Strong Coupling Constants of the Doubly Heavy Spin-1/2 Baryons with Light Pseudoscalar Mesons

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Abstract: The strong coupling constants of hadronic multiplets are fundamental parameters which carry information about the strong interactions among participating particles. These parameters can help us construct the hadron-hadron strong potential and gain information about the structure of the involved hadrons. Motivated by the recent observation of the doubly charmed Ξ_{cc} state by LHCb, we determine the strong coupling constants among the doubly heavy spin-1/2 baryons, $\Xi_{QQ'}^{(\prime)}$, $\Omega_{QQ'}^{(\prime)}$ and light pseudoscalar mesons, π, K, η and η' within the framework of the light cone QCD sum rules. The obtained results may help experimental groups in analysis of the related data at hadron colliders.

I. INTRODUCTION

The quark model [1-3] has been very successful in describing the properties of hadrons observed experimentally. Nevertheless, not all particles predicted by the quark model are experimentally well established: of the doubly heavy baryons, only the doubly charmed Ξ_{cc} state has been seen in experiments. The triply heavy baryons are also missing in experiments and the hunt for them continues. More experimental and theoretical studies on these states are required. Even in the case of Ξ_{cc} there is a puzzle in the experimental results. The first evidence for this state was reported in 2005 by the SELEX experiment, with Ξ_{cc}^+ decaying into $\Lambda_c^+ K^- \pi^+$ and $pD^+ K^-$ final states, using a $600 \text{ MeV}/c^2$ charged hyperon beam impinging on a fixed target. The mass measured by SELEX, averaged over the two decay modes, was found to be $(3518.7 \pm 1.7) \text{ MeV}/c^2$. The lifetime was measured to be less than 33 fs at 90% confidence level. It was estimated that about 20% of Λ_c^+ baryons in the SELEX experiment were produced from Ξ_{cc} decays [4, 5]. However, the FOCUS [6], BaBar [7], LHCb [8] and Belle [9] experiments were not able to confirm the SELEX results. In 2017, the doubly charmed baryon Ξ_{cc}^{++} was observed by the LHCb collaboration via the decay channel $\Xi_{cc}^{++} \to \Lambda_c^+ K^- \pi^+ \pi^+$

[10], and confirmed via measuring another decay channel, $\Xi_{cc}^{++} \to \Xi_c^+ \pi^+$ [11]. The weighted average of the Ξ_{cc}^{++} mass of the two decay modes was determined to be $3621.24 \pm 0.65(\text{stat.}) \pm 0.31(\text{syst.}) \text{ MeV}/c^2$ [11]. The lifetime of the Ξ_{cc}^{++} baryon was measured to be $0.256^{+0.024}_{-0.022}(\text{stat.}) \pm 0.014(\text{syst.}) \text{ ps}$ [12]. Recently, with a data sample corresponding to an integrated luminosity of 1.7 fb^{-1}, the $\Xi_{cc}^{++} \to D^+ pK^- \pi^+$ decay has been searched for by the LHCb collaboration, but no signal was found [13]. Certainly, experiments will continue to seek to solve the unexpected difference in parameters of these states, and will also search for other doubly heavy particles.

As can be seen, the result of the LHCb collaboration for the mass of the Ξ_{cc} is about $100 \text{ MeV}/c^2$ higher than the value reported by the SELEX collaboration. The difference between these two results has motivated theoretical research to investigate the origin of this difference. In Ref. [14], the authors have shown that the SELEX and the LHCb results for the production of doubly charmed baryons can both be correct if supersymmetric algebra is applied to hadron spectroscopy, together with the intrinsic heavy-quark QCD mechanism for the hadroproduction of heavy hadrons at large x_F.

On the theoretical side, studies on doubly heavy baryons are needed to provide many inputs to experiments.

Some aspects of doubly heavy baryons have been discussed in Refs. [15-44]. The mechanisms of production and decay of such systems have also been of interest to researchers for many years [45-58]. The production of doubly heavy baryons can be divided into two steps. The first step is the perturbative production of a heavy quark pair in the hard interaction. In the second step this pair is transformed to the baryon within the soft hadronization process. The doubly heavy baryons can participate in many interactions and processes. The fusion of two Λ_c to produce $\Xi_{cc}^{++}n$ results in an energy release of about 12 MeV, and the fusion of two Λ_b baryons to $\Xi_{bb}^0 n$ releases about 138 MeV. This suggests that an experimental setup may be designed to allow this huge released energy to be used, although the very short lifetimes of cc and bb baryons may prevent practical applications at the present time [59].

In this study, we investigate the strong coupling constants among the doubly heavy spin-1/2 baryons and light pseudoscalar mesons, π, K, η and η', which is an extension of our previous work [60]. In Ref. [60], we investigated only the symmetric $\Xi_{QQ'}$ and calculated its coupling constant with π mesons. In the present study, we investigate the strong coupling constants of the $\Xi_{QQ'}$, $\Xi'_{QQ'}$, $\Omega_{QQ'}$ and $\Omega'_{QQ'}$ doubly heavy baryons with all the light pseudoscalar mesons, π, K, η and η', with different charges. Here Q and Q' can both be b or c quarks. We use the well established non-perturbative method of light cone QCD sum rules (LCSR) in the calculations. In the framework of LCSR, which has been developed based on the standard technique of the SVZ sum rule method [61], the non-perturbative dynamics of the quarks and gluons in the baryons are described by the light-cone distribution amplitudes (DAs). The LCSR approach uses operator product expansion (OPE) near the lightcone $x^2 \approx 0$ instead of the short distance $x \approx 0$, and the nonperturbative matrix elements are parameterized by the light cone DAs, which are classified according to their twists [62-64].

The rest of the paper is organized as follows. In the next section, we describe the formalism and obtain the sum rules for the strong coupling constants under study. In Section III, the numerical analysis and results are presented. Section IV is reserved for summary and concluding notes.

II. STRONG COUPLING CONSTANTS AMONG DOUBLY HEAVY BARYONS AND PSEUDOSCALAR MESONS

Before going to the details of the calculations for the strong coupling constants, we take a look at the ground state of the doubly heavy baryons in the quark model. In the case of the doubly heavy baryons having two identical heavy quarks, i.e. $Q = Q'$, the two heavy quarks form a

diquark system with spin 1. After adding the light quark spin, the whole baryon may have spin 1/2 (Ξ_{QQ} and Ω_{QQ}) or 3/2 (Ξ_{QQ}^* and Ω_{QQ}^*). Here the interpolating current should be symmetric with respect to the exchange of heavy quarks. In the case of different heavy quarks ($Q \neq Q'$), in addition to the above case, the diquark portion can also have spin zero where together with the light quark, the total spin of the whole baryon will be 1/2, which obviously leads to anti-symmetric interpolating currents with respect to the exchange of the two heavy quarks. They are usually denoted by the primed baryons Ξ'_{bc} and Ω'_{bc}.

The main inputs in the sum rule method are interpolating currents, which are written based on the general properties of the baryons and in terms of their quark contents. In the case of doubly heavy baryons, the symmetric and anti-symmetric interpolating fields for spin-1/2 particles are given as:

$$\eta^S = \frac{1}{\sqrt{2}} \epsilon_{abc} \Big\{ (Q^{aT} C q^b)\gamma_5 Q'^c + (Q'^{aT} C q^b)\gamma_5 Q^c + t(Q^{aT} C \gamma_5 q^b)Q'^c + t(Q'^{aT} C \gamma_5 q^b)Q^c \Big\}, \qquad (1)$$

$$\eta^{\mathcal{A}} = \frac{1}{\sqrt{6}} \epsilon_{abc} \Big\{ 2(Q^{aT} C Q'^b)\gamma_5 q^c + (Q^{aT} C q^b)\gamma_5 Q'^c - (Q'^{aT} C q^b)\gamma_5 Q^c + 2t(Q^{aT} C \gamma_5 Q'^b)q^c + t(Q^{aT} C \gamma_5 q^b)Q'^c - t(Q'^{aT} C \gamma_5 q^b)Q^c \Big\}, \qquad (2)$$

where C stands for the charge conjugation operator, T denotes the transposition and t is an arbitrary mixing parameter where the case $t = -1$ corresponds to the Ioffe current. $Q^{(')}$ and q stand for the heavy and light quarks respectively and a, b, and c are the color indices. The quark contents for different members are shown in Table 1.

As an example, we demonstrate how the current of the doubly heavy baryons in its antisymmetric form is constructed considering all the quantum numbers. The simplest way of constructing a spin-1/2 baryon interpolating current is to make a diquark state of isospin and spin zero from two of three constituent quarks, with the third quark of isospin and spin-1/2. To make a diquark, we start with a meson interpolating current,

Table 1. Quark contents of the doubly heavy spin-1/2 baryons.

Baryon	q	Q	Q'
$\Xi_{QQ'}$ or $\Xi'_{QQ'}$	u or d	b or c	b or c
$\Omega_{QQ'}$ or $\Omega'_{QQ'}$	s	b or c	b or c

$$\eta_{\text{meson}} = \bar{q}_1 \Gamma q_2 , \tag{3}$$

where $\Gamma = I, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}$. Then we replace the anti-quark with its charge conjugation analog, where $q = C\bar{q}^T$. Therefore $\bar{q} = q^T C$, in which C is the charge conjugation operator and $C^T = C^{-1} = C^\dagger = -C$. This leads to the diquark interpolating currents:

$$\eta_{\text{diquark}} = q_1^T C \Gamma q_2 . \tag{4}$$

Adding the third quark spinor to make the baryon current, $[q_1^T C \Gamma q_2] \Gamma' q_3$, the generic form of the antisymmetric interpolating current for the doubly heavy baryons would be:

$$\eta^A \sim \epsilon_{abc}\{(Q^{aT}C\Gamma Q'^b)\Gamma' q^c + (Q^{aT}C\Gamma q^b)\Gamma' Q'^c$$
$$+ (q^{aT}C\Gamma Q^b)\Gamma' Q'^c - (Q \leftrightarrow Q')\}, \tag{5}$$

where ϵ_{abc} makes the whole current color singlet. To determine Γ and Γ', we focus on the diquark part of the first term of the above equation. After transposing it we have:

$$[\epsilon_{abc} Q^{aT} C \Gamma Q'^b]^T = -\epsilon_{abc} Q'^{bT} \Gamma^T C^{-1} Q^a$$
$$= \epsilon_{abc} Q'^{bT} C (C\Gamma^T C^{-1}) Q^a . \tag{6}$$

Here we consider $C^T = C^{-1}$, $C^2 = -1$ and the fact that the Grassmann numbers in the spinor components anticommute. For the quantity $C\Gamma^T C^{-1}$ we have:

$$C\Gamma^T C^{-1} = \begin{cases} \Gamma & \text{for } \Gamma = 1, \gamma_5, \gamma_\mu \gamma_5 , \\ -\Gamma & \text{for } \Gamma = \gamma_\mu, \sigma_{\mu\nu} . \end{cases} \tag{7}$$

On the LHS, we switch the color dummy indices and get:

$$[\epsilon_{abc} Q^{aT} C \Gamma Q'^b]^T = \pm \epsilon_{abc} Q'^{aT} C \Gamma Q^b , \tag{8}$$

where the $+$ and $-$ signs are for $\Gamma = \gamma_\mu$, $\sigma_{\mu\nu}$ and $\Gamma = 1, \gamma_5, \gamma_5\gamma_\mu$ respectively. For the antisymmetric interpolating current, the RHS of the above equation is antisymmetric under $Q \leftrightarrow Q'$ exchange and we have:

$$[\epsilon_{abc} Q^{aT} C \Gamma Q'^b]^T = \pm \epsilon_{abc} Q^{aT} C \Gamma Q'^b , \tag{9}$$

where the $+$ and $-$ signs are for $\Gamma = 1, \gamma_5, \gamma_5\gamma_\mu$ and $\Gamma = \gamma_\mu, \sigma_{\mu\nu}$ respectively. On the other hand, as $\epsilon_{abc} Q^{aT} C \Gamma Q'^b$ is a 1×1 matrix, it is equal to its transpose and therefore one can conclude that the only choices for Γ matrices are $\Gamma = 1, \gamma_5, \gamma_5\gamma_\mu$.

As mentioned above, the simplest way of constructing spin-1/2 baryons is to suppose that the baryon spin be equal to that of light quark q (for the first term in Eq. (5))

and thus the diquark part has a scalar structure, which implies that $\Gamma = 1, \gamma_5$. Therefore, the allowed forms of the antisymmetric interpolating current may take just the following two forms: $\epsilon_{abc}(Q^{aT}CQ'^b)\Gamma' q^c$ and $\epsilon_{abc}(Q^{aT}C\gamma_5 Q'^b)\Gamma' q^c$.

The matrices Γ'_1 and Γ'_2 can be determined considering Lorentz and parity symmetries. As the whole interpolating current is a Lorentz scalar, there are two possibilities: 1 and γ_5. The parity property of the interpolating current finally says that $\Gamma'_1 = \gamma_5$ and $\Gamma'_2 = 1$. Writing their linear combination as the most general form, the antisymmetric form of the first term of Eq. (5) is:

$$\eta^A \sim \epsilon_{abc}\{(Q^{aT}CQ'^b)\gamma_5 q^c + t(Q^{aT}C\gamma_5 Q'^b)q^c - (Q \leftrightarrow Q')\}, \tag{10}$$

where t is an arbitrary mixing parameter. Considering the Grassmann nature of the heavy quark spinor components, the antisymmetric property of ϵ_{abc} and $C^T = -C$, one can find out that the $-(Q \leftrightarrow Q')$ terms are exactly the same as the first two, which yields:

$$\eta^A \sim 2\epsilon_{abc}\{(Q^{aT}CQ'^b)\gamma_5 q^c + t(Q^{aT}C\gamma_5 Q'^b)q^c\}. \tag{11}$$

A similar argument can be used to calculate the second and third terms in Eq. (5). The symmetric interpolating current can be obtained in the same way but with the exception that in the exchange of heavy quarks in Eq. (9) no minus sign is considered.

The main goal in this section is to find the strong coupling constants among the doubly heavy baryons with spin-1/2, $\Xi_{QQ'}$, $\Xi'_{QQ'}$, $\Omega_{QQ'}$ and $\Omega'_{QQ'}$, with the light pseudoscalar mesons π, K, η and η'. To this end, we use the LCSR approach as one of the most powerful non-perturbative methods which is based on the light-cone OPE. The starting point is to write the corresponding correlation function (CF):

$$\Pi(p, q) = i \int d^4 x e^{ipx} \langle \mathcal{P}(q) | \mathcal{T}\{\eta(x)\bar{\eta}(0)\} | 0 \rangle , \tag{12}$$

where the two time-ordered interpolating currents of doubly-heavy baryons are sandwiched between the QCD vacuum and the on-shell pseudoscalar meson $\mathcal{P}(q)$. Here, p is the external four-momentum of the outgoing doubly heavy baryon. As the theory is translationally invariant we can choose one of the interpolating currents, $\bar{\eta}(0)$, at the origin. It is worth noting again that in the symmetric interpolating current (η^S), heavy quarks may be identical or different whereas in the anti-symmetric one (η^A) they must be different.

In the LCSR approach, the cornerstone is the CF. It can be calculated in two different ways. In the timelike

region, one can insert the complete set of hadronic states with the same quantum numbers as the interpolating currents to extract and isolate the ground states. It is called the phenomenological or physical side of the CF. In the spacelike region which is free of singularities, one can calculate the CF in terms of QCD degrees of freedom using OPE. It is known as the QCD or theoretical side. These two representations, which respectively are the real and imaginary parts of the CF, can be matched via a dis-

persion relation to find the corresponding sum rule. The divergences coming from the dispersion integral, as well as higher states and continuum, are suppressed using the well-known method of Borel transformation and continuum subtraction.

On the phenomenological side, after inserting the complete sets of hadronic states with the same quantum numbers as the interpolating currents and performing the Fourier integration over x, we get

$$\Pi^{\text{Phys.}}(p,q) = \frac{\langle 0|\eta|B_2(p,r)\rangle\langle B_2(p,r)\mathcal{P}(q)|B_1(p+q,s)\rangle\langle B_1(p+q,s)|\bar{\eta}|0\rangle}{(p^2-m_1^2)[(p+q)^2-m_2^2]} + \cdots , \qquad (13)$$

where the ground states are isolated and the dots represent the contribution of the higher states and continuum. $B_1(p+q,s)$ and $B_2(p,r)$ are the initial and final doubly heavy baryons with spins s and r respectively. The matrix element $\langle 0|\eta|B_i(p,s)\rangle$ is defined as:

$$\langle 0|\eta|B_i(p,s)\rangle = \lambda_{B_i} u(p,s), \qquad (14)$$

where λ_{B_i} are the residues and $u(p,s)$ is the Dirac spinor for the baryons B_i with momentum p and spin s. By the Lorentz and parity consideration, one can write the matrix element $\langle B_2(p,r)\mathcal{P}(q)|B_1(p+q,s)\rangle$ in terms of the strong coupling constant and Dirac spinors as:

$$\langle B_2(p,r)\mathcal{P}(q)|B_1(p+q,s)\rangle = g_{B_1 B_2 \mathcal{P}}\bar{u}(p,r)\gamma_5 u(p+q,s), \quad (15)$$

where $g_{B_1 B_2 \mathcal{P}}$ represents the strong coupling constant for

the strong decay $B_1 \rightarrow B_2\mathcal{P}$. The final expression for the phenomenological side of the correlation function is obtained by inserting Eqs. (14) and (15) into Eq. (13) and summing over spins:

$$\Pi^{\text{Phys.}}(p,q) = \frac{g_{B_1 B_2 \mathcal{P}}\lambda_{B_1}\lambda_{B_2}}{(p^2-m_{B_2}^2)[(p+q)^2-m_{B_1}^2]}[\not{q}\not{p}\gamma_5 + \cdots] + \cdots , \qquad (16)$$

where the ellipsis inside the bracket denote several γ-matrix structures that may appear in the final expression due to the spin summation. Here we select the structure $\not{q}\not{p}\gamma_5$ to perform analysis.

To kill the higher states and continuum contributions we apply the double Borel transformation with respect to the square of the doubly heavy baryon momenta $p_1^2 = (p+q)^2$ and $p_2^2 = p^2$, which leads to

$$\mathcal{B}_{p_1}(M_1^2)\mathcal{B}_{p_2}(M_2^2)\Pi^{\text{Phys.}}(p,q) \equiv \Pi^{\text{Phys.}}(M^2) = g_{B_1 B_2 \mathcal{P}}\lambda_{B_1}\lambda_{B_2}e^{-m_{B_1}^2/M_1^2}e^{-m_{B_2}^2/M_2^2}\not{q}\not{p}\gamma_5 + \cdots , \qquad (17)$$

where M_1^2 and M_2^2 are the Borel parameters corresponding to the square momenta p_1^2 and p_2^2 respectively, and $M^2 = M_1^2 M_2^2/(M_1^2 + M_2^2)$. As the masses of the initial and final state baryons are the same or to a good approximation equal, the Borel parameters are chosen to be equal and therefore $M_1^2 = M_2^2 = 2M^2$.

On the QCD side, choosing the corresponding structure to Eq. (16), one can express the CF function as:

$$\Pi^{\text{QCD}}(p,q) = \Pi(p,q)\not{q}\not{p}\gamma_5, \qquad (18)$$

where $\Pi(p,q)$ is an invariant function of $(p+q)^2$ and p^2. The main aim in this part is to determine this function in the Borel scheme. To this end, we insert the explicit forms of the interpolating currents (1) and (2) into the correlation function (12) and use the Wick theorem to contract all the heavy quark fields. The result for the symmetric interpolating current is as follows:

$$\Pi^{\text{QCD}}_{(S)\rho\sigma}(p,q) = \frac{i}{2}\epsilon_{abc}\epsilon_{a'b'c'}\int d^4x e^{iq\cdot x}\langle \mathcal{P}(q)|\bar{q}^{c'}_\alpha(0)q^c_\beta(x)|0\rangle\Bigg\{\Bigg[\left(\tilde{S}^{aa'}_Q(x)\right)_{\alpha\beta}\left(\gamma_5 S^{bb'}_{Q'}(x)\gamma_5\right)_{\rho\sigma} + \left(\gamma_5 S^{bb'}_{Q'}(x)C\right)_{\rho\alpha}\left(CS^{aa'}_Q(x)\gamma_5\right)_{\beta\sigma}$$
$$+ t\Big\{\left(\gamma_5\tilde{S}^{aa'}_Q(x)\right)_{\alpha\beta}\left(\gamma_5 S^{bb'}_{Q'}(x)\right)_{\rho\sigma} + \left(\tilde{S}^{aa'}_Q(x)\gamma_5\right)_{\alpha\beta}\left(S^{bb'}_{Q'}(x)\gamma_5\right)_{\rho\sigma} + \left(\gamma_5 S^{bb'}_{Q'}(x)C\gamma_5\right)_{\rho\alpha}\left(CS^{aa'}_Q(x)\right)_{\beta\sigma} - \left(S^{bb'}_{Q'}(x)C\right)_{\rho\alpha}$$
$$\times\left(\gamma_5 CS^{aa'}_Q(x)\gamma_5\right)_{\beta\sigma}\Big\} + t^2\Big\{\left(\gamma_5\tilde{S}^{aa'}_Q(x)\gamma_5\right)_{\alpha\beta}\left(S^{bb'}_{Q'}(x)\right)_{\rho\sigma} - \left(S^{bb'}_{Q'}(x)C\gamma_5\right)_{\rho\alpha}\left(\gamma_5 CS^{aa'}_Q(x)\right)_{\beta\sigma}\Big\}\Bigg] + \left(Q \longleftrightarrow Q'\right)\Bigg\}, \qquad (19)$$

where the ρ and σ are Dirac indices which run through 1 to 4, $S_Q^{aa'}(x)$ is the heavy quark propagator, $\tilde{S} = CS^T C$ and the subscript $\mathcal{S}$ denotes the symmetric part. $\langle \mathcal{P}(q)|\bar{q}_\alpha^c(x)$

$q_\beta^c(0)|0\rangle$ are the non-local matrix elements for the light quark contents of the doubly heavy baryons and are purely non-perturbative. For the anti-symmetric part we have:

$$
\begin{aligned}
\Pi_{(\mathcal{A})\rho\sigma}^{\mathrm{QCD}}(p,q) = \frac{\mathrm{i}}{6}\epsilon_{abc}\epsilon_{a'b'c'}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}q.x}\langle \mathcal{P}(q)|\bar{q}_\alpha^{c'}(0)q_\beta^c(x)|0\rangle &\Big\{ 4\mathrm{Tr}[\tilde{S}_Q^{aa'}(x)S_{Q'}^{bb'}(x)]\gamma_{\alpha\sigma}^5\gamma_{\rho\beta}^5 - 2\big(\tilde{S}_Q^{aa'}(x)S_{Q'}^{bb'}(x)\gamma^5\big)_{\alpha\sigma}\gamma_{\rho\beta}^5 \\
&- 2\big(\gamma^5 S_{Q'}^{bb'}(x)\tilde{S}_Q^{aa'}(x)\big)_{\rho\beta}\gamma_{\alpha\sigma}^5 - 2\big(\tilde{S}_{Q'}^{bb'}(x)S_Q^{aa'}(x)\gamma^5\big)_{\alpha\sigma}\gamma_{\rho\beta}^5 - 2\big(\gamma^5 S_Q^{aa'}(x)\tilde{S}_{Q'}^{bb'}(x)\big)_{\rho\beta}\gamma_{\alpha\sigma}^5 + \big(\tilde{S}_Q^{aa'}(x)\big)_{\alpha\beta}\big(\gamma^5 S_{Q'}^{bb'}(x)\gamma^5\big)_{\rho\sigma} \\
&+ \big(\tilde{S}_{Q'}^{bb'}(x)\big)_{\alpha\beta}\big(\gamma^5 S_Q^{aa'}(x)\gamma^5\big)_{\rho\sigma} + \big(\gamma^5 S_Q^{bb'}(x)C\big)_{\rho\alpha}\big(CS_Q^{aa'}(x)\gamma^5\big)_{\beta\sigma} + \big(\gamma^5 S_Q^{aa'}(x)C\big)_{\rho\alpha}\big(CS_{Q'}^{bb'}(x)\gamma^5\big)_{\beta\sigma} \\
&+ t\Big[4\mathrm{Tr}[\tilde{S}_Q^{aa'}(x)S_{Q'}^{bb'}(x)\gamma^5]\gamma_{\rho\beta}^5\delta_{\alpha\sigma} + 4\mathrm{Tr}[S_{Q'}^{bb'}(x)\tilde{S}_Q^{aa'}(x)\gamma^5]\gamma_{\alpha\sigma}^5\delta_{\rho\beta} + 2\big(\tilde{S}_{Q'}^{bb'}(x)\gamma^5 S_Q^{aa'}(x)C\gamma^5\big)_{\alpha\sigma}\delta_{\beta\rho} \\
&- 2\big(S_Q^{aa'}(x)\tilde{S}_{Q'}^{bb'}(x)\gamma^5\big)_{\beta\rho}\gamma_{\alpha\sigma}^5 - 2\big(\gamma^5\tilde{S}_Q^{aa'}(x)S_{Q'}^{bb'}(x)\big)_{\alpha\sigma}\gamma_{\rho\beta}^5 - 2\big(\gamma^5 S_{Q'}^{bb'}(x)\gamma^5\tilde{S}_Q^{aa'}(x)\big)_{\rho\beta}\delta_{\sigma\alpha} \\
&- 2\big(\gamma^5\tilde{S}_{Q'}^{bb'}(x)S_Q^{aa'}(x)\big)_{\alpha\sigma}\gamma_{\rho\beta}^5 - 2\big(\gamma^5 S_Q^{aa'}(x)\gamma^5\tilde{S}_{Q'}^{bb'}(x)\big)_{\rho\beta}\delta_{\alpha\sigma} - 2\big(\tilde{S}_Q^{aa'}(x)\gamma^5 S_{Q'}^{bb'}(x)\gamma^5\big)_{\alpha\sigma}\delta_{\rho\beta} \\
&- 2\big(S_{Q'}^{bb'}(x)\tilde{S}_Q^{aa'}(x)\gamma^5\big)_{\rho\beta}\gamma_{\alpha\sigma}^5 + \big(\gamma^5\tilde{S}_Q^{aa'}(x)\big)_{\alpha\beta}\big(\gamma^5 S_{Q'}^{bb'}(x)\big)_{\rho\sigma} + \big(\gamma^5 S_{Q'}^{bb'}(x)C\gamma^5\big)_{\rho\alpha}\big(CS_Q^{aa'}(x)\big)_{\beta\sigma} \\
&+ \big(\tilde{S}_Q^{aa'}(x)\gamma^5\big)_{\alpha\beta}\big(S_{Q'}^{bb'}(x)\gamma^5\big)_{\rho\sigma} + \big(\gamma^5 CS_Q^{aa'}(x)\gamma^5\big)_{\beta\sigma}\big(S_{Q'}^{bb'}(x)C\big)_{\rho\alpha} + \big(S_Q^{aa'}(x)C\big)_{\rho\alpha}\big(\gamma^5 CS_{Q'}^{bb'}(x)\gamma^5\big)_{\beta\sigma} \\
&+ \big(\tilde{S}_{Q'}^{bb'}(x)\gamma^5\big)_{\alpha\beta}\big(S_Q^{aa'}(x)\gamma^5\big)_{\rho\sigma} + \big(\gamma^5 S_Q^{aa'}(x)C\gamma^5\big)_{\rho\alpha}\big(CS_{Q'}^{bb'}(x)\big)_{\beta\sigma} + \big(\gamma^5\tilde{S}_{Q'}^{bb'}(x)\big)_{\alpha\beta}\big(\gamma^5 S_Q^{aa'}(x)\big)_{\rho\sigma} \Big] \\
&+ t^2\Big[4\mathrm{Tr}[\tilde{S}_Q^{aa'}(x)\gamma^5 S_{Q'}^{bb'}(x)\gamma^5]\delta_{\alpha\sigma}\delta_{\beta\rho} - 2\big(\gamma^5\tilde{S}_Q^{aa'}(x)\gamma^5 S_{Q'}^{bb'}(x)\big)_{\alpha\sigma}\delta_{\rho\beta} + 2\big(S_Q^{aa'}(x)\gamma^5\tilde{S}_{Q'}^{bb'}(x)\gamma^5\big)_{\rho\beta}\delta_{\alpha\sigma} \\
&- 2\big(S_{Q'}^{bb'}(x)\gamma^5\tilde{S}_Q^{aa'}(x)\gamma^5\big)_{\rho\beta}\delta_{\alpha\sigma} + 2\big(\gamma^5\tilde{S}_{Q'}^{bb'}(x)\gamma^5 S_Q^{aa'}(x)\big)_{\alpha\sigma}\delta_{\beta\rho} + \big(S_{Q'}^{bb'}(x)\big)_{\rho\sigma}\big(\gamma^5\tilde{S}_Q^{aa'}(x)\gamma^5\big)_{\alpha\beta} \\
&+ \big(S_Q^{aa'}(x)\big)_{\rho\sigma}\big(\gamma^5\tilde{S}_{Q'}^{bb'}(x)\gamma^5\big)_{\alpha\beta} + \big(S_{Q'}^{bb'}(x)C\gamma^5\big)_{\sigma\alpha}\big(\gamma^5 CS_Q^{aa'}(x)\big)_{\beta\rho} + \big(S_Q^{aa'}(x)C\gamma^5\big)_{\rho\alpha}\big(\gamma^5 CS_{Q'}^{bb'}(x)\big)_{\beta\sigma} \Big] \Big\}.
\end{aligned}
$$

(20)

There is also a symmetric-anti-symmetric form of the CF, $\Pi_{(\mathcal{SA})}^{\mathrm{QCD}}(p,q)$, which is the result of taking one interpolating current (say η) to be in the symmetric and the other ($\bar{\eta}$) in the anti-symmetric form, which represents the strong decays in which the initial baryon is anti-symmetric and the final one is symmetric.

The explicit expression for the heavy quark propagator is given as (see Ref. [65]):

$$
\begin{aligned}
S_Q^{aa'}(x) = \frac{m_Q^2}{4\pi^2}\frac{K_1(m_Q\sqrt{-x^2})}{\sqrt{-x^2}}\delta^{aa'} &- \mathrm{i}\frac{m_Q^2 \slashed{x}}{4\pi^2 x^2}K_2(m_Q\sqrt{-x^2})\delta^{aa'} \\
&- \mathrm{i}g_s \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}kx}\int_0^1 \mathrm{d}u\Big[\frac{\slashed{k}+m_Q}{2(m_Q^2-k^2)^2}\sigma^{\mu\nu}G_{\mu\nu}^{aa'}(ux) \\
&+ \frac{u}{m_Q^2-k^2}x^\mu\gamma^\nu G_{\mu\nu}^{aa'}(ux)\Big] + \cdots .
\end{aligned}
$$

(21)

The first two terms are the free part or perturbative contributions where K_1 and K_2 are modified Bessel functions of the second kind. Terms $\sim G_{\mu\nu}^{ab}$ are due to the expansion of the propagator on the light-cone and corres-

pond to the interaction with the gluon field. Here we use the shorthand notation

$$
G_{\mu\nu}^{aa'} \equiv G_{\mu\nu}^A t_A^{aa'},
$$

(22)

with $A = 1, 2 \cdots 8$ and $t_A = \lambda_A/2$ where λ_A are the Gell-Mann matrices.

Inserting the heavy quark propagator (21) into the CFs (19) and (20) would lead to several kinds of contributions each representing a different Feynman diagram. There are two heavy quark propagators in each term of the CFs. The leading order contribution consists of a bare loop, depicted in Fig. 1. To calculate that, every heavy quark propagator is replaced by its perturbative terms. The non-perturbative part of this contribution comes from the non-local matrix elements of the pseudoscalar meson which are defined in terms of distribution amplitudes (DAs) of twist two and higher.

Multiplication of the perturbative part of one heavy quark propagator and the gluon interaction part of another one leads to contributions which can be calculated using the pseudoscalar meson three-particle DAs. It is responsible for the exchange of one gluon between one of
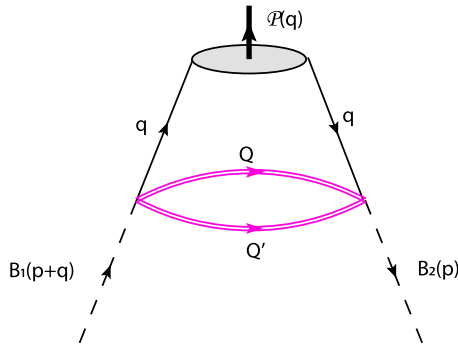
Fig. 1. (color online) The leading order diagram contributing to $\Pi(p,q)$.

Fig. 2. (color online) The one-gluon exchange diagrams.

the heavy quarks and the outgoing meson, as shown in Fig. 2. The higher order contributions corresponding to at least four-particle DAs, which are not available yet, are neglected in this work. However, we take into account the two-gluon condensates contributions $\sim g_s^2\langle GG\rangle$. To proceed we use the replacement

$$\overline{q}_\alpha^{c'}(0)q_\beta^c(x) \rightarrow \frac{1}{3}\delta^{cc'}\overline{q}_\alpha(0)q_\beta(x), \tag{23}$$

which applies the projector onto the color singlet product of quark fields. One can decompose $\overline{q}_\alpha(x)q_\beta(0)$ into terms th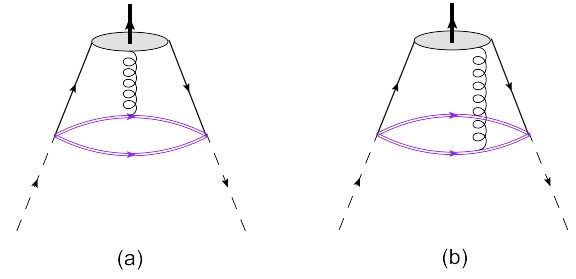at have definite transformation properties under the Lorentz group and parity, using the completeness relation to get the expansion

$$\overline{q}_\alpha(0)q_\beta(x) \equiv \frac{1}{4}\Gamma^J_{\beta\alpha}\overline{q}(0)\Gamma_J q(x), \tag{24}$$

where Γ^J runs over all possible $\gamma-$matrices with definite parity and Lorentz transformation property as

$$\Gamma^J = \mathbf{1}, \gamma_5, \gamma_\mu, i\gamma_5\gamma_\mu, \sigma_{\mu\nu}/\sqrt{2}. \tag{25}$$

This helps us to project quarks onto the corresponding distribution amplitudes.

In the following, we would like to briefly explain how the contributions of, for instance, the diagrams in Fig. 1 and Fig. 2 are calculated. For a symmetric current and Fig. 1, we get

$$\Pi^{\text{QCD}(1)}_{(S)\rho\sigma}(p,q) = \frac{i}{4}\int d^4x e^{iq\cdot x}\langle\mathcal{P}(q)|\overline{q}(0)\Gamma^J q(x)|0\rangle\Bigg\{\Bigg[\text{Tr}[\Gamma_J\tilde{S}^{(\text{pert.})}_Q(x)]\Big(\gamma_5 S^{(\text{pert.})}_{Q'}(x)\gamma_5\Big)_{\rho\sigma} + \Big(\gamma_5 S^{(\text{pert.})}_{Q'}(x)\tilde{\Gamma}_J S^{(\text{pert.})}_Q(x)\gamma_5\Big)_{\rho\sigma}$$

$$+ t\Big\{\text{Tr}[\Gamma_J\gamma_5\tilde{S}^{(\text{pert.})}_Q(x)]\Big(\gamma_5 S^{(\text{pert.})}_{Q'}(x)\Big)_{\rho\sigma} + \text{Tr}[\Gamma_J\tilde{S}^{(\text{pert.})}_Q(x)\gamma_5]\Big(S^{(\text{pert.})}_{Q'}(x)\gamma_5\Big)_{\rho\sigma} + \Big(\gamma_5 S^{(\text{pert.})}_{Q'}(x)\gamma_5\tilde{\Gamma}_J S^{(\text{pert.})}_Q(x)\Big)_{\rho\sigma}$$

$$- \Big(S^{(\text{pert.})}_{Q'}(x)\tilde{\Gamma}_J\gamma_5 S^{(\text{pert.})}_Q(x)\gamma_5\Big)_{\rho\sigma}\Big\} + t^2\Big\{\text{Tr}[\Gamma_J\gamma_5\tilde{S}^{(\text{pert.})}_Q(x)\gamma_5]\Big(S^{(\text{pert.})}_{Q'}(x)\Big)_{\rho\sigma}$$

$$- \Big(S^{(\text{pert.})}_{Q'}(x)\gamma_5\tilde{\Gamma}_J\gamma_5 S^{(\text{pert.})}_Q(x)\Big)_{\rho\sigma}\Big\}\Bigg] + \Big(Q \leftrightarrow Q'\Big)\Bigg\}, \tag{26}$$

where $\tilde{\Gamma}_J = C\Gamma^T_J C$ and

$$S^{(\text{pert.})}_Q(x) = \frac{m_Q^2}{4\pi^2}\frac{K_1(m_Q\sqrt{-x^2})}{\sqrt{-x^2}} - i\frac{m_Q^2\slashed{x}}{4\pi^2 x^2}K_2\Big(m_Q\sqrt{-x^2}\Big). \tag{27}$$

Figure 2 denotes contributions to the CF due to the exchange of one gluon between the heavy quark Q or Q' and the pseudoscalar meson $\mathcal{P}$. To be precise, taking the emission of the gluon from Q, Fig. 2(a), one can write the corresponding CF by replacing $S^{bb'}_Q(x)$ with its perturbative free part (27) and $S^{aa'}_Q(x)$ with its non-perturbative gluonic part

$$S^{aa'(\text{non-p.})}_Q(x) = -ig_s\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\int_0^1 du G^{aa'}_{\mu\nu}(ux)\Delta^{\mu\nu}_Q(x), \tag{28}$$

where $\Delta^{\mu\nu}_Q(x)$ is defined to be:

$$\Delta^{\mu\nu}_Q(x) = \frac{1}{2(m_Q^2-k^2)^2}\Big[(\slashed{k}+m_Q)\sigma^{\mu\nu} + 2u(m_Q^2-k^2)x^\mu\gamma^\nu\Big]. \tag{29}$$

After summing over the color indices one can find the following relation for the symmetric current and contribution of the exchange of the gluon between the heavy quark Q and the light pseudoscalar meson $\mathcal{P}$ (Fig. 2(a)):

$$
\begin{aligned}
\Pi^{\mathrm{QCD(2a)}}_{(S)\rho\sigma}(p,q) = -\frac{g_s}{12} \int \mathrm{d}^4x \int \frac{\mathrm{d}^4k}{(2\pi)^4} \int_0^1 \mathrm{d}u \, \mathrm{e}^{\mathrm{i}(q-k).x} \langle \mathcal{P}(q)|\bar{q}(x)\Gamma^J G_{\mu\nu}(ux)q(0)|0\rangle \times \Bigg\{ \Bigg[\mathrm{Tr}[\Gamma_J \tilde{\Delta}^{\mu\nu}_Q(x)]\Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5\Big)_{\rho\sigma} \\
+ \Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\tilde{\Gamma}_J \Delta^{\mu\nu}_Q(x)\gamma_5\Big)_{\rho\sigma} + t\Big\{ \mathrm{Tr}[\Gamma_J \gamma_5 \tilde{\Delta}^{\mu\nu}_Q(x)]\Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\Big)_{\rho\sigma} + \mathrm{Tr}[\Gamma_J \tilde{\Delta}^{\mu\nu}_Q(x)\gamma_5]\Big(S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5\Big)_{\rho\sigma} \\
+ \Big(\gamma_5 S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5 \tilde{\Gamma}_J \Delta^{\mu\nu}_Q(x)\Big)_{\rho\sigma} - \Big(S^{(\mathrm{pert.})}_{Q'}(x)\tilde{\Gamma}_J \gamma_5 \Delta^{\mu\nu}_Q(x)\gamma_5\Big)_{\rho\sigma} \Big\} + t^2 \Big\{ \mathrm{Tr}[\Gamma_J \gamma_5 \tilde{\Delta}^{\mu\nu}_Q(x)\gamma_5]\Big(S^{(\mathrm{pert.})}_{Q'}(x)\Big)_{\rho\sigma} \\
- \Big(S^{(\mathrm{pert.})}_{Q'}(x)\gamma_5 \tilde{\Gamma}_J \gamma_5 \Delta^{\mu\nu}_Q(x)\Big)_{\rho\sigma}\Big\} \Bigg] + \Big(\Delta^{\mu\nu}_Q(x) \leftrightarrow S^{(\mathrm{pert.})}_{Q'}(x)\Big) \Bigg\},
\end{aligned}
\tag{30}
$$

where $\tilde{\Delta}^{\mu\nu}_Q(x) = C\Delta^{T,\mu\nu}_Q(x)C$. The other contribution, $\Pi^{\mathrm{QCD(2b)}}_{(S)}$, which is responsible for the exchange of one gluon between the heavy quark Q' and the pseudoscalar meson $\mathcal{P}$, can simply be calculated by taking into account the perturbative part of the Q-propagator and the one-gluon emission part of the Q'-propagator. Other contributions, as well as the anti-symmetric CF, are calculated in a similar way. The results are too lengthy to present here.

From Eqs. (26) and (30) it is clear that the non-perturbative nature of the interaction is represented by the non-local matrix elements:

$$
\langle \mathcal{P}(q)|\bar{q}(x)\Gamma^J q(0)|0\rangle ,
$$
$$
\langle \mathcal{P}(q)|\bar{q}(x)\Gamma^J G_{\mu\nu}(ux)q(0)|0\rangle ,
\tag{31}
$$

which can be expressed in terms of two- and three-particle DAs of different twists for the light pseudoscalar meson $\mathcal{P}$. The expressions for the above matrix elements in terms of DAs and the explicit form of the DAs are given in Appendices A and B, respectively.

Inserting the heavy quark propagators (21) and the expressions for non-local matrix elements (31) from Appendix A into the CFs (26) and (30), one obtains the version of CF that is ready for performing the Fourier and Borel transformations as well as continuum subtraction. At this stage the CF contains several kinds of configurations with the general form:

$$
\begin{aligned}
T_{[\ ,\alpha\beta]}(p,q) = \mathrm{i} \int \mathrm{d}^4x \int_0^1 \mathrm{d}v \int \mathcal{D}\alpha \, \mathrm{e}^{\mathrm{i}p.x}(x^2)^n \\
\times [\mathrm{e}^{\mathrm{i}(\alpha_{\bar{q}}+v\alpha_g)q.x}\mathcal{G}(\alpha_i), \mathrm{e}^{\mathrm{i}q.x}f(u)][1, x_\alpha, x_\alpha x_\beta] \\
\times K_\mu\Big(m_Q\sqrt{-x^2}\Big)K_\nu(m_Q\sqrt{-x^2}).
\end{aligned}
\tag{32}
$$

Here the expressions in the brackets represent different structures that come from calculations. The blank subscript bracket indicates no x_α in the structure and n is an integer. The two and three-particle matrix elements lead to wave functions that are denoted by $f(u)$ and $\mathcal{G}(\alpha_i)$ re-

spectively. $\mathcal{D}\alpha$ is called the measure and is defined as:

$$
\int \mathcal{D}\alpha = \int_0^1 \mathrm{d}\alpha_q \int_0^1 \mathrm{d}\alpha_{\bar{q}} \int_0^1 \mathrm{d}\alpha_g \delta(1-\alpha_q-\alpha_{\bar{q}}-\alpha_g).
$$

There are several representations for the Bessel function K_ν. Here we use the cosine representation as

$$
K_\nu(m_Q\sqrt{-x^2}) = \frac{\Gamma(\nu+1/2)\,2^\nu}{\sqrt{\pi}m_Q^\nu} \int_0^\infty \mathrm{d}t \, \cos(m_Q t)\frac{(\sqrt{-x^2})^\nu}{(t^2-x^2)^{\nu+1/2}},
\tag{33}
$$

which helps us to increase the radius of convergence of the CF [66]. To perform the Fourier transformation we use the exponential representations of the $x-$structures:

$$
\begin{aligned}
(x^2)^n &= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\beta^n}(\mathrm{e}^{-\beta x^2})|_{\beta=0}, \\
x_\alpha \mathrm{e}^{\mathrm{i}P.x} &= (-\mathrm{i})\frac{\mathrm{d}}{\mathrm{d}P^\alpha}\mathrm{e}^{\mathrm{i}P.x}.
\end{aligned}
\tag{34}
$$

To be specific, one specific configuration that appears has the generic form

$$
\begin{aligned}
\mathcal{Z}_{\alpha\beta}(p,q) = \mathrm{i} \int \mathrm{d}^4x \int_0^1 \mathrm{d}v \int \mathcal{D}\alpha \mathrm{e}^{\mathrm{i}[p+(\alpha_{\bar{q}}+v\alpha_g)q].x} \\
\times \mathcal{G}(\alpha_i)(x^2)^n x_\alpha x_\beta \\
\times K_\mu\Big(m_Q\sqrt{-x^2}\Big)K_\nu\Big(m_Q\sqrt{-x^2}\Big).
\end{aligned}
\tag{35}
$$

Using some variable changes and performing the double Borel transformation by employing

$$
\mathcal{B}_{p_1}(M_1^2)\mathcal{B}_{p_2}(M_2^2)\mathrm{e}^{b(p+uq)^2} = M^2\delta\Big(b+\frac{1}{M^2}\Big)\delta(u_0-u)\mathrm{e}^{\frac{-q^2}{M_1^2+M_2^2}},
\tag{36}
$$

in which $u_0 = M_1^2/(M_1^2+M_2^2)$, one can find the final Borel-transformed result for the corresponding structure:

$$\mathcal{Z}_{\alpha\beta}(M^2) = \frac{i\pi^2 2^{4-\mu-\nu} e^{\frac{-q^2}{M_1^2+M_2^2}}}{M^2 m_{Q_1}^{2\mu} m_{Q_2}^{2\nu}} \int \mathcal{D}\alpha \int_0^1 \mathrm{d}v \int_0^1 \mathrm{d}z \frac{\partial^n}{\partial\beta^n} e^{-\frac{m_1^2 \bar{z}+m_2^2 z}{z\bar{z}(M^2-4\beta)}} z^{\mu-1} \bar{z}^{\nu-1} (M^2-4\beta)^{\mu+\nu-1}$$

$$\times \delta[u_0-(\alpha_q+v\alpha_g)]\Big[p_\alpha p_\beta + (v\alpha_g+\alpha_q)(p_\alpha q_\beta + q_\alpha p_\beta) + (v\alpha_g+\alpha_q)^2 q_\alpha q_\beta + \frac{M^2}{2} g_{\alpha\beta}\Big]. \tag{37}$$

The details of the calculation can be found in Ref. [60].

According to Eq. (20), we choose the structure $\slashed{q}\slashed{p}\gamma_5$ on the QCD side as well. Therefore, one can write

$$\Pi_{B_1 B_2 \mathcal{P}}^{\mathrm{QCD}}(M^2) = \Pi_{B_1 B_2 \mathcal{P}}(M^2) \slashed{q}\slashed{p}\gamma_5, \tag{38}$$

where B_1, B_2 and $\mathcal{P}$ represent the initial baryon, final baryon and pseudoscalar meson, respectively. The coefficient of the structure $\slashed{q}\slashed{p}\gamma_5$, i.e. $\Pi_{B_1 B_2 \mathcal{P}}(M^2)$, is obtained considering all the contributions discussed above. As an example, we present the expression for the invariant function for the specific channel $\Omega_{bb} \to \Xi_{bb}\bar{K}^0$ after the Borel transformation, which reads:

$$\Pi_{\Omega_{bb}\Xi_{bb}\bar{K}^0}(M^2) = \frac{e^{-\frac{m_{\bar{K}^0}^2}{4M^2}}}{6912\pi^2 M^6 m_b} \int_0^1 \mathrm{d}z \frac{e^{-\frac{m_b^2}{M^2 z\bar{z}}}}{z^2 \bar{z}^2} \Big\{ 72 m_b M^6 z\bar{z}^2 \Big(3 f_{\bar{K}^0} m_{\bar{K}^0}^2 m_b (t^2-1)(m_b^2 + 2M^2 z\bar{z}) \mathbb{A}(u_0)$$

$$+ 2M^2 z \Big[-6 f_{\bar{K}^0} m_b M^2 (t^2-1)\bar{z}\varphi_{\bar{K}^0}(u_0) + \mu_{\bar{K}^0}(\tilde{\mu}_{\bar{K}^0}^2-1)\Big(2m_b^2(1+t^2) + 3M^2(t-1)^2 z\bar{z}\Big)\varphi_\sigma(u_0)\Big] \Big)$$

$$+ 432 m_b M^8 z\bar{z}^2 \int_0^1 \mathrm{d}v \int \mathcal{D}\alpha \Big(f_{\bar{K}^0} m_{\bar{K}^0}^2 m_b (t^2-1)\delta[u_0-(\alpha_q+v\alpha_g)] \times \Big((2v-1)z\mathcal{A}_\parallel(\alpha_i)$$

$$+ (2z-3)\mathcal{V}_\parallel(\alpha_i) \Big) + 2\bar{z}\mathcal{V}_\perp(\alpha_i) \Big) - \mu_{\bar{K}^0} M^2(t-1)^2 z\bar{z}\delta'[u_0-(\alpha_q+v\alpha_g)]\mathcal{T}(\alpha_i)\Big)$$

$$+ g_s^2\langle GG \rangle \Big[-3 f_{\bar{K}^0} m_{\bar{K}^0}^2 (t^2-1)\Big(2m_b^6 - 3m_b^4 M^2\bar{z}^2 - 6m_b^2 M^4 z\bar{z}^3 - 6M^6 z^2\bar{z}^4 \mathbb{A}(u_0)\Big)$$

$$+ 4M^2\bar{z}\Big[-3 f_{\bar{K}^0}(t^2-1)z\Big(-2m_b^4 - m_b^2 M^2(5z-3)\bar{z} + 6M^4 z\bar{z}^3\Big)\varphi_{\bar{K}^0}(u_0) - \mu_{\bar{K}^0}(\tilde{\mu}_{\bar{K}^0}^2-1)m_b\Big(2m_b^4(1+t^2)$$

$$+ m_b^2 M^2[(1+t^2)(1-4z)-6t]z + M^4[(1+t^2)(1-4z)-6t]z^2\bar{z}\varphi_\sigma(u_0)\Big)\Big]$$

$$+ 6\int_0^1 \mathrm{d}v \int \mathcal{D}\alpha \Big(-f_{\bar{K}^0} m_{\bar{K}^0}^2 M^2(t^2-1)\bar{z}\delta[u_0-(\alpha_q+v\alpha_g)]$$

$$\times \Big\{ (2v-1)\Big[m_b^4 - m_b^2 M^2 z(1+2z) - M^4 z^2(1+2z)\bar{z}\Big]\mathcal{A}_\parallel(\alpha_i)$$

$$+ \Big[-4m_b^2 + m_b^2 M^2 z(1+2z) + M^4 z^2(1+z-2z^2)\Big]\mathcal{V}_\parallel(\alpha_i) + 2m_b^4 \mathcal{V}_\perp(\alpha_i) \Big\}$$

$$+ m_b M^4 \mu_{\bar{K}^0}(t-1)2(2v-1)z\bar{z}(m_b^2 + M^2 z\bar{z})\delta'[u_0-(\alpha_q+v\alpha_g)]\mathcal{T}(\alpha_i)\Big) \Big] \Big\}. \tag{39}$$

The invariant functions for other channels on the QCD side are calculated in the same manner but are not presented here because of their very lengthy expressions.

The next step is the continuum subtraction. To this end, we set the argument in $e^{-\frac{m_1^2 \bar{z}+m_2^2 z}{M^2 z\bar{z}}}$ (in the case of different heavy quarks) or $e^{-\frac{m_1^2}{M^2 z\bar{z}}}$ (in the case of identical heavy quarks) equal to s_0, with s_0 being the continuum threshold for higher states and continuum. Here, m_1 and m_2 stand for the heavy quark masses. As a result, the limits of z are changed:

$$\int_0^1 \mathrm{d}z \to \int_{z_{\min}}^{z_{\max}} \mathrm{d}z, \tag{40}$$

where for two different heavy quarks we get:

$$z_{\max(\min)} = \frac{1}{2s_0}\Big[(s_0 + m_1^2 - m_2^2)$$

$$+ (-)\sqrt{(s0 + m_1^2 - m_2^2)^2 - 4m_1^2 s_0}\Big]. \tag{41}$$

In the case of the heavy quarks being the same, we just need to put $m_1 = m_2$ in this result.

After performing continuum subtraction, the invariant function becomes s_0-dependent. We also have another auxiliary parameter t in the currents. Hence, in terms of the three auxiliary parameters, we analyze the results numerically with respect to their variations, and after equating the coefficients of the selected structures from both the physical and QCD sides, we get the strong coupling constants as:

$$g_{B_1 B_2 \mathcal{P}}(M^2, s_0, t) = \frac{1}{\lambda_{B_1} \lambda_{B_2}} e^{\frac{m_{B_1}^2 + m_{B_2}^2}{2M^2}} \Pi_{B_1 B_2 \mathcal{P}}(M^2, s_0, t). \quad (42)$$

We analyze these sum rules numerically in the next section.

III. NUMERICAL RESULTS

There are two sets of input parameters needed to perform the numerical analysis. One corresponds to the mass and decay constants of the light pseudoscalar meson as well as the non-perturbative parameters coming from the light-cone DAs of different twists calculated at the renormalization scale $\mu = 1$ GeV. These parameters are collected in Tables 2 and 3, respectively. The other set of input parameters corresponds to the doubly heavy baryons masses and residues, which are taken from Ref. [18] and are presented in Table 4.

The sum rules for the strong coupling constants also depend on three auxiliary parameters M^2, s_0 and t that should be fixed. To find the working intervals of these parameters, the standard prescriptions of the method are used: the variations of the results with respect to the changes in these parameters should be minimal. First of all we would like to fix the working region of t. Considering $t = \tan\theta$, we plot the strong coupling constants with respect to $\cos\theta$. We choose $\cos\theta$ and vary it in the interval $[-1, 1]$ in order to explore t in all regions. Our analysis shows that the variations of the results are minimal in

Table 2. Meson masses and leptonic decay constants along with heavy quark masses [67-70].

Parameters	Values/MeV
m_c	$1.275^{+0.025}_{-0.035}$ GeV
m_b	$4.18^{+0.04}_{-0.03}$ GeV
m_η	547.862 ± 0.018
$m_{\eta'}$	957.78 ± 0.06
m_{K^0}	497.648 ± 0.022
$m_{K^\pm}$	493.677 ± 0.013
f_π	131
f_η	130
$f_{\eta'}$	136
f_K	160

Table 3. Input parameters for twist 2, 3 and 4 DAs at the renormalization scale $\mu = 1$ GeV [68, 69].

meson	a_2	η_3	w_3	η_4	w_4
π	0.44	0.015	-3	10	0.2
K	0.16	0.015	-3	0.6	0.2
η	0.2	0.013	-3	0.5	0.2

Table 4. Baryon masses and residues [18].

Baryon	Mass/GeV	Residue/GeV^3
Ξ_{cc}	3621.4 ± 0.8 MeV [67]	0.16 ± 0.03
Ξ_{bc}	6.72 ± 0.20	0.28 ± 0.05
Ξ'_{bc}	6.79 ± 0.20	0.3 ± 0.05
Ξ_{bb}	9.96 ± 0.90	0.44 ± 0.08
Ω_{cc}	3.73 ± 0.20	0.18 ± 0.04
Ω_{bc}	6.75 ± 0.30	0.29 ± 0.05
Ω'_{bc}	6.80 ± 0.30	0.31 ± 0.06
Ω_{bb}	9.97 ± 0.90	0.45 ± 0.08

the intervals $0.5 \leqslant \cos\theta \leqslant 0.7$ and $-0.7 \leqslant \cos\theta \leqslant -0.5$ for all the strong decays under study.

The continuum threshold depends on the energy of the first excited state in each channel. Unfortunately, we have no experimental information on the first excited doubly heavy baryons. We choose it in the interval $(m_B + 0.3)^2 \leqslant s_0 \leqslant (m_B + 0.7)^2$ GeV^2, where dependence of the results on s_0 is weak. As examples, we show the dependence of $g_{\Omega^{(\prime)}_{QQ^{(\prime)}} \Omega^{(\prime)}_{QQ^{(\prime)}} \eta(\eta')}$ on continuum threshold in Fig. 3. From this figure, we see that the strong coupling constants depend very weakly on the variation of the s_0 in the selected intervals.

For M^2, the lower and higher limits are determined as follows. The lower bound, $M^2_{\min}$, is found by demanding the OPE convergence. The higher bound, $M^2_{\max}$, is determined by the requirement of pole dominance, i.e.,

$$R = \frac{\int_{(m_Q + m_{Q'})^2}^{s_0} ds \rho(s) e^{-s/M^2}}{\int_{(m_Q + m_{Q'})^2}^{\infty} ds \rho(s) e^{-s/M^2}} \geqslant \frac{1}{2}. \quad (43)$$

Figure 4 displays the dependence of $g_{\Omega^{(\prime)}_{QQ^{(\prime)}} \Omega^{(\prime)}_{QQ^{(\prime)}} \eta(\eta')}$ on M^2 in its working window and at average values of other auxiliary parameters. From this figure we see mild variations of the results with respect to the changes in the Borel parameter M^2. Extracted from the analysis, the working intervals for all auxiliary parameters in all strong decay channels are depicted in Table 5.

The final results for the strong coupling constants under study are also presented in Table 5. The errors in the presented results are due to the uncertainties in determinations of the working intervals for the auxiliary parameters, the intrinsic uncertainties of the method, the errors in the masses and residues of the doubly heavy baryons, and the uncertainties coming from the DA parameters as well as other inputs. As we previously said, in Ref. [60] we investigated the symmetric $\Xi_{QQ^{(\prime)}}$ couplings to the π meson, in which the continuum subtraction procedure is different than that of the present study. We extracted those coupling constants in the present study as well. Compar-
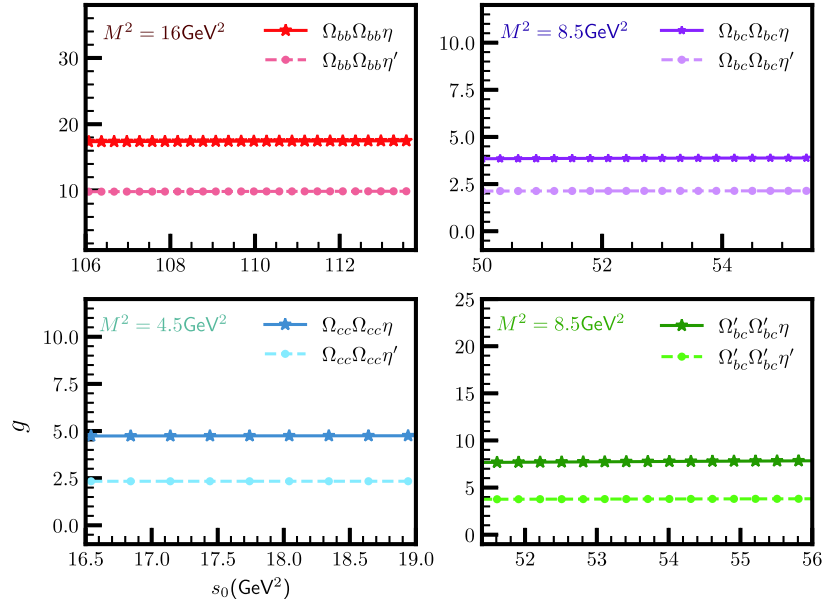
Fig. 3. (color online) The strong couplings $g_{\Omega_{QQ^{(\prime)}}^{(\prime)}\Omega_{QQ^{(\prime)}}^{(\prime)}\eta(\eta')}$ as functions of continuum threshold s_0 at the average value of $\cos\theta$.

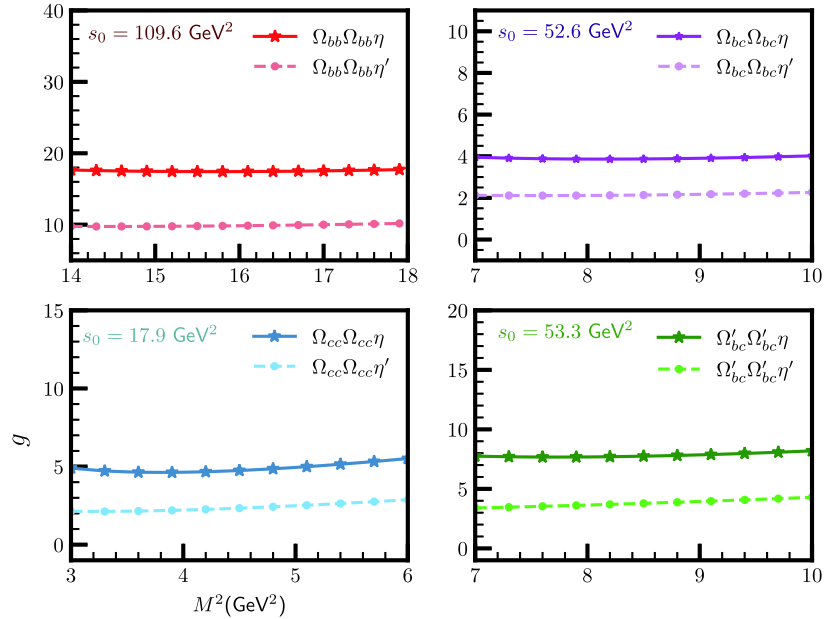**Fig. 4.** (color online) The strong couplings $g_{\Omega_{QQ^{(\prime)}}^{(\prime)}\Omega_{QQ^{(\prime)}}^{(\prime)}\eta(\eta')}$ as functions of M^2 at the average value of $\cos\theta$.

ing the results on $g_{\Xi_{bb}\Xi_{bb}\pi^0}$, $g_{\Xi_{bc}\Xi_{bc}\pi^0}$ and $g_{\Xi_{cc}\Xi_{cc}\pi^0}$ from the present study and Ref. [60], we see that the extracted values are close to each other within the presented errors. The small differences are due to the fact that the values in Ref. [60] were extracted in a single value for $\cos\theta$ inside its working window, while in the present study we take the average of many values obtained at different values of $\cos\theta$ in its working interval. The errors in the present study are small compared to the results of Ref. [60]. From Table 5, it is clear that, overall, the couplings for each symmetric/anti-symmetric case and pseudoscalar meson

in the bb channels are greater than those in the cc channels, and the latter are greater than the couplings in the bc channels. In extracting the results at η and η', we have ignored the mixing between these two states. Our results may be checked via different phenomenological approaches.

IV. SUMMARY AND CONCLUDING NOTES

Motivated by the LHCb observation of the $\Xi_{cc}^{++}(ccu)$ state, we have investigated the strong vertices of the

Table 5. Working regions of the Borel mass M^2 and continuum threshold s_0, with numerical values for different strong coupling constants extracted from the analysis.

Channel	M^2/GeV^2	s_0/GeV^2	strong coupling constant
Decays to π			
$\Xi_{bb} \to \Xi_{bb}\pi^0$	$14 \leqslant M^2 \leqslant 18$	$105.3 \leqslant s_0 \leqslant 113.6$	$17.63^{2.74}_{2.64}$
$\Xi_{bb} \to \Xi_{bb}\pi^\pm$	$14 \leqslant M^2 \leqslant 18$	$105.3 \leqslant s_0 \leqslant 113.6$	$24.93^{3.87}_{3.73}$
$\Xi_{bc} \to \Xi_{bc}\pi^0$	$7 \leqslant M^2 \leqslant 10$	$49.3 \leqslant s_0 \leqslant 55$	$3.76^{0.64}_{0.59}$
$\Xi_{bc} \to \Xi_{bc}\pi^\pm$	$7 \leqslant M^2 \leqslant 10$	$49.3 \leqslant s_0 \leqslant 55$	$5.32^{0.91}_{0.84}$
$\Xi_{cc} \to \Xi_{cc}\pi^0$	$3 \leqslant M^2 \leqslant 6$	$15.4 \leqslant s_0 \leqslant 18.7$	$5.27^{1.40}_{1.22}$
$\Xi_{cc} \to \Xi_{cc}\pi^\pm$	$3 \leqslant M^2 \leqslant 6$	$15.4 \leqslant s_0 \leqslant 18.7$	$7.45^{1.98}_{1.71}$
$\Xi'_{bc} \to \Xi'_{bc}\pi^0$	$7 \leqslant M^2 \leqslant 10$	$50.3 \leqslant s_0 \leqslant 56.1$	$7.84^{1.26}_{1.26}$
$\Xi'_{bc} \to \Xi'_{bc}\pi^\pm$	$7 \leqslant M^2 \leqslant 10$	$50.3 \leqslant s_0 \leqslant 56.1$	$11.08^{1.78}_{1.79}$
$\Xi'_{bc} \to \Xi_{bc}\pi^0$	$7 \leqslant M^2 \leqslant 10$	$50.3 \leqslant s_0 \leqslant 56.1$	$0.62^{0.18}_{0.17}$
$\Xi'_{bc} \to \Xi_{bc}\pi^\pm$	$7 \leqslant M^2 \leqslant 10$	$50.3 \leqslant s_0 \leqslant 56.1$	$0.89^{0.26}_{0.25}$
Decays to K			
$\Omega_{bb} \to \Xi_{bb}\bar{K}^0$	$14 \leqslant M^2 \leqslant 18$	$105.3 \leqslant s_0 \leqslant 113.6$	$22.36^{4.03}_{3.77}$
$\Omega_{bb} \to \Xi_{bb}K^-$	$14 \leqslant M^2 \leqslant 18$	$105.5 \leqslant s_0 \leqslant 113.8$	$22.90^{4.12}_{3.80}$
$\Omega_{bc} \to \Xi_{bc}\bar{K}^0$	$7 \leqslant M^2 \leqslant 10$	$49.7 \leqslant s_0 \leqslant 55.5$	$4.04^{0.85}_{0.73}$
$\Omega_{bc} \to \Xi_{bc}K^-$	$7 \leqslant M^2 \leqslant 10$	$49.7 \leqslant s_0 \leqslant 55.5$	$4.05^{0.85}_{0.74}$
$\Omega_{cc} \to \Xi_{cc}\bar{K}^0$	$3 \leqslant M^2 \leqslant 6$	$16.2 \leqslant s_0 \leqslant 19.6$	$5.76^{1.76}_{1.35}$
$\Omega_{cc} \to \Xi_{cc}K^-$	$3 \leqslant M^2 \leqslant 6$	$16.2 \leqslant s_0 \leqslant 19.6$	$5.78^{1.78}_{1.38}$
$\Omega'_{bc} \to \Xi'_{bc}\bar{K}^0$	$7 \leqslant M^2 \leqslant 10$	$50.4 \leqslant s_0 \leqslant 56.2$	$11.11^{2.38}_{2.08}$
$\Omega'_{bc} \to \Xi'_{bc}K^-$	$7 \leqslant M^2 \leqslant 10$	$50.4 \leqslant s_0 \leqslant 56.2$	$11.14^{2.39}_{2.08}$
Decays to η			
$\Omega_{bb} \to \Omega_{bb}\eta$	$14 \leqslant M^2 \leqslant 18$	$105.3 \leqslant s_0 \leqslant 113.6$	$17.20^{2.93}_{2.93}$
$\Omega_{bc} \to \Omega_{bc}\eta$	$7 \leqslant M^2 \leqslant 10$	$49.7 \leqslant s_0 \leqslant 55.5$	$3.36^{0.64}_{0.57}$
$\Omega_{cc} \to \Omega_{cc}\eta$	$3 \leqslant M^2 \leqslant 6$	$16.2 \leqslant s_0 \leqslant 19.6$	$4.14^{1.19}_{0.91}$
$\Omega'_{bc} \to \Omega'_{bc}\eta$	$7 \leqslant M^2 \leqslant 10$	$50.4 \leqslant s_0 \leqslant 56.2$	$8.38^{1.61}_{1.46}$
Decays to η'			
$\Omega_{bb} \to \Omega_{bb}\eta'$	$14 \leqslant M^2 \leqslant 18$	$105.3 \leqslant s_0 \leqslant 113.6$	$9.54^{1.95}_{1.86}$
$\Omega_{bc} \to \Omega_{bc}\eta'$	$7 \leqslant M^2 \leqslant 10$	$49.7 \leqslant s_0 \leqslant 55.5$	$1.78^{0.41}_{0.37}$
$\Omega_{cc} \to \Omega_{cc}\eta'$	$3 \leqslant M^2 \leqslant 6$	$16.2 \leqslant s_0 \leqslant 19.6$	$1.80^{0.80}_{0.80}$
$\Omega'_{bc} \to \Omega'_{bc}\eta'$	$7 \leqslant M^2 \leqslant 10$	$50.4 \leqslant s_0 \leqslant 56.2$	$4.43^{1.14}_{1.04}$

doubly heavy baryons of various quark contents with the light pseudoscalar π, K, η and η' mesons. We extracted the strong coupling constants at $q^2 = m_{\mathcal{P}}^2$ from the strong coupling form factors. In the calculations, we used the general forms of the interpolating currents in their symmetric and anti-symmetric forms. We also used the light cone DAs of the pseudoscalar mesons entering the calculations. The strong coupling constants are fundamental objects and their investigation can help us get useful knowledge on the nature of the strong interactions among the participating particles. Such objects can also help us in our understanding of QCD as the theory of strong in-

teractions. One of the main problems in studying the strong interactions between hadrons is to determine their interaction potential. The obtained results may be used in the construction of such strong potentials. The obtained results may also help experimental groups in analysis of the results obtained at hadron colliders. Investigation of doubly heavy baryons may help experiments in the course of searches for doubly heavy baryons. Note again that, we have only one state, Ξ_{cc}, has been detected experimentally and and listed in the PDG. However, even for this particle there is a tension between the SELEX and LHCb results on the mass and width of this particle. More

theoretical and experimental studies on the properties of doubly heavy baryons and their various interactions with other hadrons are needed.

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NOTE ADDED:

After completing this study and in the final stages of proof reading we noticed that the paper Ref. [71] appeared in arXiv, where the authors calculate the strong coupling constants among some of the doubly heavy baryons and π and K mesons.

APPENDIX A: NON-LOCAL MATRIX ELEMENTS

In this Appendix, we present explicit expressions for the non-local matrix elements up to twist 4 in terms of the DAs that enter our calculations. These expressions are well known and have been taken from Refs. [69, 70]. Here the π meson represents all pseudoscalar mesons.

$$\langle\pi(p)|\bar{q}(x)\gamma_\mu\gamma_5 q(0)|0\rangle = -\mathrm{i}f_\pi p_\mu \int_0^1 \mathrm{d}u\mathrm{e}^{\mathrm{i}\bar{u}px}\left(\varphi_\pi(u)+\frac{1}{16}m_\pi^2 x^2 \mathbb{A}(u)\right) - \frac{\mathrm{i}}{2}f_\pi m_\pi^2 \frac{x_\mu}{px}\int_0^1 \mathrm{d}u\mathrm{e}^{\mathrm{i}\bar{u}px}\mathbb{B}(u),$$

$$\langle\pi(p)|\bar{q}(x)i\gamma_5 q(0)|0\rangle = \mu_\pi \int_0^1 \mathrm{d}u\mathrm{e}^{\mathrm{i}\bar{u}px}\varphi_P(u),$$

$$\langle\pi(p)|\bar{q}(x)\sigma_{\alpha\beta}\gamma_5 q(0)|0\rangle = \frac{\mathrm{i}}{6}\mu_\pi\left(1-\tilde{\mu}_\pi^2\right)\left(p_\alpha x_\beta - p_\beta x_\alpha\right)\int_0^1 \mathrm{d}u\mathrm{e}^{\mathrm{i}\bar{u}px}\varphi_\sigma(u),$$

$$\langle\pi(p)|\bar{q}(x)\sigma_{\mu\nu}\gamma_5 g_s G_{\alpha\beta}(vx)q(0)|0\rangle = \mathrm{i}\mu_\pi\left[p_\alpha p_\mu\left(g_{\nu\beta}-\frac{1}{px}(p_\nu x_\beta+p_\beta x_\nu)\right) - p_\alpha p_\nu\left(g_{\mu\beta}-\frac{1}{px}(p_\mu x_\beta+p_\beta x_\mu)\right)\right.$$
$$\left. - p_\beta p_\mu\left(g_{\nu\alpha}-\frac{1}{px}(p_\nu x_\alpha+p_\alpha x_\nu)\right) + p_\beta p_\nu\left(g_{\mu\alpha}-\frac{1}{px}(p_\mu x_\alpha+p_\alpha x_\mu)\right)\right]$$
$$\times \int \mathcal{D}\alpha\mathrm{e}^{\mathrm{i}(\alpha_{\bar{q}}+v\alpha_g)px}\mathcal{T}(\alpha_i),$$

$$\langle\pi(p)|\bar{q}(x)\gamma_\mu\gamma_5 g_s G_{\alpha\beta}(vx)q(0)|0\rangle = p_\mu(p_\alpha x_\beta - p_\beta x_\alpha)\frac{1}{px}f_\pi m_\pi^2 \int \mathcal{D}\alpha\mathrm{e}^{\mathrm{i}(\alpha_{\bar{q}}+v\alpha_g)px}\mathcal{A}_\parallel(\alpha_i) + \left[p_\beta\left(g_{\mu\alpha}-\frac{1}{px}(p_\mu x_\alpha+p_\alpha x_\mu)\right)\right.$$
$$\left. - p_\alpha\left(g_{\mu\beta}-\frac{1}{px}(p_\mu x_\beta+p_\beta x_\mu)\right)\right]f_\pi m_\pi^2\times \int \mathcal{D}\alpha\mathrm{e}^{\mathrm{i}(\alpha_{\bar{q}}+v\alpha_g)px}\mathcal{A}_\perp(\alpha_i),$$

$$\langle\pi(p)|\bar{q}(x)\gamma_\mu ig_s G_{\alpha\beta}(vx)q(0)|0\rangle = p_\mu(p_\alpha x_\beta - p_\beta x_\alpha)\frac{1}{px}f_\pi m_\pi^2 \int \mathcal{D}\alpha\mathrm{e}^{\mathrm{i}(\alpha_{\bar{q}}+v\alpha_g)px}\mathcal{V}_\parallel(\alpha_i) + \left[p_\beta\left(g_{\mu\alpha}-\frac{1}{px}(p_\mu x_\alpha+p_\alpha x_\mu)\right)\right.$$
$$\left. - p_\alpha\left(g_{\mu\beta}-\frac{1}{px}(p_\mu x_\beta+p_\beta x_\mu)\right)\right]f_\pi m_\pi^2\times \int \mathcal{D}\alpha\mathrm{e}^{\mathrm{i}(\alpha_{\bar{q}}+v\alpha_g)px}\mathcal{V}_\perp(\alpha_i). \tag{A1}$$

APPENDIX B: DAS FOR THE LIGHT PSEUDOSCALAR MESONS

Here we present the DAs for the light pseudoscalar mesons which appear in our calculations. They are well known and taken from Refs. [69, 70].

$$\phi_\pi(u) = 6u\bar{u}\left(1 + a_1^\pi C_1(2u-1) + a_2^\pi C_2^{\frac{3}{2}}(2u-1)\right),$$

$$\mathcal{T}(\alpha_i) = 360\eta_3\alpha_{\bar{q}}\alpha_q\alpha_g^2\left(1 + w_3\frac{1}{2}(7\alpha_g-3)\right),$$

$$\phi_P(u) = 1 + \left(30\eta_3 - \frac{5}{2}\frac{1}{\mu_\pi^2}\right)C_2^{\frac{1}{2}}(2u-1)$$
$$+ \left(-3\eta_3 w_3 - \frac{27}{20}\frac{1}{\mu_\pi^2} - \frac{81}{10}\frac{1}{\mu_\pi^2}a_2^\pi\right)C_4^{\frac{1}{2}}(2u-1),$$

$$\phi_\sigma(u) = 6u\bar{u}\left[1 + \left(5\eta_3 - \frac{1}{2}\eta_3 w_3 - \frac{7}{20}\mu_\pi^2 - \frac{3}{5}\mu_\pi^2 a_2^\pi\right)C_2^{\frac{3}{2}}(2u-1)\right],$$

$$\mathcal{V}_\parallel(\alpha_i) = 120\alpha_q\alpha_{\bar{q}}\alpha_g\left(v_{00}+v_{10}(3\alpha_g-1)\right),$$

$$\mathcal{A}_\parallel(\alpha_i) = 120\alpha_q\alpha_{\bar{q}}\alpha_g\left(0 + a_{10}(\alpha_q - \alpha_{\bar{q}})\right),$$

$$\mathcal{V}_\perp(\alpha_i) = -30\alpha_g^2\left[h_{00}(1-\alpha_g) + h_{01}(\alpha_g(1-\alpha_g) - 6\alpha_q\alpha_{\bar{q}})\right.$$
$$\left. + h_{10}(\alpha_g(1-\alpha_g) - \frac{3}{2}(\alpha_{\bar{q}}^2+\alpha_q^2))\right],$$

$$\mathcal{A}_\perp(\alpha_i) = 30\alpha_g^2(\alpha_{\bar{q}}-\alpha_q)\left[h_{00}+h_{01}\alpha_g+\frac{1}{2}h_{10}(5\alpha_g-3)\right],$$

$$B(u) = g_\pi(u) - \phi_\pi(u),$$

$$g_\pi(u) = g_0 C_0^{\frac{1}{2}}(2u-1) + g_2 C_2^{\frac{1}{2}}(2u-1) + g_4 C_4^{\frac{1}{2}}(2u-1),$$

$$\mathbb{A}(u) = 6u\bar{u}\left[\frac{16}{15} + \frac{24}{35}a_2^\pi + 20\eta_3 + \frac{20}{9}\eta_4\right.$$
$$+ \left(-\frac{1}{15} + \frac{1}{16} - \frac{7}{27}\eta_3 w_3 - \frac{10}{27}\eta_4\right)C_2^{\frac{3}{2}}(2u-1)$$
$$+ \left(-\frac{11}{210}a_2^\pi - \frac{4}{135}\eta_3 w_3\right)C_4^{\frac{3}{2}}(2u-1)\Big]$$
$$+ \left(-\frac{18}{5}a_2^\pi + 21\eta_4 w_4\right)\left[2u^3(10 - 15u + 6u^2)\ln u\right.$$
$$\left.\left. + 2\bar{u}^3(10 - 15\bar{u} + 6\bar{u}^2)\ln\bar{u} + u\bar{u}(2 + 13u\bar{u})\right], \quad \text{(B1)}$$

where $C_n^k(x)$ are the Gegenbauer polynomials and

$$h_{00} = v_{00} = -\frac{1}{3}\eta_4,$$

$$a_{10} = \frac{21}{8}\eta_4 w_4 - \frac{9}{20}a_2^\pi, \quad v_{10} = \frac{21}{8}\eta_4 w_4,$$
$$h_{01} = \frac{7}{4}\eta_4 w_4 - \frac{3}{20}a_2^\pi,$$
$$h_{10} = \frac{7}{4}\eta_4 w_4 + \frac{3}{20}a_2^\pi,$$
$$g_0 = 1,$$
$$g_2 = 1 + \frac{18}{7}a_2^\pi + 60\eta_3 + \frac{20}{3}\eta_4,$$
$$g_4 = -\frac{9}{28}a_2^\pi - 6\eta_3 w_3. \quad \text{(B2)}$$

The constants inside the wavefunctions which are calculated at the renormalization scale of $\mu = 1$ GeV^2 are given as $a_1^\pi = 0$, $a_2^\pi = 0.44$, $\eta_3 = 0.015$, $\eta_4 = 10$, $w_3 = -3$ and $w_4 = 0.2$.

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