Microscopic states of Kerr black holes from boundary-bulk correspondence*

Jing-Bo Wang(王晶波)†

Institute for Gravitation and Astrophysics, College of Physics and Electronic Engineering, Xinyang Normal University, Xinyang 464000, China

Abstract: It was previously claimed by the author that black holes can be considered as topological insulators. Both black holes and topological insulators have boundary modes, and the boundary modes can be described by an effective BF theory. In this paper, the boundary modes on the horizons of black holes are analyzed using methods developed for topological insulators. BTZ black holes are analyzed first, and the results are found to be compatible with previous works. The results are then generalized to Kerr black holes, for which new results are obtained: dimensionless right- and left-temperatures can be defined and have well behavior in both the Schwarzschild limit $a \to 0$ and the extremal limit $a \to M$. Upon the Kerr/CFT correspondence, a central charge $c = 12Mr_+$ can be associated with an arbitrary Kerr black hole. Moreover, the microstates of the Kerr black hole can be identified with the quantum states of this scalar field. From this identification, the number of microstates of the Kerr black hole can be counted, yielding the Bekenstein-Hawking area law for the entropy.

Keywords: boundary modes, BTZ black hole, Kerr black hole

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I. INTRODUCTION

It is well accepted that black holes behave like thermodynamic objects. Following Bekenstein [1] and Hawking [2], a black hole has temperature and entropy

$$T_{\rm H} = \frac{\hbar\kappa}{2\pi}, \quad S_{\rm BH} = \frac{A}{4G\hbar}, \tag{1}$$

where κ is the surface gravity, and A is the area of the horizon. Understanding these properties, especially identifying the black hole microstates that account for the entropy, is a fundamental challenge of quantum gravity.

Since identifying the black hole microstates that account for entropy is a difficult problem, it is helpful to map it onto a problem that has been solved. In previous works [3, 4], the author claimed that a black hole can be considered as a kind of topological insulator. Roughly speaking, a topological insulator is a bulk insulator that has conducting boundary states. For a Bañados-Teitelboim-Zanelli (BTZ) black hole in three dimensional spacetime, this claim is tested in Refs. [5-7]. The boundary modes on the horizon of a BTZ black hole can be described by two chiral massless scalar fields with opposite chirality. The same holds for topological insulators in three dimensional spacetime (also called quantum spin Hall states) [8]. From the chiral scalar fields, one can construct the $W_{1+\infty}$ algebra that contains the near horizon symmetry algebra of a BTZ black hole as a subalgebra. The $W_{1+\infty}$ algebra can be used to classify BTZ black holes and to determine the ‘W-hairs’ of black holes, which may be essential to solving the information paradox [9-11]. The microstates of BTZ black holes are identified as the quantum states of the chiral scalar fields.

For higher dimensional black holes, such as Kerr black holes in four dimensions, the boundary modes can be described by boundary BF theory (the name refers to the Lagrangian density, $\mathcal{L} = B\wedge F$), which is also the same for higher dimensional topological insulators. From BF theory, one can construct a free scalar field theory. An essential property of topological insulators is their boundary-bulk correspondence [12, 13], which relates the topological structure of bulk states to the presence of massless boundary modes. From the boundary modes, key properties of the bulk states can be obtained [14, 15]. For BTZ black holes, we can identify the quantum states of the scalar field with the microstates of Kerr black holes.

In previous works [16, 17], it was shown that the degrees of freedom on the horizon can be described by BF theory with sources. The horizon of a black hole is a kind of isolated horizon [18, 19], which is a null hypersurface with vanishing expansion. In the framework of loop quantum gravity [20-23], the symplectic structure of general relativity is analyzed, and it is found that the boundary degrees of freedom can be described by BF theory

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* Supported by Nanhu Scholars Program for Young Scholars of XYNU

† E-mail: shuijing@mail.bnu.edu.cn

with sources. This result holds for a wide variety of cases, including general relativity in arbitrary dimensions [24-26] and Lovelock theory [27]. Starting from BF theory, BTZ black holes are analyzed first. The same methods are then applied to Kerr black holes. For Kerr black holes, new results are obtained:

• We can define dimensionless left- and right-temperatures $T_{\rm L/R}=\dfrac{r_+\pm a}{4\pi r_+}$ for all Kerr black holes (including Schwarzschild black holes).

• For an arbitrary Kerr black hole, if there exists a dual conformal field theory (CFT), which is similar to a Kerr/CFT correspondence [28-30], a central charge $c=12Mr_+$.

• We can identify the microscopic states of Kerr black holes with the quantum states of the boundary scalar field and count the number of those states, from which the Bekenstein-Hawking area law can be obtained.

The paper is organized as follows. In section II, we analyze BTZ black holes. In section III, Kerr black holes are analyzed with the same method. Section IV provides the conclusion. In what follows, we set $G=\hbar=c=1$.

II. BTZ BLACK HOLES

In this section, we consider the boundary modes on the horizon of BTZ black holes from the boundary BF theory [4]. To quantize the boundary BF theory, we mainly follow the methods in Refs. [14, 15].

The metric of a BTZ black hole is

$$\mathrm{d}s^2=-N^2\mathrm{d}v^2+2\mathrm{d}v\mathrm{d}r+r^2(\mathrm{d}\varphi+N^\varphi\mathrm{d}v)^2, \tag{2}$$

where $N^2=-8M+\dfrac{r^2}{L^2}+\dfrac{16J^2}{r^2}$, $N^\varphi=-\dfrac{4J}{r^2}$, and (M,J) are the mass and angular momentum of the black hole, respectively.

Choosing the following Newman-Penrose null co-triads

$$l=-\frac{1}{2}N^2\mathrm{d}v+\mathrm{d}r,\quad n=-\mathrm{d}v,\quad m=rN^\varphi\mathrm{d}v+r\mathrm{d}\varphi, \tag{3}$$

the corresponding spin connection, which is used later, is $A_2=\alpha m-\kappa n$ with $\alpha=N^\varphi$, $\kappa=r/L^2-r(N^\varphi)^2$.

For subsequent use, we define new coordinates $\mathrm{d}v'=\mathrm{d}v/\gamma$, where the coefficient γ is determined later. Then, on the horizon $\Delta: r=r_+$,

$$m=(rN^\varphi\mathrm{d}v+r\mathrm{d}\varphi)|_\Delta=-\frac{\gamma r_-}{L}\mathrm{d}v'+r_+\mathrm{d}\varphi. \tag{4}$$

A. The canonical formula

On the horizon $r=r_+$, we choose the coordinates $(x^0,x^1)=(v',\varphi)$. Based on our early work [4], the degrees of freedom on the horizon can be described by BF theory, which has the form

$$S=\int_\Delta BF=\int_\Delta B\mathrm{d}A, \tag{5}$$

with the constraints

$$\mathrm{d}B=\frac{1}{8\pi}m,\quad \mathrm{d}A=0, \tag{6}$$

where A is the non-rotating component of the spin connection A_2.

The canonical form is

$$S=\int_\Delta \mathrm{d}v'\mathrm{d}\varphi B(\partial_0A_1-\partial_1A_0)$$
$$=\int_\Delta \mathrm{d}v'\mathrm{d}\varphi(B\partial_0A_1+A_0\partial_1B). \tag{7}$$

Choosing the temporal gauge $A_0=0$, one can obtain $A_1=\partial_1\phi$ with scalar field ϕ. The action then becomes

$$S=-\int_\Delta \mathrm{d}v'\mathrm{d}\varphi\partial_1B\partial_0\phi. \tag{8}$$

Since the horizon is a null hypersurface, the metric is degenerate. To facilitate the work, we thus assume that the effective metric for the BF theory is

$$\mathrm{d}\tilde{s}^2=-\mathrm{d}v'^2+r_+^2\mathrm{d}\varphi^2. \tag{9}$$

Then we get

$$\sqrt{-g}=r_+,\quad \epsilon^{01}=\frac{1}{r_+}. \tag{10}$$

The action (8) can be rewritten as

$$S=-\int_\Delta \mathrm{d}^2x\sqrt{-g}\frac{1}{\sqrt{-g}}\partial_1B\partial_0\phi=\int_\Delta \mathrm{d}^2x\sqrt{-g}\pi\dot{\phi}, \tag{11}$$

where $\pi=-\dfrac{1}{\sqrt{-g}}\partial_1B$ is the canonical momentum. It iseasy to show that the Hamiltonian is zero. To describe a relativistic dynamics, one can add a Hamiltonian to get [8, 15]

$$S'=\int_\Delta \mathrm{d}^2x\sqrt{-g}(\pi\dot{\phi}-\mathcal{H}(\pi,\phi))=\int_\Delta \mathrm{d}^2x\sqrt{-g}\Big(\pi\dot{\phi}$$
$$-\frac{1}{2m_0}\pi^2-\frac{m_0}{2}g^{ij}\partial_i\phi\partial_j\phi\Big), \tag{12}$$

where m_0 is a parameter that depends on the bulk theory. It is the simplest choice that can provide the non-trivial dynamics on the boundary. Other forms of the Hamiltonian can also be chosen, as shown in Ref. [15].

The field equations from the action (12) are

$$\pi = m_0\dot{\phi}, \quad \dot{\pi} = m_0\Delta\phi, \tag{13}$$

where Δ is the Laplace-Beltrami operator. The above equation can be recast into a duality relation:

$$\epsilon^{\mu\nu}\partial_\nu B = m_0\partial^\mu\phi. \tag{14}$$

The action (12) is simply the action for a massless scalar field ϕ,

$$S' = \frac{m_0}{2}\int_\Delta \mathrm{d}^2x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{15}$$

The Hamiltonian for this massless scalar field can be given by

$$H = \frac{m_0}{2}\oint \mathrm{d}\varphi\sqrt{-g}\left((\partial_0\phi)^2 + (\frac{\partial_1\phi}{r_+})^2\right). \tag{16}$$

We can also define the angular momentum

$$J = m_0\oint \mathrm{d}\varphi\sqrt{-g}(\partial_0\phi\partial_1\phi). \tag{17}$$

These quantities will be related to the parameters (M,J) of black holes.

B. Quantization

In this section, we quantize the massless scalar field (15) in the standard way. Expanding the fields with the Fourier modes gives

$$\begin{aligned}\phi(v',\varphi) =&\phi_0 + p_v v' + p_\varphi\varphi + \sqrt{\frac{1}{m_0A}}\\ &\times\sum_{n\neq0}\sqrt{\frac{1}{2\omega_n}}[a_n\mathrm{e}^{-\mathrm{i}(\omega_n v'-k_n\varphi)} + a_n^+\mathrm{e}^{\mathrm{i}(\omega_n v'-k_n\varphi)}],\\ B(v',\varphi) =& -m_0(B_0 + \frac{p_\varphi}{r_+}v' + r_+p_v\varphi) + \sqrt{\frac{4m_0}{r_+^2A}}\\ &\times\sum_{n\neq0}\sqrt{\frac{1}{(2\omega_n)^3}}k_n[a_n\mathrm{e}^{-\mathrm{i}(\omega_n v'-k_n\varphi)} + a_n^+\mathrm{e}^{\mathrm{i}(\omega_n v'-k_n\varphi)}],\end{aligned} \tag{18}$$

where $\omega_n = \dfrac{|n|}{r_+}$, $k_n = n$, and $A = 2\pi r_+$ is the length of the circle. It is straightforward to show that the above expressions satisfy the dual relation (14).

The quantum field operators satisfy the commutative relation:

$$[\hat{\phi}(v',\varphi),\hat{\pi}(v',\varphi')] = \mathrm{i}\delta(\varphi-\varphi'), \tag{19}$$

which gives

$$[\hat{\phi}_0, m_0\hat{p}_v] = \frac{\mathrm{i}}{A}, \quad [\hat{a}_n,\hat{a}_m^+] = \delta_{n,m}. \tag{20}$$

We can also consider B and $\partial_1\phi$ as two canonical variables, leading to a further commutation relation:

$$\left[\hat{B}_0, \frac{m_0}{r_+}\hat{p}_\varphi\right] = -\frac{\mathrm{i}}{A}. \tag{21}$$

Since the zero modes ϕ_0, B_0 are constants on the cylinder, the spectra of the canonical momentum are quantized according to

$$m_0p_v = \frac{n_1}{A}, \quad \frac{m_0}{r_+}p_\varphi = \frac{n_2}{A} \quad n_1,n_2\in Z. \tag{22}$$

Next, we relate the massless scalar field with the BTZ black hole. For the BTZ black hole, the B field satisfies the constraint (6); alternatively, with the component

$$\begin{aligned}\partial_0B =& -\frac{r_-\gamma}{8\pi L},\\ \partial_1B =& \frac{r_+}{8\pi}.\end{aligned} \tag{23}$$

Submitting the expression (18) gives

$$p_v = -\frac{1}{8\pi m_0}, \quad p_\varphi = \frac{\gamma r_+r_-}{8\pi Lm_0}. \tag{24}$$

Combining this expression with Eq. (22) gives the quantization condition

$$r_+ = 4n_1, \quad r_- = 4n_2\frac{L}{\gamma r_+}, \quad n_1,n_2\in N. \tag{25}$$

The dimensionless right- and left-temperatures are defined as

$$T_{\mathrm{R/L}} = \frac{r_+\pm r_-}{2\pi L}. \tag{26}$$

To fix the coefficient γ , we make the following assumption:

$$T_{\mathrm{R/L}} \propto p_v \mp \frac{p_\varphi}{r_+}, \tag{27}$$

which gives $\gamma = \dfrac{L}{r_+}$. Then,

$$r_+ = 4n_1, \quad r_- = 4n_2, \quad n_1, n_2 \in N. \tag{28}$$

The above expression suggests that the lengths L_+, L_- of the outer and inner horizon of black hole, respectively, are both quantized according to

$$L_+ = 2\pi r_+ = 8\pi n_1, \quad L_- = 2\pi r_- = 8\pi n_2, \quad n_1, n_2 \in N. \tag{29}$$

The quantum versions of the Hamiltonian (16) and angular momentum (17) are given by

$$\begin{aligned}\hat{H} =& \pi m_0 r_+ \left(\hat{p}_v^2 + \frac{\hat{p}_\varphi^2}{r_+^2} \right) + \sum_{n\neq 0} \frac{|n|}{r_+} \hat{a}_n^+ \hat{a}_n, \\ \hat{J} =& 2\pi m_0 r_+ \hat{p}_v \hat{p}_\varphi + \sum_{n\neq 0} n \hat{a}_n^+ \hat{a}_n,\end{aligned} \tag{30}$$

where we omit the zero-point energy.

The parameter m_0 should relate to the parameters of the black hole. To fix this parameter, we assume that the angular momentum of the scalar field equals the angular momentum of the BTZ black hole; that is,

$$2\pi m_0 r_+ p_v p_\varphi = J = \frac{r_+ r_-}{4L}, \tag{31}$$

which gives

$$m_0 = \frac{L}{8\pi}. \tag{32}$$

The dimensionless temperatures (26) can then be expressed as

$$T_{\rm R/L} = -\frac{r_+}{2\pi}\left(p_v \mp \frac{p_\varphi}{r_+} \right), \tag{33}$$

and they satisfy

$$\frac{2}{T_{\rm H}} = \frac{\gamma r_+}{T_{\rm R}} + \frac{\gamma r_+}{T_{\rm L}}, \tag{34}$$

where $T_{\rm H}$ is Hawking temperature for the black hole.

The quantum scalar field $\phi(v', \varphi)$ can be considered as a collective of harmonic oscillators, and a general quantum state can be represented as $|p_v, p_\varphi; \{n_k\} > \sim (a_1^+)^{n_1} \cdots (a_k^+)^{n_k} |p_v, p_\varphi>$, where p_v, p_φ are zero mode parts, and $\{n_k\}$ are oscillating parts. As shown in Refs. [6, 7], the BTZ black hole ground state corresponds to the zero mode part; thus,

$$\begin{aligned}&< p_v, p_\varphi; \{0\} | \hat{J} | p_v, p_\varphi; \{0\} >= J, \\ &< p_v, p_\varphi; \{0\} | \hat{H} | p_v, p_\varphi; \{0\} >= \gamma M = ML/r_+.\end{aligned} \tag{35}$$

The parameter γ appears because the energy M is associated with the time coordinate v, and for v', one has

$$M' \sim \frac{\partial}{\partial v'} = \gamma \frac{\partial}{\partial v} \sim \gamma M. \tag{36}$$

In contrast, the microstates of the BTZ black hole can be represented by the oscillating part $|0,0;\{n_k\}>$. They satisfy

$$\begin{aligned}&\frac{1}{c} < 0,0; \{n_k\} | \hat{J} | 0,0; \{n_k\} >= J, \\ &\frac{1}{c} < 0,0; \{n_k\} | \hat{H} | 0,0; \{n_k\} >= ML/r_+,\end{aligned} \tag{37}$$

where $c = 3L/2G$ is the Brown-Henneaux central charge, which can be written as

$$\sum_{k\neq 0} k n_k = cJ, \quad \sum_{k\neq 0} |k| n_k = cML, \quad n_k \in N^+. \tag{38}$$

The above equations are equivalent to

$$\begin{aligned}&\sum_{k>0} k n_k^+ = \frac{c}{2}(ML + J), \\ &\sum_{k<0} (-k) n_k^- = \frac{c}{2}(ML - J), \\ &n_k^\pm \in N^+.\end{aligned} \tag{39}$$

Different sequences $\{n_k\}$ correspond to different microstates of the BTZ black hole with fixed (M, J). For non-extremal black holes, the total number of microstates for the BTZ black hole with parameters (M, J) can be calculated through the Hardy-Ramanujan formula:

$$p(N) \simeq \frac{1}{4N\sqrt{3}} \exp\left(2\pi \sqrt{\frac{N}{6}} \right). \tag{40}$$

The result is

$$\begin{aligned}N(M,J) \simeq& \frac{1}{c(ML+J)c(ML-J)} \exp\left(2\pi \sqrt{c\frac{ML+J}{12}} \right. \\ &\left. + 2\pi \sqrt{c\frac{ML-J}{12}} \right).\end{aligned} \tag{41}$$

The entropy of the BTZ black hole is thus given by

$$S = \ln N(M,J) = \frac{2\pi r_+}{4} - 2\ln(r_+^2 - r_-^2) + \cdots, \tag{42}$$

which is simply the Bekenstein-Hawking entropy formula with some low order corrections.

For extremal BTZ black holes, with $J = ML$, the constraints (39) become

$$\sum_{k>0} k n_k^+ = cML, \quad n_k^+ \in N^+. \tag{43}$$

The entropy is thus given by

$$\begin{aligned} S =& \ln N(M) = \ln\left(\frac{1}{cML}\exp\left(2\pi\sqrt{c\frac{ML}{6}}\right)\right) \\ =& \frac{2\pi r_+}{4} - 2\ln r_+ + \cdots, \end{aligned} \tag{44}$$

which is the same result as in the "horizon fluff" proposal [31].

III. KERR BLACK HOLES

In this section, we analyze Kerr black holes with the same methods. The metric of a Kerr black hole can be written as [32]

$$\begin{aligned} \mathrm{d}s^2 =& -\left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r - 2a\sin^2\theta\mathrm{d}r\mathrm{d}\varphi \\ & - \frac{4aMr\sin^2\theta}{\rho^2}\mathrm{d}v\mathrm{d}\varphi + \rho^2\mathrm{d}\theta^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2}\mathrm{d}\varphi^2, \end{aligned} \tag{45}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta^2 = r^2 - 2Mr + a^2$, $\Sigma^2 = (r^2 + a^2)\rho^2 + 2a^2Mr\sin^2\theta$, and $(M, J = Ma)$ are the mass and angular momentum of the Kerr black hole. The horizon is localized at $r = r_+$.

A suitable null co-tetrad $(l, n, m, \bar{m})$ can be chosen as

$$\begin{aligned} l =& -\frac{\Delta}{2(r^2+a^2)}\mathrm{d}v + \frac{\rho^2}{r^2+a^2}\mathrm{d}r + \frac{\Delta a\sin^2\theta}{2(r^2+a^2)}\mathrm{d}\varphi, \\ n =& \frac{r^2+a^2}{\rho^2}(-\mathrm{d}v + a\sin^2\theta\mathrm{d}\varphi), \\ m =& -\frac{a\sin\theta}{\sqrt{2}\tilde{\rho}}\mathrm{d}v + \frac{(r^2+a^2)\sin\theta}{\sqrt{2}\tilde{\rho}}\mathrm{d}\varphi + \frac{\mathrm{i}}{\sqrt{2}}\bar{\tilde{\rho}}\mathrm{d}\theta, \end{aligned} \tag{46}$$

where $\tilde{\rho} = r + \mathrm{i}a\cos\theta$. It is easy to show that area element of the horizon Δ is

$$\begin{aligned} -\mathrm{i}m\wedge\bar{m} =& a\sin\theta\mathrm{d}v\wedge\mathrm{d}\theta + (r_+^2 + a^2)\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi \\ =& a\gamma\sin\theta\mathrm{d}v'\wedge\mathrm{d}\theta + r_0^2\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi, \end{aligned} \tag{47}$$

with the new coordinate $\mathrm{d}v' = \mathrm{d}v/\gamma$ and $r_0^2 = r_+^2 + a^2$.

A. The canonical formula

On the horizon $\Delta: r = r_+$, we choose $(x^0, x^1, x^2) = (v', \theta, \varphi)$. Similar to the BTZ black hole, the boundary BF theory on the horizon Δ is

$$S = \int_\Delta BF = \int_\Delta B\mathrm{d}A, \tag{48}$$

with the constraints

$$\mathrm{d}B = -\mathrm{i}\frac{1}{8\pi}m\wedge\bar{m}, \quad \mathrm{d}A = 0. \tag{49}$$

Assuming that the effective metric for the BF theory is

$$\mathrm{d}\tilde{s}^2 = -\mathrm{d}v'^2 + r_+^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \tag{50}$$

one can get

$$\begin{aligned} \epsilon_{012} =& \sqrt{-g} = r_+^2\sin\theta, \\ \epsilon^{012} =& \epsilon^{12} = \frac{1}{r_+^2\sin\theta}. \end{aligned} \tag{51}$$

Following the same methods as in the BTZ black hole case, we choose the gauge $B_0 = A_0 = 0$ to give $A_i = \partial_i\phi$, and the action then becomes

$$\begin{aligned} S =& -\int_\Delta \mathrm{d}^3x(\partial_2B_1 - \partial_1B_2)\partial_0\phi \\ =& \int_\Delta \mathrm{d}^3x\sqrt{-g}\pi\dot{\phi}, \end{aligned} \tag{52}$$

where $\pi = -\epsilon^{ij}\partial_iB_j$ is the canonical momentum. We add a Hamiltonian to get

$$\begin{aligned} S' =& \int_\Delta \mathrm{d}^3x\sqrt{-g}\left(\pi\dot{\phi} - \frac{1}{2m_0}\pi^2 - \frac{m_0}{2}g^{ij}\partial_i\phi\partial_j\phi\right) \\ =& \frac{m_0}{2}\int_\Delta \mathrm{d}^3x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \end{aligned} \tag{53}$$

where m_0 is a mass parameter used to adjust the dimension mismatch between the boundary and bulk. This is a free scalar field theory.

The field equations can thus be recast into a duality relation:

$$\epsilon^{\mu\nu\rho}\partial_\nu B_\rho = m_0\partial^\mu\phi. \tag{54}$$

For the massless scalar field, one can obtain the Hamiltonian and angular momentum:

$$H = \frac{m_0}{2}\int_\Sigma d^2x\sqrt{-g}(\partial_0\phi\partial_0\phi + g^{ij}\partial_i\phi\partial_j\phi),$$

$$J = m_0\int_\Sigma d^2x\sqrt{-g}(\partial_0\phi\partial_2\phi), \tag{55}$$

where we omit the zero-point energy.

B. Quantization

Next, we quantize the massless scalar field (53). Expanding the fields with the Fourier modes gives

$$\phi = \phi_0 + p_v v' + p_\theta \ln(\cot\frac{\theta}{2}) + p_\varphi\varphi + \sqrt{\frac{1}{m_0 A}}\sum_{l>0}\sum_{m=-l}^{m=l}\sqrt{\frac{1}{2\omega_l}}[a_{l,m}e^{-i\omega_l v'}Y_l^m(\theta,\varphi) + a_{l,m}^+ e^{i\omega_l v'}(Y_l^m)^*(\theta,\varphi)],$$

$$B_1 = m_0(B_{10}r_+ + \frac{p_\varphi}{\sin\theta}v' + \frac{1}{2}r_+^2\sin\theta p_v\varphi - 2\sqrt{\frac{1}{m_0A}}\sum_{l>0}\sum_{m=-l}^{m=l}\sqrt{\frac{1}{(2\omega_l)^3}}\frac{m}{\sin\theta}[a_{l,m}e^{-i\omega_l v'}Y_l^m(\theta,\varphi) + a_{l,m}^+ e^{i\omega_l v'}(Y_l^m)^*(\theta,\varphi)]),$$

$$B_2 = m_0(B_{20}r_+ + p_\theta v' - \frac{1}{2}r_+^2\sin\theta p_v\theta - 2i\sqrt{\frac{1}{m_0A}}\sum_{l>0}\sum_{m=-l}^{m=l}\sqrt{\frac{1}{(2\omega_l)^3}}\sin\theta\frac{\partial}{\partial\theta}[a_{l,m}e^{-i\omega_l v'}Y_l^m(\theta,\varphi) - a_{l,m}^+ e^{i\omega_l v'}(Y_l^m)^*(\theta,\varphi)]), \tag{56}$$

where $\omega_l^2 = \dfrac{l(l+1)}{r_+^2}$, $Y_l^m(\theta,\varphi)$ are spherical harmonics, and $A = 4\pi r_+^2$. It is straightforward to show that the above expressions satisfy the dual relation (54).

The commutative relation between $\hat{\phi}$ and $\hat{\pi}$ is

$$[\hat{\phi}(v',\vec{x}), \epsilon^{ij}\partial_i\hat{B}_j(v',\vec{y})] = -i\delta^2(\vec{x}-\vec{y}), \tag{57}$$

which yields

$$[\hat{\phi}_0, m_0\hat{p}_v] = \frac{i}{A}, \quad [\hat{a}_{l,m}, \hat{a}^+_{l',m'}] = \delta_{l,l'}\delta_{m,m'}. \tag{58}$$

We can also consider B_i and $\epsilon^{ij}\partial_j\phi$ as two canonical variables, leading to two further commutation relations:

$$\left[\hat{B}_{10}, m_0 r_+\frac{\hat{p}_\varphi}{r_+^2\sin\theta}\right] = \frac{i}{A}, \quad \left[\hat{B}_{20}, m_0 r_+\frac{\hat{p}_\theta}{r_+^2}\right] = -\frac{i}{A}. \tag{59}$$

For the Kerr black hole, the B field satisfies the constraint (49) or has the components

$$\partial_0 B_1 = \frac{a\gamma\sin\theta}{8\pi},$$
$$\partial_1 B_2 - \partial_2 B_1 = \frac{r_0^2\sin\theta}{8\pi},$$
$$\partial_0 B_2 = 0, \tag{60}$$

which gives

$$p_v = -\frac{r_0^2}{8\pi m_0 r_+^2}, \quad p_\varphi = \frac{a\sin^2\theta\gamma}{8\pi m_0}, \quad p_\theta = 0. \tag{61}$$

To eliminate the trigonometric functions, we define $|\hat{p}_\varphi| = \dfrac{a\gamma}{8\pi m_0}$, which can be considered as a quantum operator of p_φ.

Then, the commutation relations give

$$\left[\hat{\phi}_0, -\frac{4\pi\hat{r}_0^2}{8\pi}\right] = i, \quad \left[\hat{B}_{10}, \frac{\hat{a}r_+\gamma}{2}\right] = i. \tag{62}$$

Since the zero modes ϕ_0, B_{i0} are constant on the horizon, the canonical momentum is quantized with

$$\frac{4\pi r_0^2}{8\pi} = n_1, \quad \frac{ar_+\gamma}{2} = n_2, \quad n_1, n_2 \in N. \tag{63}$$

To fix the parameters γ, m_0, we define the dimensionless left- and right-temperatures as (33)

$$T_{R/L} = -\frac{r_+}{2\pi}\left(p_v \mp \frac{|p_\varphi|}{r_+}\right) = \frac{r_0^2}{16\pi^2 m_0 r_+}\left(1 \pm \frac{a\gamma r_+}{r_0^2}\right). \tag{64}$$

We require that, in the extremal limit $a \to M$, $T_L \to 0$. A natural choice should thus be

$$\gamma = \frac{r_0^2}{r_+^2}. \tag{65}$$

We assume that they also satisfy the condition (34); that is,

$$\frac{2}{T_H} = \frac{\gamma r_+}{T_R} + \frac{\gamma r_+}{T_L} \Rightarrow m_0 = \frac{M}{2\pi}. \tag{66}$$

The dimensionless temperatures are then given by

$$T_{R/L} = \frac{1}{4\pi}\left(1 \pm \frac{a}{r_+}\right). \tag{67}$$

Notice that in the Schwarzschild case $a=0$, the temperatures reduce to $T=\dfrac{1}{4\pi}$. In the extremal limit $a\to M$, they coincide with the temperatures in the Kerr/CFT correspondence $T_{\rm R/L}=\dfrac{1}{4\pi a}(r_+\pm r_-)$.

If we assume that a dual CFT exists for an arbitrary Kerr black hole, the central charge c can be obtained through the Cardy formula:

$$S_{\rm Cardy}=\frac{\pi^2}{3}c(T_{\rm R}+T_{\rm L})=S_{\rm BH}=2\pi Mr_+, \qquad (68)$$

which gives

$$c=12Mr_+=\frac{6S_{\rm BH}}{\pi}. \qquad (69)$$

The quantized condition (63) gives

$$\begin{aligned}&A_{\rm H}=4\pi r_0^2=8\pi n_1, \quad S_{\rm BH}=2\pi n_1,\\ &c=12n_1, \quad J=n_2, \quad n_1,n_2\in N.\end{aligned} \qquad (70)$$

That is, the area and angular momentum of the Kerr black hole are both quantized. The area spectrum has the same form as in Refs. [33, 34].

The scalar field $\phi(v',\theta,\varphi)$ can be considered as a collective of harmonic oscillators, and a general state can be represented as

$$|p_v,p_\varphi;\{n_{l,m}\}>\sim(\hat{a}^+_{1,m})^{n_{1,m}}\cdots(\hat{a}^+_{l,m})^{n_{l,m}}|p_v,p_\varphi>,$$

where $|p_v,p_\varphi>$ are zero mode parts, and $|\{n_{l,m}\}>$ are occupation numbers for the oscillator part. The quantum Hamiltonian operator and the angular momentum operator for the free scalar field (55) can be written as

$$\begin{aligned}\hat{H}_{\rm free}=&\frac{m_0}{2}A\left(\hat{p}_v^2+\frac{|\hat{p}_\varphi|^2}{r_+^2}\right)+\sum_{l>0}\sum_{m=-l}^{m=l}\frac{\sqrt{l(l+1)}}{r_+}\hat{a}^+_{l,m}\hat{a}_{l,m},\\ \hat{J}=&m_0A\hat{p}_v|\hat{p}_\varphi|+\sum_{l>0}\sum_{m=-l}^{m=l}m\hat{a}^+_{l,m}\hat{a}_{l,m}.\end{aligned} \qquad (71)$$

Unlike for the BTZ black hole, in high dimensional spacetime $D\geqslant 4$, general relativity has local degrees of freedom. It is thus natural to consider the scalar field with the interaction

$$\hat{H}_{\rm full}=\hat{H}_{\rm free}+\hat{H}_{\rm int}. \qquad (72)$$

The calculation of entropy for higher dimensional Kerr black holes [35] suggests the consideration of the full Hamiltonian, which has the following spectrum:

$$\begin{aligned}\hat{H}_{\rm full}=&\frac{m_0}{2}A(\hat{p}_v^2+\frac{|\hat{p}_\varphi|^2}{r_+^2})+\sum_{l>0}\sum_{m=-l}^{m=l}\frac{|m|}{r_+}\hat{a}^+_{l,m}\hat{a}_{l,m}\\ =&\hat{H}_0+\sum_{l>0}\sum_{m=-l}^{m=l}\frac{|m|}{r_+}\hat{n}_{l,m}=\hat{H}_0+\sum_m\frac{|m|}{r_+}\hat{n}_m, \quad m\neq 0,\end{aligned} \qquad (73)$$

where $\hat{n}_m=\sum_{l>0}\hat{n}_{l,m}$, $\hat{n}_{l,m}=\hat{a}^+_{l,m}\hat{a}_{l,m}$ are number operators. The angular momentum operator can also be rewritten as

$$\hat{J}=\hat{J}_0+\sum_m m\hat{n}_m. \qquad (74)$$

It is easy to show that the zero mode part has half the value of the angular momentum and that of the energy for the Kerr black hole; that is,

$$\begin{aligned}&<p_v,p_\varphi;\{0\}|\hat{J}|p_v,p_\varphi;\{0\}>=m_0Ap_v|p_\varphi|=\frac{J}{2}=\frac{aM}{2},\\ &<p_v,p_\varphi;\{0\}|\hat{H}|p_v,p_\varphi;\{0\}>=\frac{m_0}{2}A\left(p_v^2+\frac{|p_\varphi|^2}{r_+^2}\right)=\frac{\gamma}{2}\frac{M}{2}.\end{aligned} \qquad (75)$$

One may get confused about the term $\dfrac{M}{2}$ in (75). This can be explained by associating the Hamiltonian with the enthalpy [36] or with the thermodynamic potential [35, 37] of the black hole.

The microscopic states of the Kerr black hole are represented by $|0,0;\{n_m\}>$. They contribute the other half of the value of the angular momentum and that of the energy for the Kerr black hole; that is,

$$\begin{aligned}&\frac{1}{c}<0,0;\{n_m\}|\hat{J}|0,0;\{n_m\}>=\frac{J}{2},\\ &\frac{1}{c}<0,0;\{n_m\}|\hat{H}|0,0;\{n_m\}>=\gamma\frac{M}{4},\end{aligned} \qquad (76)$$

where $c=12Mr_+$ is the central charge. Different sequences $\{n_m\}$ correspond to different microstates of Kerr black holes with the same (M,J).

The constraints (76) are equivalent to

$$\sum_m|m|n_m=\frac{cM^2}{2}, \quad \sum_m mn_m=\frac{caM}{2}, \quad m\neq 0. \qquad (77)$$

Calculating the Kerr black hole entropy is thus transformed into a mathematical problem: it suffices to count the number of unique sequences $\{n_m\}$ that satisfy the constraints (77). The constraints are the same as those for BTZ black holes. The entropy of Kerr black holes can thus be obtained:

$$\begin{aligned} N(M,J) \simeq & \frac{1}{c(M^2+J)c(M^2-J)} \exp\left(2\pi\sqrt{c\frac{M^2+J}{24}}\right. \\ & \left. +2\pi\sqrt{c\frac{M^2-J}{24}}\right), \\ S =& \ln N(M,J) = 2\pi M r_+ - \ln[M^4 r_+^2(r_+^2-a^2)] + \cdots, \end{aligned} \tag{78}$$

which is simply the Bekenstein-Hawking entropy formula with some low order corrections.

For extremal Kerr black holes with $a = M$, the entropy is given by

$$\begin{aligned} S =& \ln N(M,J) = \ln\left(\frac{1}{caM}\exp\left(2\pi\sqrt{\frac{caM}{12}}\right)\right) \\ =& 2\pi M^2 - 2\ln(M^2) + \cdots, \end{aligned} \tag{79}$$

which matches the result in Ref. [38] for log-corrections.

IV. CONCLUSION

In this paper, we analyze the boundary modes on the horizons of black holes with the methods developed for topological insulators. BTZ black holes are analyzed first, and the results are found to be compatible with previous works. The same methods are then applied to Kerr black holes. Several new results are obtained: dimensionless right- and left-temperatures can be defined and have well behavior in both the Schwarzschild limit $a\to 0$ and the extremal limit $a \to M$. Moreover, a central charge $c = 12Mr_+$ is associated with an arbitrary Kerr black hole. We also identify the microstates of the Kerr black hole with the quantum states of the scalar field on the horizon. Based on this identification, the number of microstates of the Kerr black hole can be counted. Calculating the Kerr black hole entropy is thus transformed into a counting problem of the number of unique sequences $\{n_m\}$ that satisfy some constraints. The final result is the Bekenstein-Hawking entropy formula with some low order corrections.

Due to the compactness of the scalar field, one can obtain the interesting result that the area of both BTZ (29) and Kerr (70) black holes is quantized with an equally spaced spectrum. This result is consistent with earlier works [33, 39].

Since this approach is related to the Kerr/CFT correspondence approach, we should compare our results with those from the Kerr/CFT correspondence approach. In the Kerr/CFT correspondence, for an arbitrary Kerr black hole, the dimensionless right- and left-temperatures are given by $T_{L/R} = \frac{r_+ \pm r_-}{4\pi a}$, and the associated central charges are $c_L = c_R = 12J = 12Ma$. In our approach, the related quantities are $T_{L/R} = \frac{r_+ \pm a}{4\pi r_+}$ and $c_L = c_R = 12Mr_+$. In the extremal limit $r_+ \to a$, the two results coincide. In contrast, for the Schwarzschild black hole with $a = 0$, the Kerr/CFT correspondence gives $T_L \to \infty$ and $c_L = c_R = 0$, which are un-physical. However, our results have well behavior in this case, and we thus find them to be more reasonable.

Based on the "horizon fluff" proposal, the microstates of the extremal Kerr black hole are identified in Ref. [38]. The number of microstates gives the Bekenstein-Hawking area law. This approach relies on the $U(1)$ Kac-Moody algebra and Virasoro algebra, which is closely related to the Kerr/CFT correspondence. In our approach, we identify the microstates of a Kerr black hole with the quantum states of the massless scalar field, and the algebra is the commutation algebra for harmonic oscillators. Whether the Virasoro algebra can be obtained from our approach is under investigation. A related issue is the value of the central charge in the entropy calculation formulae (37) and (76). We cannot derive this value from first principles since we do not have a conformal field theory and the relevant Virasoro algebra. It is determined manually in order to obtain the right entropy. Nevertheless, it takes the same form for all Kerr black holes, which can be considered as a non-trivial check.

References

[1] J. D. Bekenste, Physical Review D **7**(8), 2333-2346 (1973)
[2] S. W. Hawking, Nature **248**(5443), 30-31 (1974)
[3] J. Wang. Black hole as topological insulator (I): the BTZ black hole case. 2017
[4] J. Wang. Black hole as topological insulator (II): the boundary modes. 2017
[5] J. Wang, Classification of black holes in three dimensional spacetime by the $W_{1+\infty}$ symmetry. 2018
[6] J. Wang, Chin. Phys. C **43**(9), 095104 (2019)
[7] J. Wang, Phys. Lett. B **792**, 56-59 (2019)
[8] G. Y. Cho and J. E. Moore, Annals Phys. **326**, 1515-1535 (2011)
[9] S. W. Hawking, Phys. Rev. D **14**, 2460-2473 (1976)
[10] S. Chakraborty and K. Lochan, Universe **3**(3), 55 (2018)
[11] D. Marolf, Rept. Prog. Phys. **80**(9), 092001 (2017)
[12] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010)
[13] X. L. Qi and S. C. Zhang, Rev. Mod. Phys. **83**(4), 1057-1110 (2011)
[14] X. Chen, A. Tiwari, and S. Ryu. Bulk-boundary correspondence in (3+1)-dimensional topological phases. Phys. Rev., B **94**(4), 045113(2016). [Addendum: Phys. Rev.B **94**(7), 079903(2016)]
[15] A. Cappelli, E. Randellini, and J. Sisti, JHEP **05**, 135 (2017)
[16] J. Wang, Y. Ma, and X.-A. Zhao, Phys. Rev. D **89**, 084065 (2014)
[17] C.-G. Huang and J. Wang, Gen. Rel. Grav. **48**(8), 115 (2016)

[18] A. Ashtekar, J. Baez, A. Corichi *et al*., Phys. Rev. Lett. **80**, 904-907 (1998)
[19] A. Ashtekar, John C. Baez, and Kirill Krasnov, Adv. Theor. Math. Phys. **4**, 1-94 (2000)
[20] C. Rovelli. *Quantum Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2004
[21] T. Thiemann. *Modern Canonical Quantum General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2008
[22] A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **21**(15), R53-R152 (2004)
[23] M. Han, Y. Ma, and W. Huang, Int. J. Mod. Phys. D **16**(9), 1397-1474 (2007)
[24] J. Wang. The Entropy of BTZ Black Hole from Loop Quantum Gravity. 2014
[25] J. Wang and C.-G. Huang, Class. Quant. Grav. **32**, 035026 (2015)
[26] J. Wang and C.-G. Huang, Int. J. Mod. Phys. D **25**(14), 1650100 (2016)
[27] J. Wang, C.-G. Huang, and L. Li, Chin. Phys. C **40**(8), 083102 (2016)
[28] M. Guica, T. Hartman, W. Song *et al*., Phys. Rev. D **80**, 124008 (2009)
[29] A. Castro, A. Maloney, and A. Strominger, Phys. Rev. D **82**, 024008 (2010)
[30] G. CompYre. The Kerr/CFT correspondence and its extensions. Living Rev. Rel., **15**,11(2012). [Living Rev. Rel. **20**(1), 1(2017)]
[31] H. Afshar, D. Grumiller, M. M. Sheikh-Jabbari *et al*., JHEP **08**, 087 (2017)
[32] B. Krishnan. Quasi-local black hole horizons. In A. Ashtekar and V. Petkov, editors, *Springer Handbook of Spacetime*, pages 527-555. 2014
[33] J. D. Bekenstein, Lett. Nuovo Cim. **11**, 467 (1974)
[34] M. Maggiore, Phys. Rev. Lett. **100**, 141301 (2008)
[35] J. Wang. Central charges for Kerr and Kerr-AdS black holes in diverse dimensions. 2019
[36] T. Padmanabhan, Mod. Phys. Lett. A **30**(03n04), 1540007 (2015)
[37] G. W. Gibbons, M. J. Perry, and C. N. Pope, Class. Quant. Grav. **22**, 1503-1526 (2005)
[38] K. Hajian, M. M. Sheikh-Jabbari, and H. Yavartanoo, Phys. Rev. D **98**(2), 026025 (2018)
[39] D. Kothawala, T. Padmanabhan, and S. Sarkar, Phys. Rev. D **78**, 104018 (2008)