Conformal field theory on the horizon of a BTZ black hole*

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Abstract: In a three-dimensional spacetime with negative cosmological constant, general relativity can be written as two copies of $SO(2,1)$ Chern-Simons theory. On a manifold with a boundary, the Chern-Simons theory induces a conformal field theory—Wess-Zumino-Witten theory on the boundary. In this paper, it is shown that with suitable boundary conditions for a Banados-Teitelboim-Zanelli black hole, the Wess-Zumino-Witten theory can reduce to a chiral massless scalar field on the horizon.

Keywords: BTZ black hole, Chern-Simons theory, WZW theory

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1 Introduction

In a three-dimensional spacetime, general relativity becomes much simpler, since it has no local degrees of freedom [1]. Indeed, the theory is equivalent to a Chern-Simons theory with a suitable gauge group [2, 3]. Even so, there exists a black hole solution in the theory [4] when the cosmological constant Λ is negative. In such a case, the gauge group is $SO(2,1)\times SO(2,1)$. The black hole, named after Banados, Teitelboim and Zanelli (BTZ), can have very large entropy if the radius of the horizon is large. The entropy cannot be explained by counting the local degrees of freedom, because a 3D gravitational theory has no local degrees of freedom.

There are several ways to explain the entropy of a BTZ black hole with the help of conformal theories. For a good review, see Ref. [5]. It is well known that a Chern-Simons (CS) theory on a manifold with a boundary induces a Wess-Zumino-Witten (WZW) theory on the boundary, which is a conformal field theory. By use of the $SO(2,1)\times SO(2,1)$ WZW theory, Carlip explained the entropy of the BTZ black hole [6]. However, due to the non-compactness of $SO(2,1)$, it is difficult to quantize the $SO(2,1)$ WZW theory. With a slightly stronger boundary condition, the $SO(2,1)\times SO(2,1)$ WZW theory can reduce to a Liouville theory on the conformal boundary [7, 8]. The central charge of the Liouville theory with the Cardy formula [9, 10] can be used to explain the entropy of the BTZ black hole [5]. On the other hand, in their seminal work [11], Brown and Henneaux showed that the asymptotic symmetry group of AdS_3 is generated by two copies of Virasoro algebra, which also correspond to a conformal field theory. Based on this result, the entropy of a BTZ black hole can be calculated [12, 13], and matches the Bekenstein-Hawking formula.

In the above approaches, the conformal theories are defined either at conformal infinity or at the horizon. For the former case, it is difficult to distinguish a black hole from a 'star', since they have the same infinity behavior. So, the more interesting approach is that the conformal field theories are located at the horizon of a black hole, and the following discussion is limited to this approach.

From a very different point of view, it has been shown [14–20] that the boundary degrees of freedom on an isolated horizon in any dimensional spacetime (including in a 3-dimensional spacetime) can be described by a boundary BF theory on an isolated horizon, so that the entropy of a black hole can be explained by use of the boundary BF theory. In this paper, with the same boundary conditions as boundary BF theory, we want to see whether an explicit form of conformal field theories (CFTs) on the horizon can be obtained or not. Our strategy is to start from the Chern-Simons theory with suitable boundary conditions set by the boundary BF theory, and then show that the WZW theory reduces to a chiral massless scalar field on the 2-dimensional isolated horizon. Since 3D general relativity contains two copies of Chern-Simons theory, there are, correspondingly, two chiral massless

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scalar fields on the BTZ horizon.

The paper is organized as follows. In Section 2, we summarize the relations between gravity, Chern-Simons theory and WZW theory. In Section 3, the BTZ black hole is considered. With suitable boundary condition, the boundary WZW theory reduces to a chiral massless scalar field theory. Section 4 gives our conclusion.

2 Gravity, Chern-Simons theory and WZW theory

As first shown in Ref. [2], (2+1)−dimensional general relativity can be written as a Chern-Simons theory. For the case of negative cosmological constant $\Lambda=-1/L^2$, one can define two $SO(2,1)$ connection 1-forms

$$A^{(\pm)a}=\omega^a\pm\frac{1}{L}e^a, \tag{1}$$

where e^a and ω^a are orthonormal co-triads and spin connection 1-forms, respectively, and $a=0,1,2$ is the gauge group index. Then, up to a boundary term, the first order action of gravity can be rewritten as

$$\begin{aligned}I_{\mathrm{GR}}[e,\omega]=&\frac{1}{8\pi G}\int[e^a\wedge(\mathrm{d}\omega_a+\frac{1}{2}\epsilon_{abc}\omega^b\wedge\omega^c)\\&-\frac{1}{6L^2}\epsilon_{abc}e^a\wedge e^b\wedge e^c]\\=&I_{\mathrm{CS}}[A^{(+)}]-I_{\mathrm{CS}}[A^{(-)}],\end{aligned} \tag{2}$$

where $A^{(\pm)}=A^{(\pm)a}T_a$ are $SO(2,1)$ gauge potentials, T_a are generators of the $SO(2,1)$ group, and the Chern-Simons action is

$$I_{\mathrm{CS}}[A]=\frac{k}{4\pi}\int\mathrm{Tr}\{A\wedge\mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\}, \tag{3}$$

with

$$k=\frac{L}{4G}. \tag{4}$$

Similarly, the CS equation

$$F^{(\pm)}=\mathrm{d}A^{(\pm)}+A^{(\pm)}\wedge A^{(\pm)}=0 \tag{5}$$

is equivalent to the requirement that the connection is torsion-free and the metric has a constant negative Riemann curvature. The equation implies that the potential A can be locally written as

$$A=g^{-1}\mathrm{d}g. \tag{6}$$

When the manifold has a boundary, a boundary term must be added and this term depends on our choice of boundary conditions. Assume that the boundary has topology $\partial M=R\times S^1$. The standard boundary condition is chosen to be [5]

$$\delta A_u|_{\partial M}=0,\quad \text{or}\quad \delta A_{\tilde{u}}|_{\partial M}=0, \tag{7}$$

where u and $\tilde{u}$ are two coordinates on the boundary. Then the boundary term is

$$I_{bd}=\frac{k}{4\pi}\int_{\partial M}\mathrm{d}u\mathrm{d}\tilde{u}\mathrm{Tr}(A_uA_{\tilde{u}}). \tag{8}$$

With the boundary term, the total action, $I_{\mathrm{CS}}[A]+I_{bd}[A]$, is not gauge-invariant under the gauge transformation

$$\bar{A}=g^{-1}Ag+g^{-1}\mathrm{d}g. \tag{9}$$

To restore the gauge-invariance, the Wess-Zumino-Witten term is introduced for the first boundary condition [21, 22]:

$$\begin{aligned}I_{\mathrm{WZW}}[g^{-1},A_u]=&\frac{1}{4\pi}\int_{\partial M}\mathrm{d}u\mathrm{d}\tilde{u}\mathrm{Tr}(g^{-1}\partial_u gg^{-1}\partial_{\tilde{u}}g+2g^{-1}\partial_{\tilde{u}}gA_u)\\&+\frac{1}{12\pi}\int_M\mathrm{Tr}(g^{-1}\mathrm{d}g)^3,\end{aligned} \tag{10}$$

which is a chiral WZW action for a field g coupled to a background gauge potential A_u.

With the WZW term, the full action is gauge-invariant,

$$\begin{aligned}&(I_{\mathrm{CS}}+I_{bd})[\bar{A}]+kI_{\mathrm{WZW}}[\mathrm{e}^{-1},\bar{A}]\\&=(I_{\mathrm{CS}}+I_{bd})[A]+kI_{\mathrm{WZW}}[g^{-1},A],\end{aligned} \tag{11}$$

where e is the unit element of the group.

Thus, the gauge transformation g becomes a dynamical variable at the boundary, and is described by the WZW action, which is a conformal field theory. Those 'would-be gauge degrees of freedom' [23] are present because the gauge invariance is broken at the boundary.

3 Boundary action on the horizon of a BTZ black hole

In the previous section, the boundary of the manifold could be arbitrary. If the horizon of the BTZ black hole is considered, more reductions can be made due to the special properties of the horizon.

3.1 BTZ black hole

To study the physics at a horizon Δ, it is more suitable to use advanced Eddington coordinates (v,r,φ). The metric of a BTZ black hole can be written as

$$\mathrm{d}s^2=-N^2\mathrm{d}v^2+2\mathrm{d}v\mathrm{d}r+r^2(\mathrm{d}\varphi+N^\varphi\mathrm{d}v)^2. \tag{12}$$

We choose the following null co-triads [24],

$$l=-\frac{1}{2}N^2\mathrm{d}v+\mathrm{d}r,\quad n=-\mathrm{d}v,\quad m=rN^\varphi\mathrm{d}v+r\mathrm{d}\varphi, \tag{13}$$

which relate to the orthonormal co-triads by

$$l\equiv e^-=\sqrt{\frac{1}{2}}(e^0-e^1),\quad n\equiv e^+=\sqrt{\frac{1}{2}}(e^0+e^1),\quad e^2=m. \tag{14}$$

After some calculations, the Chern-Simons connections (1) are:

$$\begin{aligned}A^{(\pm)-}&=-(N^{\varphi}\mp\frac{1}{L})\mathrm{d}r-\frac{N^2}{2}\mathrm{d}(\varphi\pm\frac{v}{L}),\\A^{(\pm)+}&=-\mathrm{d}(\varphi\pm\frac{v}{L}),\\A^{(\pm)2}&=r(N^{\varphi}\pm\frac{1}{L})\mathrm{d}(\varphi\pm\frac{v}{L}),\end{aligned}\tag{15}$$

where $A^{\pm}=(A^0\pm A^1)/\sqrt{2}$. Those connections also define the boundary BF theory on the isolated horizon.

We define new variables,

$$u=\varphi-\frac{v}{L},\quad \tilde{u}=\varphi+\frac{v}{L},\tag{16}$$

which gives

$$A=A_r\mathrm{d}r+A_u\mathrm{d}u+A_{\tilde{u}}\mathrm{d}\tilde{u}.\tag{17}$$

A crucial property of the connections (15) is that, on the whole manifold, one has

$$A_u^{(+)\pm}=A_u^{(+)2}\equiv0,\quad A_{\tilde{u}}^{(-)\pm}=A_{\tilde{u}}^{(-)2}\equiv0.\tag{18}$$

Another crucial property is that when approaching the horizon Δ [24],

$$A_{\tilde{u}}^{(+)-}=A_u^{(-)-}=\frac{N^2}{2}\to0.\tag{19}$$

These two properties are important for later calculation.

Since the topology of the space-section is cylindrical, which is non-trivial, the vacuum Chern-Simons equation $F=0$ will be solved by a non-periodic group element Q with [8]

$$A=Q^{-1}\mathrm{d}Q.\tag{20}$$

A general $SO(2,1)$ group element $Q(\tilde{u},u,r)$, using the Gauss decomposition, can be written as

$$Q=\begin{pmatrix}1&\frac{1}{\sqrt{2}}x_1\\0&1\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\Psi_1/2}&0\\0&\mathrm{e}^{-\Psi_1/2}\end{pmatrix}\begin{pmatrix}1&0\\-\frac{1}{\sqrt{2}}y_1&1\end{pmatrix}.\tag{21}$$

With those parameters, the WZW action is [7]

$$kI_{\mathrm{WZW}}=\frac{k}{4\pi}\int_{\Delta}\mathrm{d}u\mathrm{d}\tilde{u}\frac{1}{2}(\partial_u\Psi\partial_{\tilde{u}}\Psi-\mathrm{e}^{\Psi}(\partial_ux\partial_{\tilde{u}}y+\partial_uy\partial_{\tilde{u}}x)).\tag{22}$$

3.2 Gauge transformation

Now we consider the gauge transformation (9) with group element g_1 for the $A^{(+)}$.

To preserve the boundary condition (18)

$$\delta A_u^{(+)}|_{\partial M}=0,\tag{23}$$

the gauge transformation should be $g_1=g_1(r,\tilde{u})$. However, this boundary condition is still not enough to determine whether the system we are dealing with is a black hole or not. So, more restricted boundary conditions are needed. A crucial property of horizon Δ is non-expansion. In other words, the expansion θ_l of the null normal vector l, which is defined by $\theta_l\triangleq m^am^b\nabla_al_b$, vanishes on Δ. The $\triangleq$ means that the equality is valid on the horizon Δ. This property reflects on the vanishing of the connection $A^{(+)-}$, since after some calculation it can be shown that

$$A^{(+)-}=-\theta_lm+(\frac{1}{L}-\tau)l\triangleq0,\tag{24}$$

where $\tau=n^al^b\nabla_am_b$ is the spin coefficient in Newman-Penrose formalism [25]. We want the gauge transformation to keep this condition.

We assume the gauge transformation is given by the $SO(2,1)$ group element,

$$\begin{aligned}&g_1(x_1,y_1,\Psi_1)\\&=\begin{pmatrix}1&\frac{1}{\sqrt{2}}x_1\\0&1\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\Psi_1/2}&0\\0&\mathrm{e}^{-\Psi_1/2}\end{pmatrix}\begin{pmatrix}1&0\\-\frac{1}{\sqrt{2}}y_1&1\end{pmatrix}.\end{aligned}\tag{25}$$

Under the gauge transformation (9),

$$\bar{A}^{(+)-}=\mathrm{e}^{\Psi_1}(A^{(+)-}-A^{(+)2}x_1+A^{(+)+}x_1^2/2+\mathrm{d}x_1).\tag{26}$$

Since $A^{(+)2},A^{(+)+}$ are both finite at the horizon, to keep the boundary condition (24), one needs

$$x_1(r,\tilde{u})\triangleq0,\tag{27}$$

and $\Psi_1(r)$ is finite at the horizon.

The other components transform into

$$\begin{aligned}\bar{A}^{(+)2}=&A^{(+)2}(1-\mathrm{e}^{\Psi_1}y_1x_1)-A^{(+)+}x_1(1-\mathrm{e}^{\Psi_1}y_1x_1/2)\\&+A^{(+)-}\mathrm{e}^{\Psi_1}y_1+\mathrm{d}\Psi_1+\mathrm{e}^{\Psi_1}y_1\mathrm{d}x_1,\\\bar{A}^{(+)+}=&A^{(+)+}\mathrm{e}^{-\Psi_1}(1-\mathrm{e}^{\Psi_1}y_1x_1/2)^2\\&+A^{(+)2}y_1(1-\mathrm{e}^{\Psi_1}y_1x_1/2)+A^{(+)-}\mathrm{e}^{\Psi_1}y_1^2/2\\&+y_1\mathrm{d}\Psi_1+\mathrm{d}y_1+y_1^2\mathrm{e}^{\Psi_1}\mathrm{d}x_1/2.\end{aligned}\tag{28}$$

Those components are required to be finite at the horizon, so that

$$y_1(r_+)=\text{finite},\quad \Psi_1(r_+)=\text{finite},\tag{29}$$

where r_+ is the outer radius of the black hole. Due to Eq. (27), the derivatives of x_1 with respect to $(u,\tilde{u})$ are 0. So the second term in the action (22) vanishes on the horizon

$$\mathrm{e}^{\Psi_1}(\partial_ux_1\partial_{\tilde{u}}y_1+\partial_uy_1\partial_{\tilde{u}}x_1))\triangleq0.\tag{30}$$

So the final action on the horizon is then

$$\begin{aligned}kI_{\mathrm{WZW}}&=\frac{k}{4\pi}\int_{\Delta}\mathrm{d}u\mathrm{d}\tilde{u}\frac{1}{2}\partial_u\Psi_1\partial_{\tilde{u}}\Psi_1\\&=\frac{k}{4\pi L}\int_{\Delta}\mathrm{d}v\mathrm{d}\varphi[L^2(\partial_v\Psi_1)^2-(\partial_{\varphi}\Psi_1)^2],\end{aligned}\tag{31}$$

with Ψ_1 depending only on $\tilde{u}=\varphi+\frac{v}{L}$. This is a chiral massless scalar field theory which is very similar to the edge theory for the quantum Hall effect [26]. Even the on-shell action is zero; it has non-trivial dynamics. After quantizing this theory, the Hilbert space forms the representation of the Abelian Kac-Moody algebra.

Similar results can be obtained for the $A^{(-)}$, which gives another chiral massless scalar field Ψ_2 depending only on u.

4 Conclusion

In this paper, field theory on the horizon of a BTZ black hole was investigated. Starting from the Chern-Simons theory, one can get a chiral WZW theory on any boundary. Restricting to the horizon, this WZW theory reduces further to a chiral massless scalar field theory. Since general relativity is equivalent to two copies of CS theory, the final theory on the horizon is two chiral massless scalar field theories with opposite chirality. The sum of those two chiral boson actions can be written in Liouville form [27].

Compared with the conformal field theories on the conformal boundary, the massless scalar field theory–which is also a conformal field theory–is more relevant to black hole physics. It is just on the horizon. In this paper, we give an explicit action for the scalar field. The scalar field was also obtained in Refs. [28, 29], from near-horizon symmetry algebra. The Fourier modes of its energy-momentum tensor form a Virasoro algebra with central charge $c=1$. From this near horizon Virasoro algebra $\mathcal{L}_n$ one can construct a Virasoro algebra L_n with $c=3L/2G$, which is the algebra on the conformal boundary [30, 31]. The map is given by

$$L_n=\frac{1}{c}\left(\mathcal{L}_{cn}+\left(\frac{c^2}{24}-\frac{1}{24}\right)\delta_{n,0}\right),\quad n\in Z,c\in N. \tag{32}$$

We assume that the central charge c is integer-valued. Roughly speaking, this can be understood by the fact [32] that the central charge is $c=6k$ where k is the level of the Chern-Simons theory (4). To have a well-defined quantized Chern-Simons theory, k should be an integer, and so is c.

The difference between those two central charges might be understood as the red-shift effect. In Ref. [31] it was shown that the energy scale at the stretched horizon and the asymptotic one are related by a factor $c=3L/2G$. The massless free scalar field (which has $c=1$) has little entropy by the Cardy formula, so only captures part of the degrees of freedom in the bulk. There might be some possible way to solve the "mismatching of entropy" issue.

As a final remark, the conformal symmetry used here is different from that which appears in Carlip's effective description of the black hole entropy in arbitrary dimensions [33]. As noticed in Ref. [34], the symmetry of this paper is on the "$\varphi-v$ cylinder", while the symmetry of Ref. [33] is on the "$r-v$ plane".

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