The symmetry group of Feynman diagrams and consistency of the BPHZ renormalization scheme*

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Abstract: We study the relation between the symmetry group of a Feynman diagram and its reduced diagrams. We then prove that the counterterms in the BPHZ renormalization scheme are consistent with adding counterterms to the interaction Hamiltonian in all cases, including that of Feynman diagrams with symmetry factors.

Keywords: BPHZ renormalization, Feynman diagram, symmetry group, perturbation Hamiltonian

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I. INTRODUCTION

The principle of renormalization [1-6] in quantum field theory [7-10] is to introduce divergent counterterms in the interaction (perturbation) Hamiltonian (or Lagrangian). These terms naturally yield new terms in the S matrix [11]. They are the counterterms to eliminate the ultraviolet divergence of Feynman integrals for all Feynman diagrams to any given order [4].

Thus there are two concepts: 1. What are the counterterms in the Feynman integrals for a given Feynman diagram [12], and are they enough to give a convergent result [4]? 2. What are the counterterms in the perturbation Hamiltonian? Can they precisely give the counterterms for the Feynman diagrams to a given order? The first question was answered by the BPHZ renormalization scheme [1,4]. The second question can easily be answered for Feynman diagrams without a symmetry factor. However, for symmetric Feynman diagrams, the situation becomes more involved. To the best of our knowledge, the consistency of these two concepts would still need an explicit proof, since the non-trivial symmetry factor is an important issue in perturbation field theory [13-15].

In this paper, we study the symmetry group G_Γ of a Feynman diagram Γ. We find that there is a subgroup $G_\Gamma^{(1)}$ associated with a reduced Feynman diagram $\widetilde{\Gamma}$ which keeps the union of the reduced part of $\Gamma : \bigcup_\tau \gamma_\tau$ invariant. Furthermore, the symmetry group of the reduced Feynman diagrams $\widetilde{\Gamma}$ is the quotient group $G_\Gamma^{(1)}/\prod_\tau G_{\gamma_\tau}$. We

then explicitly give the counterterms in the perturbation Hamiltonian. We further prove that they give the counterterms of Feynman integrals for Feynman diagrams with symmetry factors. We remark that the idea of the current paper arose when the authors were editing a textbook on quantum field theory [16], and this paper is a further discussion of the material regarding the BPHZ scheme in that textbook.

The paper is organized as follows. In the next section, we introduce the symmetry group of a Feynman diagram and derive the symmetry factor appearing in the S matrix via Wick's theorem. In Section III, we review the theory of the BPHZ renormalization scheme, then give the symmetry group of $\widetilde{\Gamma}$ in Section IV. In Section V, we prove that we can consistently introduce new interaction terms (details are given in Appendix A) corresponding to each renormalization part of the Feynman diagram of a field theory, which will naturally give the counterterms required in the BPHZ scheme.

II. SYMMETRY FACTOR AND SYMMETRY GROUP OF A FEYNMAN DIAGRAM

The S matrix of a perturbation field theory is

$$S_{fi} = \sum_n \frac{(-i)^n}{n!} \langle f|T\Big\{ \int d^4 x_1 \cdots d^4 x_n \mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_n) \Big\}|i\rangle \equiv \sum_n S_{fi}^{(n)} . \tag{1}$$

Here $\mathcal{H}_I(x)$ is the perturbation Hamiltonian density, and it is a polynomial of derivatives of the fields. By Wick's theorem, the quantity

$$\langle f|T\left\{\int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_n\mathcal{H}_I(x_1)\cdots\mathcal{H}_I(x_n)\right\}|i\rangle \qquad (2)$$

is expressed as a sum of the product of field (and derivative) contractions for all possible combinations. Each combination of a pairing of the fields corresponds to a Feynman diagram, where $\mathcal{H}_I(x)$ is related to some vertices. For each Feynman diagram, one can calculate a Feynman integral in either coordinate space or momentum space [17].

Since in a vertex of $\mathcal{H}_I(x)$ several fields may be equivalent, the corresponding vertex lines are equivalent. For definiteness, we may assign numbers to the fields, i.e. give a number to each vertex line. Changing the assignment will, in general, give a new term in the Wick's expansion. Its value is equal to the original one. In this way, for a vertex with a group of l equivalent lines in ϕ^l theory, we get a factorial $l!$. In Eq. (2), changing the assignment of $x_1\cdots x_n$ for a given contraction in Wick's expansion will give a new term in this expansion. After integrating over $x_1\cdots x_n$, these terms give the same result. In this way, we get a factorial $n!$. This means we have $n!$ equal terms, each of them corresponding to an assignment of vertices x_i for a Feynman diagram.

A given perturbation Hamiltonian $\mathcal{H}_I(x)$ may have several vertices. Each vertex is composed of a vertex point x and several vertex lines. Each of them represents a field operator. Some vertex lines of the same vertex may be equivalent (Fig. 1). Once we assign a coordinate number to a vertex point: $x = x_i$, and assign numbers $1, 2, \cdots$ to each set of equivalent vertex lines (of the chosen ver-

tex), we actually fix all of the field operators in a vertex in $\mathcal{H}_I(x)$ in Eqs. (1) and (2). Thus when all these vertex lines are connected pair-wise (including those in "in" and "out" states), we get a certain term in Wick's expansion. In this way, we get a Feynman diagram [12] with all vertices given coordinate numbers and all equivalent vertex lines of each vertex given line numbers.

Alternatively, for a blank Feynman diagram, we first assign a coordinate number $(x_1, \cdots, x_m)$ to each vertex point $(p_1, \cdots, p_m)$. Say, $p_1 = x_1, p_2 = x_2 \cdots, p_m = x_m$. Next, for each vertex p_a (and p_b), we assign numbers $1, 2, \cdots$ to each end (attached to each vertex) of those equivalent vertex lines $L_{ab\sigma}$. Then we get a "marked" Feynman diagram, which corresponds to one term in Wick's expansion. Notice that only for topologically different "marked" Feynman diagram, we get different terms in Wick's expansion. While for different assignment of a blank Feynman diagram but topologically equivalent, they correspond to the same term in Wick's expansion (Fig. 2).

The operation of marking a blank Feynman diagram can be related to a group G. Changing the assignment (marking) of a blank (unmarked) Feynman diagram can be expressed as a group element
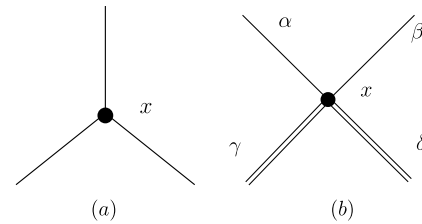

Fig. 1. Vertex with vertex point and vertex lines. (*a*) Three vertex lines are equivalent. (*b*) The two vertex lines α and β are equivalent, while γ and δ are equivalent.
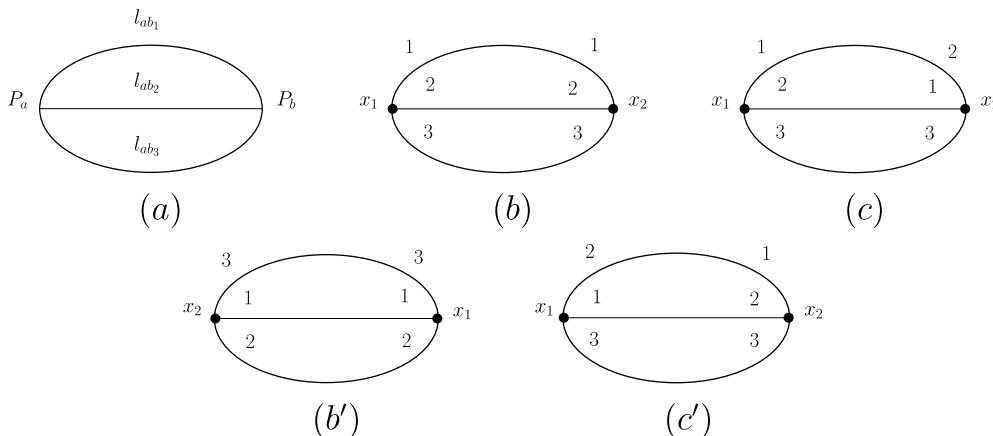
Fig. 2. Marking a Feynman diagram Γ. (*a*) A blank Feynman diagram Γ before marking. (*b*), (*b'*), (*c*), and (*c'*) are marked Feynman diagrams of (*a*). (*b*) and (*b'*) are topologically equivalent (lines 1, 2, and 3 of x_1 connect to lines 1, 2, and 3 of x_2, respectively). (*c*) and (*c'*) are topologically equivalent (lines 1, 2, and 3 of x_1 connect to lines 2, 1, and 3 of x_2, respectively), while (*b*) and (*c*) are not equivalent.

$$g = g_n \times \prod_i g^i. \tag{3}$$

Here g_n is an element of the permutation group S_n, which corresponds to the reassignment of the coordinates of vertices (the permutation group is a symmetry of the Feynman diagram only after the integration over all coordinates is performed; this is also true for vertex lines). g^i is an element of the symmetry group of each vertex which corresponds to the reassignment of the equivalent vertex lines of vertex point p_i. The operations of reassignment which do not change the topological structure of a marked Feynman diagram form a set G_Γ, and it satisfies the properties of a group. G_Γ is the symmetry group of a blank Feynman diagram Γ.

We have $G_\Gamma \subset G$. For element $g_\Gamma \in G_\Gamma$, $g \in G$, and the element of G: $g' = g_\Gamma g$ gives a topologically equivalent marked Feynman diagram to that given by g. Thus the

number of coset $m = |G|/|G_\Gamma|$ is the actual number of different terms in Wick's expansion which give the same blank Feynman integral form (2).

We use the following (Figs. 3 and 4) to further illustrate the action of g, $g_\Gamma g$, and gg_Γ.

We denote $\mathcal{L}(\Gamma)$ as the set of internal lines of Γ and $\mathcal{V}(\Gamma)$ as the set of vertices of Γ. By Wick's expansion of Eq. (2) we get

$$\sum_{\Gamma'(n)} \int d^4x_1 \cdots d^4x_n \prod_{L_{ab\sigma} \in \mathcal{L}(\Gamma)} \widetilde{\Delta}_F^{ab\sigma}(x_a - x_b) \prod_{V_a \in \mathcal{V}(\Gamma)} P_a. \tag{4}$$

Here P_a is a numerical function depending on vertex V_a, and $\widetilde{\Delta}_F^{ab\sigma}(x_a - x_b)$ is a polynomial of derivatives of the Feynman propagator with respect to 4-coordinates x_a and x_b. The summation is over all topologically different marked n point Feynman diagrams $\Gamma'(n)$. After integrat-

(a)

P_1	α_1	β_1	γ_1	
P_2	α_2	β_2	γ_2	
P_3	α_3	β_3	γ_3	δ_3
P_4	α_4	β_4	γ_4	δ_4

(b)

x_1	2	1	3	
x_2	1	2	3	
x_4	1	2	4	3
x_3	1	2	3	4

(c)

x_1	2	1	3	
x_4	2	3	1	
x_2	1	2	3	4
x_3	1	4	2	3

(e) = $g \cdot$ (c)

(d)

(f)

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$
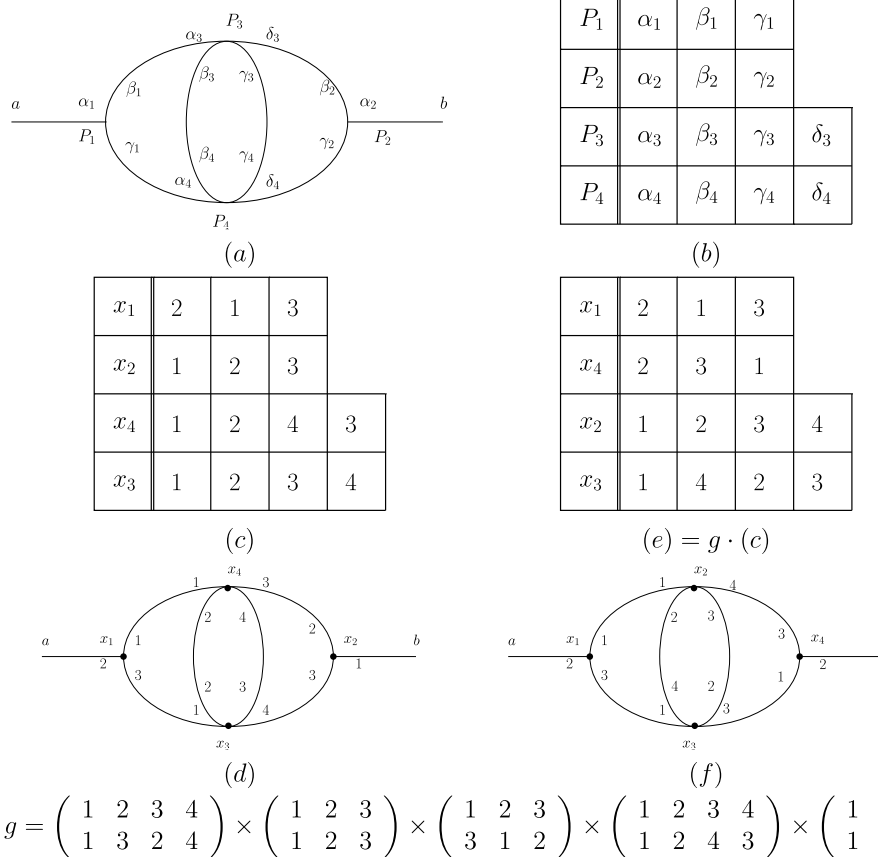
Fig. 3. The action of g (Eq. (3)). The first factor of g acts on vertices, and the rest of factors act on attached vertex lines. (a) A blank Feynman diagram. (b) The vertex points (p_1, p_2, p_3, and p_4) and vertex lines ($p_1 : \alpha_1\beta_1\gamma_1$, $p_2 : \alpha_2\beta_2\gamma_2$, $p_3 : \alpha_3\beta_3\gamma_3\delta_3$, $p_4 : \alpha_4\beta_4\gamma_4\delta_4$), where all vertex lines are equivalent for each vertex. (c) An assignment of (a), which gives a marked Feynman diagram (d). (e) Another assignment of (a), which gives a marked Feynman diagram (f). The group element g is an operation changing the assignment, and it acts on (d) giving (f). The group element g acts on (d), while the indices 1, 2, ⋯ in the above expression of g refer to the indices of the original blank Feynman diagram (a) (and (b)). For example, the first factor of g swaps P_2 with P_3 in (a), which means exchanging x_2 and x_4 in (c) (also (d)). The aim of this way of labeling the Feynman diagram is to better classify different kinds of vertices and vertex lines, and to avoid confusion. This concept is in accordance with the first paragraph in Section 4.
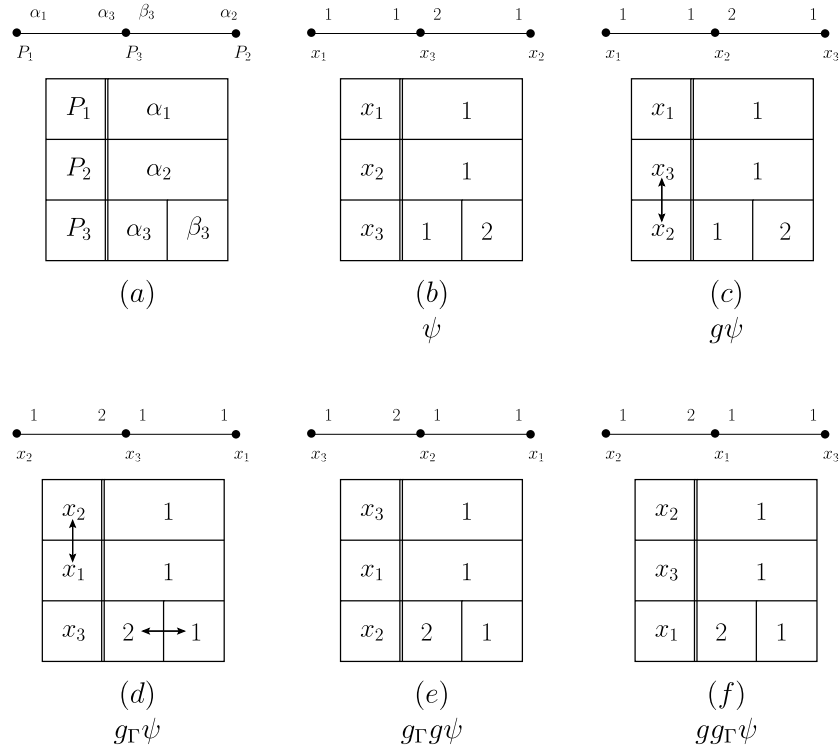
$$g = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \times id \times id \times id, \; g\psi \text{ gives } (c)$$

$$g_\Gamma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \times id \times id \times \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \; g_\Gamma \psi \text{ gives } (d)$$

Fig. 4. The actions of $g_\Gamma g, g$, and gg_Γ. This is similar to the case in Fig. 3. Their first factors act on vertices, and the rest act on vertex lines. Similarly, the indices 1, 2, $\cdots$ in the above expression for g and g_Γ refer to the indices (P_1, P_2, etc.) of the original blank Feynman diagram (a). (a) Blank Feynman diagram (b) A marked Feynman diagram Γ (c) The action of g on ψ: it swaps P_2 with P_3 in (a), which swaps x_2 with x_3 in (b). Still 1, 2, and 3 in the first factor of g correspond to P_1, P_2, and P_3, respectively. (d) The action of g_Γ on ψ: the first factor exchanges P_1 with P_2 (x_2 and x_1 in (d)), and the last factor exchanges vertex line α_3 with line β_3 (2 and 1 of x_3 in (d)). (e) The action of $g_\Gamma g$ on ψ. Based on (c), further action of g_Γ on $g\psi$, which means swapping x_1 with x_3 in (c) (P_1 and P_2 in (a)), and swapping 1 with 2 of x_3 in (c) (vertex line α_3 and line β_3 in (a)). (f) The action of gg_Γ on ψ. Based on (d), further action of g on $g_\Gamma \psi$ means to swap x_1 and x_3 in (d) (P_2 and P_3 in (a)). Here (c) and (e) are topologically equivalent, while (c) and (f) are topologically unequivalent. Thus $g_\Gamma g\psi$ and $g\psi$ correspond to the same term in Wick's expansion, but $(gg_\Gamma)\psi$ and $g\psi$ do not.

ing over all $x_i's$, one obtains an integral expression in momentum space:

$$S = \sum_n \frac{1}{n!} \sum_{\Gamma'(n)} \int \prod_{L_{ab\sigma} \in \mathcal{L}(\Gamma)} \frac{\mathrm{d}^4 l_{ab\sigma}}{(2\pi)^4} \Delta_F^{ab\sigma}(l_{ab\sigma}) \prod_{V_a \in \mathcal{V}(\Gamma)}$$

$$\times \{ P_a(\{l_{ab'\sigma'}\}) \times (2\pi)^4 \delta^4 (\sum_{b'\sigma'} l_{ab'\sigma'} - q_a) \}, \qquad (5)$$

where P_a is a polynomial of $l_{ab'\sigma'}$ (the momentum of line $L_{ab'\sigma'}$), q_a is the total incoming momentum at V_a, while Δ_F is the Feynman propagator in momentum space. In Eq. (4) the $(-i)^n$ factor is absorbed into the $\prod P_a$ term. After integration of the momenta to eliminate δ functions, each term of connected $\Gamma'(n)$ leaves an overall

$(2\pi)^4 \delta^4(\Sigma q_a)$ function and an integral over independent momenta, since for the same blank Feynman diagram, integrals corresponding to different marked Feynman diagrams are equal. The contribution of the S matrix for a connected blank Feynman diagram $\Gamma(n)$ with n vertices is:

$$S_{\Gamma(n)} = \frac{1}{n!} \frac{|G|}{|G_\Gamma|} \int \frac{\mathrm{d}k_1 \cdots \mathrm{d}k_l}{(2\pi)^l} I_\Gamma^0(k)(2\pi)^4 \delta^4 \left(\sum_a q_a \right)$$

$$= \frac{1}{n!} \frac{|G_n| \times |\prod_i G^i|}{|G_\Gamma|} \int \frac{\mathrm{d}k_1 \cdots \mathrm{d}k_l}{(2\pi)^l}$$

$$\times I_\Gamma^0(k)(2\pi)^4 \delta^4 \left(\sum_a q_a \right). \qquad (6)$$

Thus one has:

$$S_{\Gamma(n)} = \frac{1}{|G_\Gamma|} \times \int \frac{dk_1 \cdots dk_l}{(2\pi)^l} |\prod_i G^i|$$
$$\times I_\Gamma^0(k)(2\pi)^4 \delta^4\left(\sum_a q_a\right)$$
$$\equiv \frac{1}{|G_\Gamma|} \times \int \frac{dk_1 \cdots dk_l}{(2\pi)^l} I_\Gamma(k)(2\pi)^4 \delta^4\left(\sum_a q_a\right). \quad (7)$$

The factor $(-i)$ is absorbed into each vertex of $I_\Gamma^0(k)$, while the term $|\prod_i G^i|$ is absorbed into each vertex of $I_\Gamma(k)$.

The total S matrix is then:

$$S = \sum_n \sum_{\Gamma(n)} S_{\Gamma(n)}. \quad (8)$$

The second summation is over all blank Feynman diagrams with n vertices.

In Eq. (6), the momenta $k_1 \cdots k_l$ are essentially those remaining after eliminating the δ functions. The set $\{k_1 \cdots k_l\}$ is denoted by k.

III. BPHZ RENORMALIZATION SCHEME AND VERTEX OF A REDUCED FEYNMAN DIAGRAM

In the following, we deal mainly with diagrams which are connected. If we cut any internal line of a connected diagram and the diagram is still connected, we call such a diagram a proper diagram. A proper diagram γ with superficial dimension [7-9] $d(\gamma) \geqslant 0$ is called a renormalization part [4].

We have for a blank proper Feynman diagram Γ (7):

$$S_{\Gamma(n)} = \frac{1}{|G_\Gamma|} \int \frac{dk_1 \cdots dk_l}{(2\pi)^l} I_\Gamma(k)(2\pi)^4 \delta^4\left(\sum q_a\right). \quad (9)$$

A Feynman integral associated with a proper Feynman diagram is defined as:

$$J_\Gamma = \int \prod_{L_{ab\sigma} \in \mathcal{L}(\Gamma)} \frac{d^4 l_{ab\sigma}}{(2\pi)^4} \prod_{V_a \in \mathcal{V}(\Gamma)} \{(2\pi)^4 \delta^4(\sum l_{ab\sigma} - q_a)\} I_\Gamma(q,l)$$
$$= \int \frac{dk_1 \cdots dk_l}{(2\pi)^l} I_\Gamma(k,q) \times (2\pi)^4 \delta^4\left(\sum_a q_a\right), \quad (10)$$

where $\mathcal{V}(\Gamma)$ is the set of vertices in Γ, q_a is the total incoming momentum at V_a, $\mathcal{L}(\Gamma)$ is the set of internal lines of Γ, and the line $L_{ab\sigma}$ is one of the lines from a vertex V_a to a vertex V_b with 4-momenta $l_{ab\sigma}$. The integrand I_Γ is

$$I_\Gamma = \prod_{L_{ab\sigma} \in \mathcal{L}(\Gamma)} \Delta_F^{ab\sigma} \prod_{V_a \in \mathcal{V}(\Gamma)} P_a, \quad (11)$$

where Δ_F is the Feynman propagator, and P_a is a polynomial of $\{l_{ab\sigma}\}$ determined by the vertex of $\mathcal{H}_I(x)$.

For further derivation, we need to specify the integral parameters $k_i's$. Due to the δ functions at each vertex, we have nonhomogeneous linear equations for the incoming 4-momenta q_a at the vertex V_a,

$$\sum_{b\sigma} l_{ab\sigma} = q_a, \quad a = 1, \cdots, n, \quad (12)$$

where

$$l_{ab\sigma} = -l_{ba\sigma} \quad (13)$$

is the 4-momentum on one of the lines $L_{ab\sigma}$ from the vertex V_a to the vertex V_b. The solution is not unique for

$$\sum_{V_a \in \Gamma} q_a = 0. \quad (14)$$

When Γ is connected, we define a solution for which the quantity [4]

$$M = \sum_{L_{ab\sigma} \in \mathcal{L}(\Gamma)} l_{ab\sigma}^2 = \sum (q_{ab\sigma})^2 \quad (15)$$

is minimal. We call it a canonical distribution of incoming momenta $\{q_a\}$ for Γ. One can prove that $q_{ab\sigma}$ is a linear combination of $q_a's$. Then generally $l_{ab\sigma}$ consists of two parts. One part comes from q_a, and the other part comes from integral variables like $k_{ab\sigma}$ (they are momenta that form inner loop flows in the connected diagram)

$$l_{ab\sigma} = q_{ab\sigma} + k_{ab\sigma}.$$

So here $k_{ab\sigma}$ satisfies the homogeneous linear equations (compared with Eqs. (12) and (13)):

$$\sum_{b\sigma} k_{ab\sigma} = 0, \quad k_{ab\sigma} = -k_{ba\sigma}, \quad a = 1, \cdots, n. \quad (16)$$

It is important to point out that for a subdiagram $\gamma \subset \Gamma$ with a line $L_{ab\sigma} \in \gamma$ (also $L_{ab\sigma} \in \Gamma$), $q_{ab\sigma}^\gamma$ and $q_{ab\sigma}^\Gamma$ (we use the superscript to indicate the belonging) are different. From the linearity of $q_{ab\sigma}$ with respect to q_a, we have the following statement.

Proposition 1. *The difference of $q_{ab\sigma}^\gamma$ and $q_{ab\sigma}^\Gamma$ is the canonical distribution*

$$\Delta q_a = -\sum_{b\sigma} k_{ab\sigma}^\Gamma, \quad L_{ab\sigma} \in \mathcal{L}(\Gamma), L_{ab\sigma} \notin \mathcal{L}(\gamma), \quad (17)$$

and thus they are linear functions of $k_{ab\sigma}^\Gamma$ of lines in Γ.

We can choose (not arbitrarily) some lines of Γ to fix the parameters $k_{ab\sigma}^\Gamma$ and to further fix the solution. When Γ has m loops, we can fix m lines. Then the integrals $k_1 \cdots k_l$ can be chosen as $(k_{ab\sigma}^\Gamma)^\mu$, $\mu = 0, 1, 2, 3$ (for ϕ^4 theory) of these lines. For $\gamma_1, \cdots \gamma_c \subset \Gamma$ and when $\gamma_1, \cdots \gamma_c$ are disjoint diagrams, we can choose them in such a way that when the chosen lines are in γ_τ, they also form independent lines in each γ_τ. In this case, each subdiagram γ_τ can be reduced to a vertex. We denote the lines in the reduced diagram $\widetilde{\Gamma}$ as:

$$L_{ab\sigma}^{\widetilde{\Gamma}} \in \Gamma/\gamma, \quad L_{a'b'\sigma'}^{\gamma_\tau} \in \gamma_\tau.$$

Then due to proposition 1, we can properly choose independent lines such that the integral over independent momenta is:

$$\int \prod^{ind} \mathrm{d}^4 \widetilde{l}_{ab\sigma}^{\widetilde{\Gamma}} \prod_{\tau=1}^c \prod^{ind} \mathrm{d}^4 l_{a'b'\sigma'}^{\gamma_\tau} = \int \prod^{ind} \mathrm{d}^4 k_{ab\sigma}^{\widetilde{\Gamma}} \prod_{\tau=1}^c \prod^{ind} \mathrm{d}^4 k_{a'b'\sigma'}^{\gamma_\tau}. \tag{18}$$

The Feynman integral (10) is generally divergent at large k for Γ with loops. The BPHZ renormalization scheme [7,8] gives a finite part $R_\Gamma = R_\Gamma(k, q_a)$ such that the renormalized integral converges:

$$F_\Gamma = \int \frac{\mathrm{d}k_1 \cdots \mathrm{d}k_l}{(2\pi)^l} R_\Gamma(k, q)(2\pi)^4 \delta^4\left(\sum_a q_a\right). \tag{19}$$

Zimmermann proved the convergence of Eq. (19) [4] by an application of Weinberg's theorem [7]. The integrand $R_\Gamma(k, q)$ of Eq. (19) is defined by:

$$R_\Gamma(k, q) = I_\Gamma(k, q) + \sum_{\gamma_1 \cdots \gamma_c} I_{\Gamma/\gamma_1 \cdots \gamma_c}(kq) \prod_{\tau=1}^c O_{\gamma_\tau}(k^{\gamma_\tau}, q^{\gamma_\tau}). \tag{20}$$

The sum is over all sets of renormalization parts γ_τ, $\tau = 1, \cdots, c$, which are mutually disjoint. The parameters $k = \{k_1 \cdots k_l\}$ are essentially the remaining momenta after integration eliminates the δ functions in Eq. (4). In Eq. (20) we have defined $I_{\Gamma/\gamma_1 \cdots \gamma_\tau} = I_\Gamma / \prod_{\tau=1}^c I_{\gamma_\tau}$, which is only determined by vertices and lines contained in Γ but not in

any γ_τ. The functions O_γ are recursively defined as

$$O_\gamma(k^\gamma, q^\gamma) = -t_{q^\gamma}^{d(\gamma)} \Bigg\{ I_\gamma(k^\gamma, q^\gamma) + \sideset{}{'}\sum_{\gamma_1' \cdots \gamma_{c'}'} I_{\gamma/\gamma_1' \cdots \gamma_{c'}'}(k^\gamma, q^\gamma) \\ \times \prod_{\tau=1}^c O_{\gamma_\tau}(k^{\gamma_\tau}, q^{\gamma_\tau}) \Bigg\}. \tag{21}$$

The sum extends over all sets of mutually disjoint renormalization parts $\{\gamma_1', \cdots, \gamma_{c'}'\}$ (assuming they are subdiagrams of γ), but does not include $\{\gamma\}$. The function O_γ is a Taylor series with respect to the incoming independent momentum $q^\gamma \equiv \{q_a\}$,

$$t_q^d f = \sum_{l=0}^d \frac{1}{l!} \sum_{q_1 \cdots q_l} q_1 \cdots q_l \left(\frac{\partial}{\partial q_1} \cdots \frac{\partial}{\partial q_l} f(g)\right)_{q=0}. \tag{22}$$

The sum of q_j extends over all components of independent incoming 4-momenta of $\{q_a\}$.

Since O_γ is recursively defined, one must calculate it step by step, from smaller interior renormalization parts to bigger outer ones. Due to the fact that for $\gamma' \subset \gamma$, we have proposition 1, we get:

$$q_{a'b'\sigma'}^{\gamma'} = q_{a'b'\sigma'}^\gamma + \text{linear combinations of } \{k_{ab\sigma}^\gamma\}$$

$$k_{a'b'\sigma'}^{\gamma'} = k_{a'b'\sigma'}^\gamma + \text{linear combinations of } \{k_{ab\sigma}^\gamma\}. \tag{23}$$

One can conclude that O_γ is a polynomial of independent components of $\{q_a^\gamma\}$. The coefficients are functions of $\{k_{ab\sigma}^\gamma\}$. Equation (19) can be rewritten as

$$F_\Gamma = \Bigg\{ \int \frac{\mathrm{d}k_1 \cdots \mathrm{d}k_l}{(2\pi)^l} I_\Gamma + \sum_{\gamma_1 \cdots \gamma_c} \int \frac{\mathrm{d}k_1 \cdots \mathrm{d}k_l}{(2\pi)^l} I_{\Gamma/\gamma_1 \cdots \gamma_c}(k, q) \\ \times \prod_{\tau=1}^c O_{\gamma_\tau}(k^{\gamma_\tau}, q^{\gamma_\tau}) \Bigg\} (2\pi)^4 \delta^4\left(\sum_a q_a\right) \equiv J_\Gamma + \sum_{\gamma_1 \cdots \gamma_c} \widetilde{J}_{\Gamma/\gamma_1 \cdots \gamma_c}, \tag{24}$$

where $\widetilde{J}_{\Gamma/\gamma_1 \cdots \gamma_c}$ are counterterms of J_Γ. From Eq. (18) we have:

$$\widetilde{J}_{\Gamma/\gamma_1 \cdots \gamma_c} = \int \frac{\mathrm{d}k_1 \cdots \mathrm{d}k_l}{(2\pi)^l} I_{\Gamma/\gamma_1 \cdots \gamma_c}(k, q) \prod_{\tau=1}^c O_{\gamma_\tau}(k^{\gamma_\tau}, q^{\gamma_\tau})(2\pi)^4 \delta^4\left(\sum_a q_a\right) = \int \prod_{L_{ab\sigma} \in \Gamma/\gamma_1 \cdots \gamma_c}^{ind} \frac{\mathrm{d}^4 k_{ab\sigma}^\Gamma}{(2\pi)^4} I_{\Gamma/\gamma_1 \cdots \gamma_c}(k^\Gamma, q) \prod_{\tau=1}^c \int \prod_{L_{a'b'\sigma'} \in \gamma_\tau}^{ind}$$

$$\times \frac{\mathrm{d}^4 k_{a'b'\sigma'}^{\gamma_\tau}}{(2\pi)^4} O_{\gamma_\tau}(k^{\gamma_\tau}, q^{\gamma_\tau})(2\pi)^4 \delta^4\left(\sum_a q_a\right) \equiv \int \prod_{L_{ab\sigma} \in \Gamma/\gamma_1 \cdots \gamma_c}^{ind} \frac{\mathrm{d}^4 k_{ab\sigma}^\Gamma}{(2\pi)^4} I_{\Gamma/\gamma_1 \cdots \gamma_c}(k^\Gamma, q) \times \prod_{\tau=1}^c Q_{\gamma_\tau}(2\pi)^4 \delta^4\left(\sum_a q_a\right), \tag{25}$$

where $\prod^{ind}$ extends to all independent lines. The integral Q_{γ_τ} is divergent, but it is convergent after regularization. Thus we may regard Q_{γ_τ} as a new vertex which corresponds to a renormalization part γ_τ. We call it a vertex of reduction.

Equation (25) is a Feynman integral for Feynman diagram $\widetilde{\Gamma}$. It is a reduced diagram from Γ, in which all lines and vertices of γ_τ are shrunk to a vertex $\widetilde{Q}_{\gamma_\tau}$.

Can we introduce new interactions (with divergent coefficients) into interaction Lagrangian $\Delta \mathcal{L}_I$ which gives precisely these counterterms to a given order? This is a consistency problem for the BPHZ renormalization scheme. The answer is affirmative. We would like to prove this in the coming sections. It can be proved that the function Q_{γ_τ} has symmetry of γ_τ (Appendix A). We next study the symmetry group of $\widetilde{\Gamma}$ denoted by $G_{\widetilde{\Gamma}}$.

IV. SYMMETRY GROUP OF A REDUCED FEYNMAN DIAGRAM $\widetilde{\Gamma}$

In the derivation in Section II, one group element "g^i" in $g \in G$ for changing the number of equivalent vertex lines in a vertex is attached to a certain point P_a of a blank Feynman diagram. Generally, the vertices are of different type at different points. For the symmetry group of a Feynman diagram Γ, we can alternatively attach such a group element to a running coordinate x_j. This is because in $g_\Gamma \in G_\Gamma$, we always have the same type of vertices for the same x_j before and after reassigning. In this way, we may regard the element of $g_\Gamma \in G_\Gamma$ as a continuous mapping of a Feynman diagram into itself. This maps a vertex (including the vertex point together with all its vertex lines) into another vertex of the same type. In the following, we always refer g_Γ to the second meaning.

We define a set A as the union of all $\gamma_\tau's$

$$A = \bigcup_{\tau=1}^{c} \gamma_\tau , \qquad (26)$$

which includes all internal lines and vertex points of all $\gamma_\tau's$. We define sets B_τ as all internal lines and vertex points of γ_τ; actually it is γ_τ itself.

In the group G_Γ, those elements which do not change the set A (i.e. which map A to A) form a subgroup of G_Γ. We denote it as $G_\Gamma^{(1)}(\gamma_1 \cdots \gamma_c)$. This is because the set of such elements is closed under multiplication and inversion. Similarly, we have a subgroup $G_\Gamma^{(2)}(\gamma_1 \cdots \gamma_c) = \prod_{\tau=1}^{c} G_\Gamma^{(2)}(\gamma_\tau)$, where $G_\Gamma^{(2)}(\gamma_\tau) \subset G_\Gamma$ maps each $B_\tau = \gamma_\tau$ to itself, and $G_\Gamma^{(2)}(\gamma_\tau)$ keeps all lines and vertex points outside γ_τ invariant (identity). Since the mapping keeps the connection relation between the vertex points and the vertex lines of any vertex, $G_\Gamma^{(2)}(\gamma_\tau)$ also keeps the "boundary points" which connect exterior lines of γ_τ invariant. We

can prove that $G_\Gamma^{(2)}$ is a normal subgroup of $G_\Gamma^{(1)}$ since an element of $G_\Gamma^{(2)}$ does not change the exterior of A. Then we can further prove that the symmetry group of the reduced Feynman diagram $\widetilde{\Gamma}$ is the quotient group

$$G_{\widetilde{\Gamma}} = G_\Gamma^{(1)} / G_\Gamma^{(2)} . \qquad (27)$$

This is because it maps the reduced vertices to themselves with their symmetry. We use (Fig. 5) to illustrate the relation between a Feynman diagram Γ and its reduced Feynman diagrams. The configuration of Γ with $\gamma_1 \cdots \gamma_c$ can be mapped to Γ with $\gamma_1', \cdots, \gamma_c'$ (in the same blank configuration) by each element of G_Γ. Those elements which keep the set A invariant form subgroup $G_\Gamma^{(1)}$. Let $g_\Gamma \in G_\Gamma$, and $h \in G_\Gamma^{(1)}$. Then for a certain g_Γ and any $h \in G_\Gamma^{(1)}$, the element $g' = g_\Gamma h$ maps the set A to a set $A' = \bigcup_\tau \gamma_\tau'$. Thus the coset $G_\Gamma / G_\Gamma^{(1)}$ is characterized by different "A'"s which are also sets of disjoint renormalization parts of Γ. Therefore the renormalized integrand (20) can be written as:

$$R_\Gamma(k,q) = I_\Gamma(k,q) + \sum_{\check{\Gamma}}{}_1 \sum_{\text{coset}}{}_2 I_{\Gamma/\gamma_1 \cdots \gamma_c}(k,q) \prod_{\tau=1}^{c} O_{\gamma_\tau}(k^{\gamma_\tau}, q^{\gamma_\tau}) . \qquad (28)$$

The first sum extends over all configurations of $\check{\Gamma} : \Gamma$ with different $\gamma_1 \cdots \gamma_c$, which are topologically different from each other. The second sum extends over configurations produced by coset elements $G_\Gamma / G_\Gamma^{(1)}$ (acting on Γ). After integration one has the renormalized integral from Eq. (24):

$$F_\Gamma = J_\Gamma + \sum_{\check{\Gamma}}{}_1 m_{\check{\Gamma}} \int \prod_{L_{ab\sigma} \in \Gamma/\{\gamma_1 \cdots \gamma_c\}}^{ind} \frac{\mathrm{d}^4 k_{ab\sigma}^\Gamma}{(2\pi)^4} I_{\Gamma/\gamma_1 \cdots \gamma_c}(k^\Gamma, q)$$
$$\times \prod_{\tau=1}^{c} O_{\gamma_\tau}(2\pi)^4 \delta^4 \left(\sum_a q_a \right), \qquad (29)$$

where $m_{\check{\Gamma}} = \dfrac{|G_\Gamma|}{|G_\Gamma^{(1)}|}$. In the derivation, we need the fact that Q_{γ_τ} is symmetric under the mapping of G_Γ.

V. CONTRIBUTION OF REDUCED FEYNMAN DIAGRAM $\widetilde{\Gamma}$ TO THE S MATRIX

When we introduce new reduced vertices $\hat{Q}_\gamma$ to $\Delta \mathcal{L}_I = -\Delta \mathcal{H}_I$, the S matrix gets many new terms due to these new vertices. In a reduced vertex, the order of the original perturbation constant is more than one since it contains more than one original vertex. Thus if we collect terms according to the orders of original perturbation constants in the S matrix, say, n-th order, the term with respect to a reduced diagram $\widetilde{\Gamma}$ may be with a "perturba-
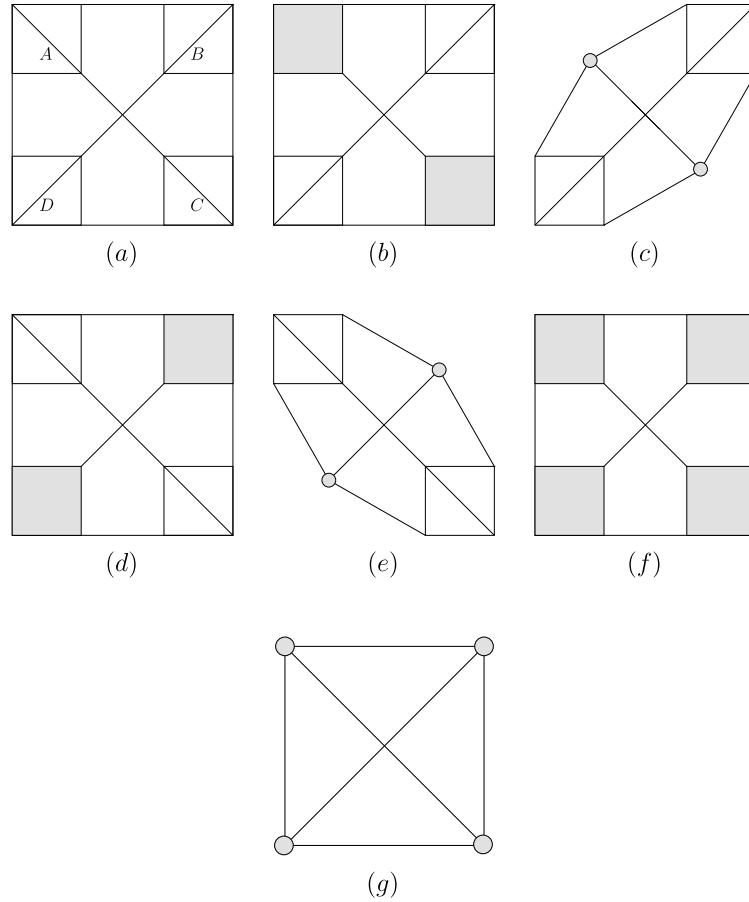
Fig. 5.   Diagram of Γ with $\widetilde{\Gamma}'s$. (*a*) Diagram of Γ where A, B, C, D are renormalization parts. (*b*) Choose A and C as $\gamma_\tau's$ to get $\check{\Gamma}_1$. (*c*) $\widetilde{\Gamma}_1$ from $\check{\Gamma}_1$ (shrink all lines and points of A (and C) into a point). (*d*) Another configuration equivalent to (*b*), which gives $\check{\Gamma}'_1$. (*e*) $\widetilde{\Gamma}'_1$ from $\check{\Gamma}'_1$, equivalent to $\widetilde{\Gamma}_1$. (*f*) $\{\gamma_\tau\} = \{A, B, C, D\}$ gives $\check{\Gamma}_2$. (*g*) $\widetilde{\Gamma}_2$.

tion order" m, $m < n$. We denote such a term $S_{\widetilde{\Gamma}(n,m)}$. Assume we have all reduced vertices associated with all γ_τ (renormalization part) for $n_{\gamma_\tau} \leqslant n$, and assume these vertices have the symmetry of γ_τ. We denote the exact symmetry group for exterior lines of such vertices as $g_{O_{\gamma_\tau}}$ (see Appendix A for details). Due to the perturbation theory, there will be a term associated with $\widetilde{\Gamma}$ in the S matrix. From Eq. (6) we have:

$$S_{\widetilde{\Gamma}(n,m)} = \frac{1}{m!} \frac{|G_m| \times |\prod^1 G_{O_{\gamma_\tau}}| \times |\prod^2 G^i|}{|G_{\widetilde{\Gamma}}|}$$
$$\times \int \frac{dk_1 \cdots dk_s}{(2\pi)^s} I^0_{\widetilde{\Gamma}} (2\pi)^4 \delta^4 \left(\sum_a q_a \right), \qquad (30)$$

where G_m is the permutation group S_m, $\prod^1$ extends over all reduced vertices, and $\prod^2$ extends over the remaining original vertices in $\widetilde{\Gamma}$. Since $|G_m| = m!$, we obtain:

$$S_{\widetilde{\Gamma}(n,m)} = \frac{|\prod^1 G_{O_{\gamma_\tau}}| \times |\prod^2 G^i|}{|G_{\widetilde{\Gamma}}|}$$
$$\times \int \frac{dk_1 \cdots dk_s}{(2\pi)^s} I^0_{\widetilde{\Gamma}} (2\pi)^4 \delta^4 \left(\sum_a q_a \right). \qquad (31)$$

Together with $S_{\Gamma(n)}$, this gives:

$$S_{\Gamma(n)} + S_{\widetilde{\Gamma}(n,m)} = \frac{|\prod^2 G^i| \times |\prod^3 G^i|}{|G_\Gamma|} \int \frac{dk_1 \cdots dk_l}{(2\pi)^l} I^0_\Gamma (2\pi)^4 \delta^4 \left(\sum_a q_a \right) + \frac{|\prod^2 G^i| \times |\prod^3 G^i|}{|G_\Gamma|} \times \frac{|G_\Gamma| \times |\prod^1 G_{O_{\gamma_\tau}}|}{|G_{\widetilde{\Gamma}}| \times |\prod^3 G^i|}$$
$$\times \int \frac{dk_1 \cdots dk_s}{(2\pi)^l} I^0_{\widetilde{\Gamma}} (2\pi)^4 \delta^4 \left(\sum_a q_a \right), \qquad (32)$$

where $\prod^3$ extends over all vertices in $\bigcup_\tau \gamma_\tau \subset \Gamma$. Thus one has $|\prod^2 G^i| \times |\prod^3 G^i| = \prod G^i$, which extends over all vertices of Γ. The term $S_{\widetilde{\Gamma}}^{(n,m)}$ should match the coun-

terterms in the BPHZ renormalization scheme. We denote the vertex of γ_τ as $\int \prod_{ind} k_{ab\sigma}^{\gamma_\tau} O_{\gamma_\tau}^0$. Based on Eq. (25), Eq. (32) can be written as:

$$
\begin{aligned}
S_{\Gamma(n)} + S_{\widetilde{\Gamma}(n,m)} &= \frac{|\prod G^i|}{|G_\Gamma|} \int \frac{dk_1 \cdots dk_l}{(2\pi)^l} \left\{ I_\Gamma^0 + \frac{|G_\Gamma| \prod^1 |G_{O_{\gamma_\tau}}|}{|G_{\widetilde{\Gamma}}| \prod^3 |G^i|} I_{\Gamma/\{\gamma_1 \cdots \gamma_c\}}^0 \prod_\tau O_{\gamma_\tau}^0 \right\} (2\pi)^4 \delta^4 \left(\sum_a q_a \right) \\
&= \frac{|\prod G^i|}{|G_\Gamma|} \int \frac{dk_1 \cdots dk_l}{(2\pi)^l} \left\{ I_\Gamma^0 + \frac{|G_\Gamma||G_\Gamma^{(2)}| \prod_\tau |G_{O_{\gamma_\tau}}|}{|G_\Gamma^{(1)}| \prod^3 |G^i|} I_{\Gamma/\{\gamma_1 \cdots \gamma_c\}}^0 \prod O_{\gamma_\tau}^0 \right\} (2\pi)^4 \delta^4 \left(\sum_a q_a \right) \\
&= \frac{|\prod G^i|}{|G_\Gamma|} \int \frac{dk_1 \cdots dk_l}{(2\pi)^l} \left\{ I_\Gamma^0 + \frac{|G_\Gamma| \prod_\tau |G_{\gamma_\tau}| \prod_\tau |G_{O_{\gamma_\tau}}|}{|G_\Gamma^{(1)}| \prod^3 |G^i|} I_{\Gamma/\{\gamma_1 \cdots \gamma_c\}}^0 \prod O_{\gamma_\tau}^0 \right\} (2\pi)^4 \delta^4 \left(\sum_a q_a \right),
\end{aligned}
\tag{33}
$$

where $|\dfrac{G_\Gamma^{(2)}}{G_\Gamma^{(1)}}| = \dfrac{1}{|G_{\widetilde{\Gamma}}|}$, and $|\dfrac{G_\Gamma}{G_\Gamma^{(1)}}|$ is the number of cosets, which indicates the number of different $\widetilde{\Gamma}'s$ whose con-

figurations are the same as $\widetilde{\Gamma}$.

Taking into account all reduced diagrams of Γ, we have:

$$
S_{\Gamma(n)} + \sum_l S_{\widetilde{\Gamma}_l(n,m)} = \frac{\prod |G^i|}{|G_\Gamma|} \int \frac{dk_1 \cdots dk_l}{(2\pi)^l} \left\{ I_\Gamma^0 + \sum_l \frac{\prod_\tau |G_{O_{\gamma_\tau}}||G_{\gamma_\tau}|}{\prod^3 |G^i|} \right\} \times I_{\Gamma/\{\gamma_1 \cdots \gamma_c\}}^0 \prod_\tau O_{\gamma_\tau}^0 (2\pi)^4 \delta^4 \left(\sum_a q_a \right),
\tag{34}
$$

where l is specified by the set $\{\gamma_1 \cdots \gamma_c\}$ of mutually disjoint renormalization parts in Γ.

Next we absorb all factors $|G^i|$ of symmetry groups of the original vertices into these vertex constants, and require

$$
\frac{1}{|G_{\gamma_\tau}||G_{O_{\gamma_\tau}}|} O_{\gamma_\tau} = \frac{1}{|G_{\hat{\gamma}_\tau}|} O_{\gamma_\tau} = O_{\gamma_\tau}^0,
\tag{35}
$$

which corresponds to $O_{\gamma_\tau}^0$ in Eq. (34). We have:

$$
\begin{aligned}
&S_{\Gamma(n)} + \sum_l S_{\widetilde{\Gamma}_l(n,m)} \\
&= \frac{1}{|G_\Gamma|} \int dk_1 \cdots dk_l \left\{ I_\Gamma + \sum_{\{\gamma_1 \cdots \gamma_c\}} I_{\Gamma/\{\gamma_1 \cdots \gamma_c\}} \prod_\tau O_{\gamma_\tau} \right\} \\
&\times (2\pi)^4 \delta^4 \left(\sum_a q_a \right) = \frac{1}{|G_\Gamma|} F_\Gamma.
\end{aligned}
\tag{36}
$$

This is just the BPHZ formula. The corresponding operator of $\hat{O}_{\gamma_\tau}^0$ in Eq. (35) is given in Appendix A, and it only depends on the renormalization part γ_τ.

VI. CONCLUSION

In conclusion, we have considered the procedure for introducing a new vertex $\hat{Q}_{\gamma_\tau}^0$ into $\Delta\mathcal{L}_I$ for all renormaliz-

ation parts γ_τ (proper diagram with $d(\gamma_\tau) \geq 0$) with number of vertices $m \leq n$. When we collect all terms of original parameter order $m' \leq n$ (the order for original perturbation parameters) in the S matrix, then it will automatically give the counterterms of Feynman integrals as the BPHZ scheme requires for any Feynman diagram Γ.

Thus the BPHZ scheme is consistent with adding counterterms in $\Delta\mathcal{L}_I$.

APPENDIX A. Symmetry of vertex Q_γ

Similar to G_Γ, we define a mapping group $G_{\hat{\gamma}}$ for the Feynman diagram γ. In this mapping, we allow the external lines to be mapped to equivalent lines of another vertex of the same type. This is different from G_Γ, where the external lines always keep fixed. We find that the group G_γ is a normal subgroup of $G_{\hat{\gamma}}$. We denote $\hat{\gamma}$ with external lines of γ belonging to $\mathcal{L}(\hat{\gamma})$. We have:

$$
O_\gamma = -t^\gamma \left(I_\gamma + \sum_{\{\gamma_1 \cdots \gamma_c\}} {}' I_{\gamma/\{\gamma_1 \cdots \gamma_c\}} \prod_\tau O_{\gamma_\tau} \right),
\tag{A1}
$$

and

$$Q_\gamma = \int \prod^{ind} \frac{\mathrm{d}^4 k_{ab\sigma}^\gamma}{(2\pi)^4} O_\gamma, \tag{A2}$$

due to Eq. (25). Assume for all proper renormalization parts $\gamma_\tau \subset \gamma$, O_{γ_τ} are symmetric functions with the symmetry of γ_τ. That means, under $G_{\hat\gamma_\tau}$, the value of Q_{γ_τ} is invariant. We can write Eq. (A2) as:

$$Q_\gamma = -t^\gamma V_\gamma . \tag{A3}$$

Here V_γ is defined as:

$$V_\gamma \delta^4 \left(\sum q_a^\gamma \right) = \int \prod_{L_{ab\sigma} \in \mathcal{L}(r)} \frac{\mathrm{d}^4 k_{ab\sigma}^\gamma}{(2\pi)^4} \check{I}_\gamma^0$$

$$+ \sum{}' \int \prod_{L_{ab\sigma} \in \mathcal{L}(\gamma/\{\gamma_1 \cdots \gamma_c\})} \frac{\mathrm{d}^4 k_{ab\sigma}^\gamma}{(2\pi)^4} \check{I}_{\gamma/\{\gamma_1 \cdots \gamma_c\}}^0$$

$$\times \prod_\tau \left\{ Q_{\gamma_\tau} (2\pi)^4 \delta^4 \left(\sum_{V_a \in \gamma_\tau} q_a^{\gamma_\tau} \right) \right\}, \tag{A4}$$

in which:

$$\check{I}_\gamma^0 = \prod_{L_{ab\sigma} \in \mathcal{L}(\gamma)} \Delta_F^{ab\sigma} \prod_{V_a \in \mathcal{V}(\gamma)} \left\{ P_a (2\pi)^4 \delta^4 \left(\sum_{L_{ab'\sigma'} \in \mathcal{L}(\hat\gamma)} l_{ab'\sigma'} \right) \right\} . \tag{A5}$$

The situation is similar for $\check{I}_{\gamma/\{\gamma_1 \cdots \gamma_c\}}$, and Eq. (A4) is invariant under the mapping of $G_{\hat\gamma}$. Thus we can always choose $V_\gamma(s)$ which is invariant under $G_{\hat\gamma}$. On the other hand, we can also choose $V_\gamma(ind)$ which is only a function of independent q_a^γ. Due to $\delta^4(\sum^a q_a)$ in Eq. (A4), the number of independent q_a^γ is less than the number of all q_a^γ. We have:

$$V_\gamma(ind) \delta^4 \left(\sum q_a^\gamma \right) = V_\gamma(s) \delta^4 \left(\sum q_a^\gamma \right). \tag{A6}$$

One can show by truncating the Taylor series in Eq. (22) that:

$$-t^\gamma(ind) V_\gamma(ind) \delta^4 \left(\sum q_a^\gamma \right) = -t^\gamma V_\gamma(s) \delta^4 \left(\sum q_a^\gamma \right), \tag{A7}$$

where $-t^\gamma(ind) V_\gamma(ind)$ is the Q_γ in Eq. (A2). We then prove that Q_γ can also be chosen as a symmetric (invariant) function under the mapping of $G_{\hat\gamma}$. It is fixed under G_γ. Thus as a reduced vertex, we have the symmetry group,

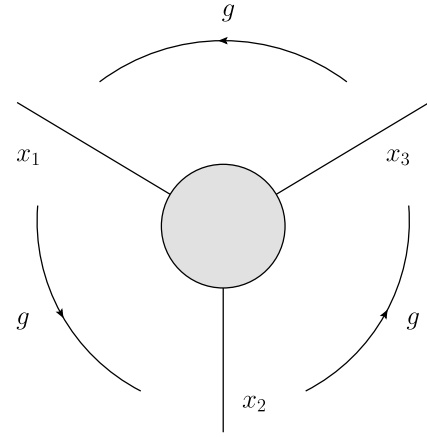$$G_{O_\gamma} = G_{\hat\gamma}/G_\gamma . \tag{A8}$$

Fig. A1. Mapping leading to an invariant function ($x'_1 = x_2$, $x'_2 = x_3, x'_3 = x_1$).

Here we make a remark about invariants. If the momenta of external lines change following the mapping, the function Q_γ is invariant. For example, under the mapping depicted in Fig. A1, if the function $f(x_1, x_2, x_3) = f(x'_1, x'_2, x'_3)$, we say function f is invariant under the mapping g.

We denote Q_γ as

$$\delta^4 \left(\sum_a q_a^\gamma \right) Q_\gamma = -t^\gamma V_\gamma(\{q_a^\gamma\}) \delta^4 \left(\sum_a q_a^\gamma \right). \tag{A9}$$

In the integral equations (1) and (2), the derivative $\left(-i \frac{\partial}{\partial x_a} \right)$ operator produces a momentum q_a factor. Thus the function Q_γ can be realized in $\Delta \mathcal{L}_I$ by the operator proportional to

$$\hat{Q}_\gamma = i \sum_{l=0}^{d(\gamma)} \frac{1}{l!} \sum_{i_1 \cdots 1_l} \left(-i \frac{\partial}{\partial x_{i_1}} \right) \cdots \left(-i \frac{\partial}{\partial x_{i_l}} \right)$$

$$\times \left(\prod_i \varphi_{a_i}(x_a) \prod_j \varphi_{b_j}(x_b) \cdots \right) \Big|_{x_a = x_b = \cdots = x}$$

$$\times \frac{\partial}{\partial q_{i_1}} \cdots \frac{\partial}{\partial q_{i_l}} V_\gamma(q_a^\gamma)|_{q=0}, \tag{A10}$$

where $i_1, \cdots, i_l$ in summation run over all $a = 1, \cdots, n_\gamma$, (n_γ = number of vertices in γ), and $\mu = 0, 1, 2, 3$. In Eq. (A10), fields $\{\varphi_{a_i}, i = 1, \cdots, l_a\}$ produce external lines at the vertex $V_a, \cdots$.

From Eq. (35) the operator of the reduced vertex introduced in $\Delta \mathcal{L}_I$ is then

$$\hat{Q}_{\gamma_\tau}^0 = \frac{1}{|G_{O_{\gamma_\tau}}||G_{\gamma_\tau}|} \hat{Q}_{\gamma_\tau} = \frac{1}{|G_{\hat\gamma_\tau}|} \hat{Q}_{\gamma_\tau} . \tag{A11}$$

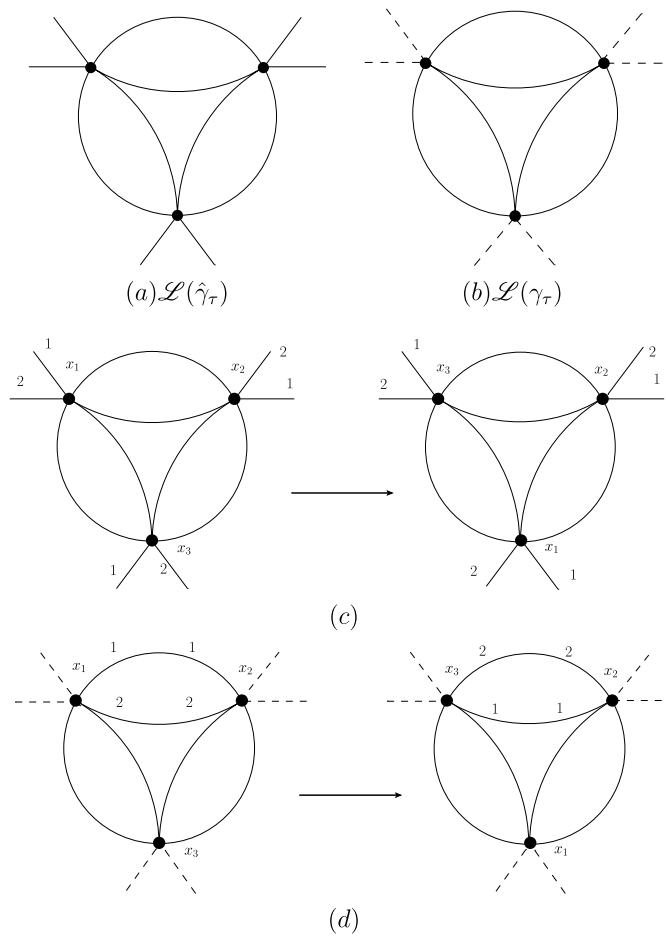G_{γ_τ} is a normal subgroup of $G_{\hat\gamma_\tau}$, see Fig. A2 below.

Fig. A2.   A renormalization part γ_τ and associated $G_{\hat{\gamma}_\tau}$ and G_{γ_τ}. (*a*) The lines belonging to $\hat{\gamma}_\tau$. (*b*) The lines (only internal lines!) belonging to γ_τ. (*c*) A mapping $g_{\hat{\gamma}_\tau}$, including the change of points and external lines of γ_τ. This mapping does not belong to G_{γ_τ}. (*d*) A mapping $g_{\gamma_\tau} \in G_{\gamma_\tau} \subset G_{\hat{\gamma}_\tau}$.

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