we can rewrite Eqn. (4) as the radial equation of motion,

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2=\frac{[E+qA_t(r)]^2}{m^2}-f(r)\left(1+\frac{L^2}{m^2r^2}\right), \tag{6}$$

where we use $P^r(r)=m\mathrm{d}r/\mathrm{d}\tau$, and τ is the affine parameter along the worldline. Note that a particle can exist in the region where $(\mathrm{d}r/\mathrm{d}\tau)^2\geqslant 0$, and $\mathrm{d}r/\mathrm{d}\tau=0$ corresponds to a turning point. Specifically, for a particle existing at the event horizon, Eqn. (6) gives that $(\mathrm{d}r/\mathrm{d}\tau)^2|_{r=r_+}\geqslant 0$, and hence, $E\geqslant q\Phi$. Furthermore, if the particle falls into the black hole, one has $(\mathrm{d}r/\mathrm{d}\tau)^2>0$ at the event horizon, which leads to $E>q\Phi$. In short, for a particle of energy E and charge q around a black hole of potential Φ, $E>q\Phi$ provides a lower bound E_{low} on E, which ensures that the particle exists near the event horizon and is absorbed by the black hole.

For an RN-AdS black hole, the metric function $f(r)$ and electric potential Φ are

$$f(r;M,Q)=1-\frac{2M}{r}+\frac{Q^2}{r^2}+\frac{r^2}{l^2}\quad\text{and}\quad\Phi=\frac{Q}{r_+}, \tag{7}$$

respectively, where M and Q are the mass and charge of the black hole, respectively, and l is the AdS radius. Here, the parameters M and Q are put explicitly as arguments of the function $f(r)$ for subsequent convenience. With fixed charge Q, the mass $M_e(Q)$ and horizon radius $r_e(Q)$ of the extremal RN-AdS black hole are determined by $f(r_e(Q);M,Q) = \mathrm{d}f(r;M,Q)/\mathrm{d}r|_{r=r_e(Q)} = 0$, which gives

$$M_e(Q) = \frac{\sqrt{6}l}{18}\left(2+\sqrt{1+12Q^2/l^2}\right)\sqrt{\sqrt{1+12Q^2/l^2}-1},$$
$$r_e(Q) = \frac{l}{\sqrt{6}}\sqrt{\sqrt{1+12Q^2/l^2}-1}. \tag{8}$$

Like an RN black hole, when $M \geqslant M_e(Q)$, the RN-AdS black hole solution has an event horizon at $r = r_+(M,Q)$, which is obtained by solving $f(r;M,Q) = 0$ [73]. Otherwise, the event horizon disappears, and a naked singularity emerges, which leads to violation of the WCCC. If the event horizon exists, one can define the black hole temperature T and entropy S as

$$T = \frac{1}{4\pi}\frac{\partial f(r;M,Q)}{\partial r}\bigg|_{r=r_+(M,Q)} \quad \text{and} \quad S = \pi r_+^2(M,Q). \tag{9}$$

Suppose one starts with an initial black hole of mass M and charge Q with $M \geqslant M_e(Q)$ and throws a test particle of energy $E \ll M$ and charge $q \ll Q$ into the black hole. After the particle is absorbed, the final configuration has mass $M+\mathrm{d}M$ and charge $Q+\mathrm{d}Q$. The energy and charge conservation of the absorbing process gives

$$\mathrm{d}M = E \quad \text{and} \quad \mathrm{d}Q = q, \tag{10}$$

where E and q are related via Eqn. (5). If $M+E \geqslant M_e(Q+q)$, there exists an event horizon in the final black hole solution, which hides the naked singularity. However, for $M+E < M_e(Q+q)$, the naked singularity can be seen by distant observers due to the absence of an event horizon.

We first check the first and second laws of thermodynamics for an RN-AdS black hole during the absorption. In this case, the final black hole solution should have an event horizon at $r = r_+(M,Q)+\mathrm{d}r_+$ such that thermodynamic variables are well defined. Therefore, the horizon radius, mass, and charge of the final black hole satisfy

$$f(r_+(M,Q)+\mathrm{d}r_+;M+\mathrm{d}M,Q+\mathrm{d}Q) = 0, \tag{11}$$

which leads to

$$\frac{\partial f(r;M,Q)}{\partial r}\bigg|_{r=r_+(M,Q)}\mathrm{d}r_+ + \frac{\partial f(r;M,Q)}{\partial M}\bigg|_{r=r_+(M,Q)}\mathrm{d}M + \frac{\partial f(r;M,Q)}{\partial Q}\bigg|_{r=r_+(M,Q)}\mathrm{d}Q = 0. \tag{12}$$

Using Eqns. (5), (7), (9), and (10), one finds that the above equation gives

$$\left|P^r(r_+(M,Q))\right| = T\mathrm{d}S. \tag{13}$$

For an extremal black hole with $T=0$, because $P^r(r_+(M,Q)) \propto T$, Eqn. (13) is trivial. Nevertheless, for a non-extremal RN-AdS black hole with $T>0$, the variation of entropy is

$$\mathrm{d}S = \frac{|P^r(r_+(M,Q))|}{T} > 0, \tag{14}$$

which means the second law of thermodynamics is satisfied. Moreover, plugging Eqns. (10) and (13) into Eqn. (5) yields the first law of thermodynamics:

$$\mathrm{d}M = \Phi\mathrm{d}Q + T\mathrm{d}S. \tag{15}$$

To test the WCCC, we consider an extremal or near-extremal RN-AdS black hole and check whether throwing a charged particle can overcharge the black hole. To overcharge the black hole, the final configuration should exceed extremality:

$$M+E < M_e(Q+q), \tag{16}$$

which, together with Eqn. (5), gives the constraints on the energy of the particle

$$E_{\text{low}} \equiv \frac{qQ}{r_+(M,Q)} < E < M_e(Q+q) - M \equiv E_{\text{up}}. \tag{17}$$

As $q \ll Q$, we can expand $M_e(Q+q)$ and obtain

$$E_{\text{up}} \simeq M_e(Q) - M + M_e'(Q)q + \frac{M_e''(Q)}{2}q^2, \tag{18}$$

where

$$M_e'(Q) = \frac{Q}{r_e(Q)} \quad \text{and} \quad M_e''(Q) = \frac{\sqrt{\sqrt{1+12Q^2/l^2}-1}}{l\sqrt{2/3+8Q^2/l^2}} > 0. \tag{19}$$

If the initial black hole is extremal, the lower and upper bounds on E become

$$E_{\text{low}} = \frac{qQ}{r_e(Q)} \quad \text{and} \quad E_{\text{up}} \simeq M_e'(Q)q + \frac{M_e''(Q)}{2}q^2 > E_{\text{low}}. \tag{20}$$

Therefore, there always exists a test charged particle with $E_{\text{low}} < E < E_{\text{up}}$ that can overcharge the extremal RN-AdS black hole. For a near-extremal RN-AdS black hole with Q and $M = M_e(Q)+\epsilon^2$, the lower and upper bounds on E

become

$$E_{\rm low} \simeq M_e'(Q)q - A(Q)q\epsilon \quad \text{and}$$
$$E_{\rm up} \simeq M_e'(Q)q + \frac{M_e''(Q)}{2}q^2 - \epsilon^2, \tag{21}$$

where $A(Q) > 0$ is some function of Q, and ϵ is a small parameter. To have $E_{\rm up} > E_{\rm low}$, we find

$$q \equiv a\epsilon > \frac{-A(Q) + \sqrt{A^2(Q) + 2M_e''(Q)}}{M_e''(Q)}\epsilon. \tag{22}$$

The constraints expressed in (17) give the energy E of the particle,

$$E = M_e'(Q)a\epsilon + b\epsilon^2 \quad \text{with} \quad -A(Q)a < b < \frac{M_e''(Q)}{2}a^2 - 1. \tag{23}$$

Therefore, a charged particle with its charge and energy satisfying Eqns. (22) and (23), respectively, can overcharge the near-extremal black hole. In summary, the WCCC is always violated for extremal and near-extremal RN-AdS black holes.

III. RN BLACK HOLE IN A CAVITY

In this section, we throw a charged particle into an RN black hole enclosed in a cavity and test the first and second laws of thermodynamics and the WCCC. We now consider a thermodynamic system with an RN black hole enclosed in a cavity, the wall of which is at $r = r_B$. The four-dimensional RN black hole solution is

$$\mathrm{d}s^2 = -f(r;M,Q)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r;M,Q)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right),$$
$$f(r;M,Q) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad A = A_t(r)\mathrm{d}t = -\frac{Q}{r}\mathrm{d}t, \tag{24}$$

where M and Q are the black hole charge and mass, respectively. The Hawking temperature $T_{\rm BH}$ of the black hole is given by

$$T_{\rm BH} = \frac{1}{4\pi}\left.\frac{\partial f(r;M,Q)}{\partial r}\right|_{r=r_+(M,Q)} = \frac{1}{4\pi r_+(M,Q)}\left(1 - \frac{Q^2}{r_+^2(M,Q)}\right) \tag{25}$$

where $r_+(M,Q) = M + \sqrt{M^2 - Q^2}$ is the radius of the outer event horizon. Suppose that the wall of the cavity is maintained at a temperature T. It was shown in [75] that the system temperature T can be related to the black hole temperature $T_{\rm BH}$ as

$$T = \frac{T_{\rm BH}}{\sqrt{f(r_B;M,Q)}}, \tag{26}$$

which means that T,measured at $r = r_B$, is blueshifted from $T_{\rm BH}$, measured at $r = \infty$. The thermal energy $\mathcal{E}$ and potential Φ of this system are [75]

$$\mathcal{E} = r_B\left[1 - \sqrt{f(r_B;M,Q)}\right],$$
$$\Phi = \frac{A_t(r_B) - A_t(r_+)}{\sqrt{f(r_B;M,Q)}}. \tag{27}$$

The physical space of $r_+(M,Q)$ is bounded by

$$r_e(Q) \leqslant r_+(M,Q) \leqslant r_B, \tag{28}$$

where $r_e(Q) = Q$ is the horizon radius of the extremal black hole.

After we throw a particle of energy E and charge q into the RN black hole, the thermal energy and charge of the system are changed from $(\mathcal{E},Q)$ to $(\mathcal{E}+\mathrm{d}\mathcal{E}, Q+\mathrm{d}Q)$. The energy and charge conservation give

$$\mathrm{d}Q = q,$$
$$\mathrm{d}\mathcal{E} = \frac{1}{\sqrt{f(r_B;M,Q)}}\left(\mathrm{d}M - \frac{Q\mathrm{d}Q}{r_B}\right) = E, \tag{29}$$

where we use Eqn. (27) to express $\mathrm{d}\mathcal{E}$ in terms of $\mathrm{d}M$ and $\mathrm{d}Q$. Here, we assume that the radius r_B of the cavity is fixed during the absorption. Eqn. (29) leads to variation of the black hole mass M,

$$\mathrm{d}M = \sqrt{f(r_B;M,Q)}E + \frac{qQ}{r_B}. \tag{30}$$

For the purpose of discussing the thermodynamic laws, the final RN black hole after absorbing the particle is assumed to have an event horizon, which is located at $r = r_+(M,Q) + \mathrm{d}r_+$. Similar to the RN-AdS case, we have

$$\left.\frac{\partial f(r;M,Q)}{\partial r}\right|_{r=r_+(M,Q)}\mathrm{d}r_+ + \left.\frac{\partial f(r;M,Q)}{\partial M}\right|_{r=r_+(M,Q)}\mathrm{d}M$$
$$+ \left.\frac{\partial f(r;M,Q)}{\partial Q}\right|_{r=r_+(M,Q)}\mathrm{d}Q = 0. \tag{31}$$

Eqns. (5), (25), (26), and (30) then give

$$T\mathrm{d}S = \left|P^r(r_+(M,Q))\right| + \frac{qQ}{r_+(M,Q)} - q\Phi, \tag{32}$$

where $S = \pi r_+^2(M,Q)$ is the entropy of the system. Using

Eqn. (27), we rewrite Eqn. (32) as

$$TdS = \left|P^r(r_+(M,Q))\right| + \left(\frac{1}{r_+} - \frac{1}{r_+\sqrt{f(r_B;M,Q)}} + \frac{1}{r_B\sqrt{f(r_B;M,Q)}}\right)QdQ, \tag{33}$$

where the prefactor of QdQ is positive. This shows that, when $T=0$ (i.e., $Q=M$), both sides of (33) are zero, which cannot provide any information about dS. For a non-extremal black hole, Eqn. (33) gives that the entropy increases when $dQ>0$. However, when $dQ<0$, the entropy can increase or decrease depending on the value of dQ. Therefore, the second law of thermodynamics is indefinite for an RN black hole in a cavity. Substituting Eqn. (5) into Eqn. (32), we obtain

$$d\mathcal{E} = \Phi dQ + TdS, \tag{34}$$

which is the first law of thermodynamics.

To overcharge an RN black hole of mass M and charge Q in a cavity by a test particle of energy E and charge q, the mass $M+dM$ and charge $Q+q$ of the final configuration must satisfy

$$M + dM < Q + q, \tag{35}$$

which, due to Eqn. (30), puts an upper bound on E,

$$E < E_{\rm up} \equiv \frac{Q+q-M-\frac{qQ}{r_B}}{\sqrt{f(r_B;M,Q)}}. \tag{36}$$

In contrast, Eqn. (5) also puts a lower bound on E,

$$E > \frac{qQ}{r_+(M,Q)} \equiv E_{\rm low}. \tag{37}$$

For an extremal black hole with $M=Q$, we find

$$E_{\rm up} = E_{\rm low} = q, \tag{38}$$

which indicates that the extremal black hole cannot be overcharged. Considering a near-extremal RN black hole with Q and $M=Q+\epsilon^2$, one can show that

$$E_{\rm low} \simeq q - \frac{\sqrt{2}q\epsilon}{\sqrt{Q}} \quad \text{and} \quad E_{\rm up} \simeq q - \frac{\epsilon^2}{1-Q/r_B}. \tag{39}$$

If $E_{\rm up} > E_{\rm low}$, the charge q of the test particle should be bounded from below,

$$q \equiv a\epsilon > \frac{\epsilon}{(1-Q/r_B)}\sqrt{\frac{Q}{2}}. \tag{40}$$

Moreover, the corresponding energy E of the particle is

$$E = a\epsilon + b\epsilon^2 \quad \text{with} \quad -\frac{\sqrt{2}a}{\sqrt{Q}} < b < -\frac{1}{1-Q/r_B}. \tag{41}$$

Therefore, a test particle with (q,E) in the parameter regions (40) and (41) can overcharge the near-extremal RN black hole in a cavity, which invalidates the WCCC.

IV. CONCLUSION

In this paper, via absorbing a test charged particle, we calculated the variations of thermodynamic quantities of RN black holes with two boundary conditions, namely the asymptotically AdS boundary and the Dirichlet boundary in the asymptotically flat spacetime. With these variations, we checked the first and second laws of thermodynamics and WCCC in these two cases. Our results are summarized in Table 1. In the limit of $l\to\infty$, an RN-AdS black hole becomes an RN black hole. Similarly, an RN black hole in a cavity also reduces to an RN black hole when $r_B\to\infty$. We find that taking the limits of our results in both AdS and cavity cases gives the same result regarding the validity of the thermodynamic laws and WCCC for an RN black hole, which is also presented in Table 1.

In [76, 77], the thermodynamic and phase structure of an RN-AdS black hole and RN black hole in a cavity were shown to be strikingly similar. However, it was found that the thermodynamic geometry in these two cases behaves rather differently [95], which implies that there may be a connection between the black hole microstates and the boundary condition. In this paper, we showed that the validity of the second law of thermodynamics and WCCC in the AdS and cavity cases are quite different, which further motivated us to explore the connection between the internal microstructure of black holes

Table 1. Results for the first and second laws of thermodynamics and weak cosmic censorship conjecture (WCCC), which are tested for an RN-AdS black hole, RN black hole in a cavity, and RN black hole absorbing a test charged particle.

	RN-AdS BH	RN BH in a cavity	RN BH
1st Law	Satisfied.	Satisfied.	Satisfied.
2nd Law	Satisfied.	Satisfied for $dQ>0$. Indefinite for $dQ<0$.	Satisfied.
WCCC	Violated for extremal and near-extremal BHs.	Satisfied for extremal BH. Violated for near-extremal BH.	Satisfied for extremal BH. Violated for near-extremal BH.

and the boundary condition.

To understand the scale of energy required for a particle to overcharge a black hole, we can convert physical quantities in Planck units to SI units. In fact, for a particle of charge q and energy E in Planck units, the charge and energy in SI units are qq_p and Em_pc^2, respectively, where $q_p \equiv \sqrt{4\pi\epsilon_0\hbar c} = e/\sqrt{\alpha}$ is the Planck charge, ϵ_0 is the permittivity of free space, e is the elementary charge, α is the fine structure constant, and $m_p = \sqrt{\hbar c/G}$ is the Planck mass. From Eqns. (22), (23), (40), and (41), we find that a test particle that can overcharge a charged black hole should have

$$E \sim q \text{ (in Planck units)}, \tag{42}$$

which leads to

$$E \sim \frac{|q|}{e}\left(\sqrt{\alpha}m_pc^2\right) \sim \frac{|q|}{e} \times 10^{18} \text{ GeV (in SI units)}. \tag{43}$$

For example, if one throws an ionized hydrogen nucleus of $m \sim 1 \text{ GeV}/c^2$ and $q = e$ to overcharge a charged black hole, the nucleus is ultrarelativistic with an enormous kinetic energy $\sim 10^{18}$ GeV to overcome the electrostatic repulsion between the particle and the black hole.

In this paper, we discussed the WCCC in the test limit, in which the interaction between test particles and the black hole background is neglected. Although this method is simple and straightforward, it is subject to several limitations. For example, our results showed that the final mass of the black hole needs to have a second-order correction in q to overcharge the black hole. However, if one calculates the mass consistently to order q^2, the test limit, which is valid only to linear order in q, is not enough, and hence, all second-order effects, e.g., self-force and finite size effects, should be considered. In [43], a general formula for the full second-order correction to mass was proposed, and it was found that the WCCC is valid for Kerr-Newman black holes up to the second-order perturbation of the matter fields. In this new version of the gedanken experiment, the WCCC was tested and found to be valid for various black holes [47, 48, 100, 101]. In particular, the WCCC was investigated for RN-AdS black holes in the extended phase space in [102], which showed that the WCCC cannot be violated for RN-AdS black holes under the second-order approximation of matter field perturbations. In contrast, we used the test limit to study the WCCC for RN-AdS black holes in the normal phase space with fixed cosmological constant and found that the WCCC is violated. Apart from the different phase spaces considered, differences between our results and those in [102] suggest that corrections beyond the test limit can play an important role in the analysis of the WCCC for charged AdS black holes and black holes in a cavity. We leave this for future work.

ACKNOWLEDGMENTS

We are grateful to Haitang Yang and Zhipeng Zhang for useful discussions and valuable comments.

References

[1] R. Penrose, Riv. Nuovo Cim. **1**, 252 (1969) [Gen. Rel. Grav. **34**, 1141 (2002)]
[2] D. Christodoulou, Phys. Rev. Lett. **25**, 1596 (1970)
[3] J. M. Bardeen, Nature **226**, 64 (1970)
[4] D. Christodoulou and R. Ruffini, Phys. Rev. D **4**, 3552 (1971)
[5] J. D. Bekenstein, Lett. Nuovo Cim. **4**, 737 (1972)
[6] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973)
[7] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973)
[8] J. M. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] doi: 10.1023/A:1026654312961, 10.4310/ATMP.1998.v2.n2.a1 [hep-th/9711200]
[9] A. Chamblin, R. Emparan, C. V. Johnson *et al.*, Phys. Rev. D **60**, 064018 (1999), arXiv:hep-th/9902170
[10] A. Chamblin, R. Emparan, C. V. Johnson *et al.*, Phys. Rev. D **60**, 104026 (1999), arXiv:hep-th/9904197
[11] M. M. Caldarelli, G. Cognola, and D. Klemm, Class. Quant. Grav. **17**, 399 (2000), arXiv:hep-th/9908022
[12] R. G. Cai, Phys. Rev. D **65**, 084014 (2002), arXiv:hep-th/0109133
[13] D. Kubiznak and R. B. Mann, JHEP **1207**, 033 (2012), arXiv:1205.0559 [hep-th]
[14] S. Gunasekaran, R. B. Mann, and D. Kubiznak, JHEP **1211**, 110 (2012), arXiv:1208.6251 [hep-th]
[15] S. W. Wei and Y. X. Liu, Phys. Rev. D **87**(4), 044014 (2013), arXiv:1209.1707 [gr-qc]
[16] R. G. Cai, L. M. Cao, L. Li *et al.*, JHEP **1309**, 005 (2013), arXiv:1306.6233 [gr-qc]
[17] W. Xu and L. Zhao, Phys. Lett. B **736**, 214 (2014), arXiv:1405.7665 [gr-qc]
[18] A. M. Frassino, D. Kubiznak, R. B. Mann *et al.*, JHEP **1409**, 080 (2014), arXiv:1406.7015 [hep-th]
[19] M. H. Dehghani, S. Kamrani, and A. Sheykhi, Phys. Rev. D **90**(10), 104020 (2014), arXiv:1505.02386 [hep-th]
[20] R. A. Hennigar, W. G. Brenna, and R. B. Mann, JHEP **1507**, 077 (2015), arXiv:1505.05517 [hep-th]
[21] P. Wang, H. Wu, and H. Yang, JCAP **1904**, 052 (2019), arXiv:1808.04506 [gr-qc]
[22] D. Wu, P. Wu, H. Yu *et al.*, Phys. Rev. D **101**(2), 024057 (2020)
[23] R. Wald, Ann. Phys. **82**, 548 (1974)
[24] V. E. Hubeny, Phys. Rev. D **59**, 064013 (1999), arXiv:gr-qc/9808043
[25] T. Jacobson and T. P. Sotiriou, Phys. Rev. Lett. **103**, 141101 (2009) Erratum: [Phys. Rev. Lett. **103**, 209903 (2009)] doi: 10.1103/PhysRevLett.103.209903, 10.1103/PhysRevLett.103.141101 [arXiv: 0907.4146 [gr-

qc]]
[26] A. Saa and R. Santarelli, Phys. Rev. D **84**, 027501 (2011), arXiv:1105.3950 [gr-qc]
[27] S. Hod, Phys. Rev. Lett. **100**, 121101 (2008), arXiv:0805.3873 [gr-qc]
[28] E. Barausse, V. Cardoso, and G. Khanna, Phys. Rev. Lett. **105**, 261102 (2010), arXiv:1008.5159 [gr-qc]
[29] E. Barausse, V. Cardoso, and G. Khanna, Phys. Rev. D **84**, 104006 (2011), arXiv:1106.1692 [gr-qc]
[30] P. Zimmerman, I. Vega, E. Poisson *et al*., Phys. Rev. D **87**(4), 041501 (2013), arXiv:1211.3889 [gr-qc]
[31] M. Colleoni and L. Barack, Phys. Rev. D **91**, 104024 (2015), arXiv:1501.07330 [gr-qc]
[32] M. Colleoni, L. Barack, A. G. Shah *et al*., Phys. Rev. D **92**(8), 084044 (2015), arXiv:1508.04031 [gr-qc]
[33] G. E. A. Matsas and A. R. R. da Silva, Phys. Rev. Lett. **99**, 181301 (2007), arXiv:0706.3198 [gr-qc]
[34] M. Richartz and A. Saa, Phys. Rev. D **78**, 081503 (2008), arXiv:0804.3921 [gr-qc]
[35] S. Isoyama, N. Sago, and T. Tanaka, Phys. Rev. D **84**, 124024 (2011), arXiv:1108.6207 [gr-qc]
[36] S. Gao and Y. Zhang, Phys. Rev. D **87**(4), 044028 (2013), arXiv:1211.2631 [gr-qc]
[37] S. Hod, Phys. Rev. D **87**(2), 024037 (2013), arXiv:1302.6658 [gr-qc]
[38] K. Duztas and I. Semiz, Phys. Rev. D **88**(6), 064043 (2013), arXiv:1307.1481 [gr-qc]
[39] H. M. Siahaan, Phys. Rev. D **93**(6), 064028 (2016), arXiv:1512.01654 [gr-qc]
[40] J. Natario, L. Queimada, and R. Vicente, Class. Quant. Grav. **33**(17), 175002 (2016), arXiv:1601.06809 [gr-qc]
[41] K. Duztas, Phys. Rev. D **94**(12), 124031 (2016), arXiv:1701.07241 [gr-qc]
[42] K. S. Revelar and I. Vega, Phys. Rev. D **96**(6), 064010 (2017), arXiv:1706.07190 [gr-qc]
[43] J. Sorce and R. M. Wald, Phys. Rev. D **96**(10), 104014 (2017), arXiv:1707.05862 [gr-qc]
[44] V. Husain and S. Singh, Phys. Rev. D **96**(10), 104055 (2017), arXiv:1709.02395 [gr-qc]
[45] T. Crisford, G. T. Horowitz, and J. E. Santos, Phys. Rev. D **97**(6), 066005 (2018), arXiv:1709.07880 [hep-th]
[46] B. Gwak, JHEP **1711**, 129 (2017), arXiv:1709.08665 [gr-qc]
[47] J. An, J. Shan, H. Zhang *et al*., Phys. Rev. D **97**(10), 104007 (2018), arXiv:1711.04310 [hep-th]
[48] B. Ge, Y. Mo, S. Zhao *et al*., Phys. Lett. B **783**, 440 (2018), arXiv:1712.07342 [hep-th]
[49] T. Y. Yu and W. Y. Wen, Phys. Lett. B **781**, 713 (2018), arXiv:1803.07916 [gr-qc]
[50] B. Gwak, JHEP **1809**, 081 (2018), arXiv:1807.10630 [grqc]
[51] Y. Gim and B. Gwak, Phys. Lett. B **794**, 122 (2019), arXiv:1808.05943 [gr-qc]
[52] D. Chen, Chin. Phys. C **44**(1), 015101 (2020), arXiv:1812.03459 [grqc]
[53] X. X. Zeng and H. Q. Zhang, Chin. Phys. C **45**(2), 025112 (2021), arXiv:1901.04247 [hep-th]
[54] D. Chen, W. Yang, and X. Zeng, Nucl. Phys. B **946**, 114722 (2019), arXiv:1901.05140 [hep-th]
[55] B. Gwak, JCAP **1908**, 016 (2019), arXiv:1901.05589 [gr-qc]
[56] X. X. Zeng, Y. W. Han, and D. Y. Chen, Chin. Phys. C **43**(10), 105104 (2019), arXiv:1901.08915 [gr-qc]
[57] D. Chen, Eur. Phys. J. C **79**(4), 353 (2019), arXiv:1902.06489 [hep-th]
[58] P. Wang, H. Wu, and H. Yang, Eur. Phys. J. C **79**(7), 572 (2019)
[59] X. X. Zeng, X. Y. Hu, and K. J. He, Nucl. Phys. B **949**, 114823 (2019), arXiv:1905.07750 [hep-th]
[60] W. Hong, B. Mu, and J. Tao, Nucl. Phys. B **949**, 114826 (2019), arXiv:1905.07747 [gr-qc]
[61] T. T. Hu, Y. Song, S. Sun *et al*., Eur. Phys. J. C **80**(2), 147 (2020), arXiv:1906.00235 [hep-th]
[62] S. Q. Hu, Y. C. Ong, and D. N. Page, Phys. Rev. D **100**(10), 104022 (2019), arXiv:1906.05870 [gr-qc]
[63] B. Mu and J. Tao, *Minimal Length Effect on Thermodynamics and Weak Cosmic Censorship Conjecture in anti-de Sitter Black Holes via Charged Particle Absorption*, arXiv: 1906.10544 [gr-qc]
[64] K. J. He, X. Y. Hu, and X. X. Zeng, Chin. Phys. C **43**(12), 125101 (2019), arXiv:1906.10531 [gr-qc]
[65] Q. Gan, G. Guo, P. Wang *et al*., Phys. Rev. D **100**(12), 124009 (2019), arXiv:1907.04466 [hep-th]
[66] X. X. Zeng and X. Y. Hu, *Thermodynamics and weak cosmic censorship conjecture with pressure in the rotating BTZ black holes*, arXiv: 1908.03845 [gr-qc]
[67] K. J. He, G. P. Li, and X. Y. Hu, Eur. Phys. J. C **80**(3), 209 (2020), arXiv:1909.09956 [hep-th]
[68] S. Q. Hu, B. Liu, X. M. Kuang *et al*., Chin. Phys. C **44**, 105107 (2020), arXiv:1910.04437 [gr-qc]
[69] B. Gwak, JCAP **03**, 058 (2020), arXiv:1910.13329 [gr-qc]
[70] Q. Gan, P. Wang, H. Wu *et al*., Chin. Phys. C **45**(2), 025103 (2021), arXiv:1911.10996 [gr-qc]
[71] W. Hong, B. Mu, and J. Tao, Int. J. Mod. Phys. D **29**(12), 2050078 (2020), arXiv:2001.09008 [physics.gen-ph]
[72] B. Gwak, Chin. Phys. C **44**(12), 12 (2020), arXiv:2002.09729 [gr-qc]
[73] Y. Zhang and S. Gao, Int. J. Mod. Phys. D **23**, 1450044 (2014), arXiv:1309.2027 [gr-qc]
[74] J. W. York, Jr., Phys. Rev. D **33**, 2092 (1986)
[75] H. W. Braden, J. D. Brown, B. F. Whiting *et al*., Phys. Rev. D **42**, 3376 (1990)
[76] S. Carlip and S. Vaidya, Class. Quant. Grav. **20**, 3827 (2003), arXiv:gr-qc/0306054
[77] A. P. Lundgren, Phys. Rev. D **77**, 044014 (2008), arXiv:gr-qc/0612119
[78] J. X. Lu, S. Roy, and Z. Xiao, JHEP **1101**, 133 (2011), arXiv:1010.2068 [hep-th]
[79] C. Wu, Z. Xiao, and J. Xu, Phys. Rev. D **85**, 044009 (2012), arXiv:1108.1347 [hep-th]
[80] J. X. Lu, R. Wei, and J. Xu, JHEP **1212**, 012 (2012), arXiv:1210.0708 [hepth]
[81] J. X. Lu and R. Wei, JHEP **1304**, 100 (2013), arXiv:1301.1780 [hepth]
[82] D. Zhou and Z. Xiao, JHEP **1507**, 134 (2015), arXiv:1502.00261 [hep-th]
[83] Z. Xiao and D. Zhou, JHEP **1509**, 028 (2015), arXiv:1507.02088 [hep-th]
[84] P. Wang, H. Yang, and S. Ying, Phys. Rev. D **101**(6), 064045 (2020), arXiv:1909.01275 [gr-qc]
[85] N. Sanchis-Gual, J. C. Degollado, P. J. Montero *et al*., Phys. Rev. Lett. **116**(14), 141101 (2016), arXiv:1512.05358 [gr-qc]
[86] N. Sanchis-Gual, J. C. Degollado, C. Herdeiro *et al*., Phys. Rev. D **94**(4), 044061 (2016), arXiv:1607.06304 [gr-qc]
[87] P. Basu, C. Krishnan, and P. N. B. Subramanian, JHEP

1611, 041 (2016), arXiv:1609.01208 [hep-th]
[88] Y. Peng, B. Wang, and Y. Liu, Eur. Phys. J. C **78**(3), 176 (2018), arXiv:1708.01411 [hep-th]
[89] Y. Peng, JHEP **1707**, 042 (2017), arXiv:1705.08694 [hep-th]
[90] Y. Peng, Phys. Lett. B **780**, 144 (2018), arXiv:1801.02495 [gr-qc]
[91] Y. Peng, JHEP **1912**, 064 (2019), arXiv:1910.13718 [gr-qc]
[92] Y. Peng, Eur. Phys. J. C **80**(3), 202 (2020), arXiv:2002.01892 [gr-qc]
[93] P. Wang, H. Wu, and H. Yang, JHEP **1907**, 002 (2019), arXiv:1901.06216 [gr-qc]
[94] K. Liang, P. Wang, H. Wu *et al*., Eur. Phys. J. C **80**(3), 187 (2020), arXiv:1907.00799 [gr-qc]
[95] P. Wang, H. Wu, and H. Yang, Eur. Phys. J. C **80**(3), 216 (2020), arXiv:1910.07874 [gr-qc]
[96] F. Simovic and R. B. Mann, Class. Quant. Grav. **36**(1), 014002 (2019), arXiv:1807.11875 [gr-qc]
[97] S. Haroon, R. A. Hennigar, R. B. Mann *et al*., Phys. Rev. D **101**, 084051 (2020), arXiv:2002.01567 [gr-qc]
[98] B. Mu, P. Wang, and H. Yang, Adv. High Energy Phys. **2017**, 3191839 (2017), arXiv:1408.5055 [grqc]
[99] J. Tao, P. Wang, and H. Yang, Nucl. Phys. B **922**, 346 (2017), arXiv:1505.03045 [gr-qc]
[100] J. Jiang, B. Deng, and Z. Chen, Phys. Rev. D **100**(6), 066024 (2019), arXiv:1909.02219 [hep-th]
[101] J. Jiang, X. Liu, and M. Zhang, Phys. Rev. D **100**(8), 084059 (2019), arXiv:1910.04060 [hep-th]
[102] X. Y. Wang and J. Jiang, JCAP **07**, 052 (2020), arXiv:1911.03938 [hep-th]