Hamiltonian analysis of 4-dimensional spacetime in Bondi-like coordinates *

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Abstract: We discuss the Hamiltonian formulation of gravity in four-dimensional spacetime under Bondi-like coordinates $\{v,r,x^a,a=2,3\}$. In Bondi-like coordinates, the three-dimensional hypersurface is a null hypersurface, and the evolution direction is the advanced time v. The internal symmetry group SO(1,3) of the four-dimensional spacetime is decomposed into SO(1,1), SO(2), and $\mathrm{T}^{\pm}(2)$, whose Lie algebra $\mathfrak{so}(1,3)$ is decomposed into $\mathfrak{so}(1,1)$, $\mathfrak{so}(2)$, and $\mathrm{t}^{\pm}(2)$ correspondingly. The SO(1,1) symmetry is very obvious in this type of decomposition, which is very useful in $\mathfrak{so}(1,1)$ BF theory. General relativity can be reformulated as the four-dimensional coframe (e^I_μ) and connection (ω^{IJ}_μ) dynamics of gravity based on this type of decomposition in the Bondi-like coordinate system. The coframe consists of two null 1-forms e^-, e^+ and two spacelike 1-forms e^2, e^3. The Palatini action is used. The Hamiltonian analysis is conducted by Dirac's methods. The consistency analysis of constraints has been done completely. Among the constraints, there are two scalar constraints and one two-dimensional vector constraint. The torsion-free conditions are acquired from the consistency conditions of the primary constraints about π^μ_{IJ}. The consistency conditions of the primary constraints $\pi^0_{IJ}=0$ can be reformulated as Gauss constraints. The conditions of the Lagrange multipliers have been acquired. The Poisson brackets among the constraints have been calculated. There are 46 constraints including 6 first-class constraints $\pi^0_{IJ}=0$ and 40 second-class constraints. The local physical degrees of freedom is 2. The integrability conditions of Lagrange multipliers n_0, l_0, and e^A_0 are Ricci identities. The equations of motion of the canonical variables have also been shown.

Keywords: Hamiltonian analysis, 4d gravity, Bondi-like coordinates

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1 Introduction

The Hamiltonian analysis plays an extremely important role in the initial-value problem and canonical quantization. For a gravitational system, the Hamiltonian analysis depends on two fundamental elements, namely, the foliation of a spacetime and the choice of canonical variables.

The most frequently used foliation method is to foliate a spacetime by a series of three-dimensional spacelike hypersurfaces along a time-like vector field [1], based on which the initial-value problem is well defined. An alternative foliation method is to foliate the spacetime along two null vector fields [2], named by 2+2 formalism, based on which the initial-value problem can also be well defined.

For a better understanding of the gravitational radiation, a null foliation is proposed [3], which provides a canonical formulation of a theory on outgoing null hypersurfaces. In a neighborhood of an outgoing beam of wave near the future null infinity in an asymptotical flat spacetime, the metric can be written in a Bondi-Sachs coordinate system $\{u,r,x^a,a=2,3\}$ [4, 5],

$$\mathrm{d}s^2=g_{00}\mathrm{d}u^2+2g_{01}\mathrm{d}u\mathrm{d}r+2g_{0a}\mathrm{d}u\mathrm{d}x^a+g_{ab}\mathrm{d}x^a\mathrm{d}x^b, \quad (1.1)$$

with $g_{00}<0$, $g_{01}<0$, and $g_{0a}>0$. The metric has four Bondi conditions: g_{11}, g_{12}, $g_{13}=0$, and $\det(g_{ab})\sim r^2$. In the system, the retarded time u is a null coordinate. Each u defines a three-dimensional null hypersurface in the four-dimensional spacetime. The spatial coordinate r is regarded as the distance from the isolated gravitational source. For a given u, every r defines a two-dimensional spacelike surface in the three-dimensional null hypersurface.

For a beam of an outgoing gravitational wave, u always remains constant in its propagation direction.

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Therefore, if one wants to study the propagation properties of a given beam of a gravitational wave, the advanced time coordinate v should be used instead of the retarded time coordinate u. In the study of the geometry near an isolated horizon that is a null hypersurface, the advanced time coordinate v should also be used [6–8]. In these cases, the metric is better written in a Bondi-like coordinate system $\{v,r,x^a,a=2,3\}$ or $\{x^0,x^1,x^a,a=2,3\}$:

$$ds^2=g_{00}dv^2+2g_{01}dvdr+2g_{0a}dvdx^a+g_{ab}dx^adx^b, \quad (1.2)$$

with $g_{00}<0$, $g_{01}>0$, and $g_{0a}>0$. In the above metric, there are three Bondi-like conditions, namely, g_{11}, g_{12}, and $g_{13}=0$; therefore, the metric has only 7 variables rather than 10 variables in the general form of a metric. The 4th coordinate condition is not imposed here.

Unlike the 1+3 spacelike foliation and 2+2 foliation, which can be used in the analysis of initial-value problems in the whole spacetime, the 1+3 null foliation can only be used in a finite region of the spacetime where there is no null signal incident in the opposite direction. In fact, in the study of the propagation of a beam of a gravitational wave, the one-way propagating wave and its propagation property are focused on and, thus, it is supposed that no other null signals exist. In the case of an isolated horizon, by definition, it is a null hypersurface without the incident of ingoing signals.

To have a better knowledge of the evolution of a geometry, the three-geometry h_{ij} on a three-dimensional spacelike hypersurface in the ADM formalism [1] and the two-geometry γ_{ab} on a two-dimensional spacelike surface in the $2+2$ formalism are chosen as the canonical configuration variables. To make general relativity look like a gauge theory, having polynomial forms, $\mathfrak{su}(2)$-connection on a three-dimensional hypersurface is chosen as the canonical configuration variable [9]. The $\mathfrak{su}(2)$-connection is also constructed for the 2+2 formalism [10–12] and for the 1+3 null decomposition [13, 14] and serves as the canonical configuration variable. The reason for the choice of $\mathfrak{su}(2)$-connection comes from the fact that the Lorentz group can be decomposed as the direct product of two SO(3) subgroups, namely, SO(1,3)=SO(3)⊗SO(3), and the corresponding Lie algebra $\mathfrak{so}(3)$ is isomorphic to $\mathfrak{su}(2)$.

Because the local symmetry SO(1,$d-1$) in a d-dimensional spacetime with $d\neq4$ does not have a similar decomposition, such a type of connection dynamics cannot be generalized to other dimensional spacetimes. In order to overcome this difficulty, the $\mathfrak{so}(d)$-connection instead of the $\mathfrak{so}(d-1)$-connection as the basic configuration variable [15]. With the $\mathfrak{so}(d)$-connection, unfortunately, the Lagrangian formalism on a spacetime with a Lorentzian signature fails to be constructed, although the Hamiltonian formalism can be established [15]. In fact, the local Lorentz group SO(1, $d-1$) in a d-dimensional spacetime can always be decomposed as SO(1,d–1)=SO(1,1)×SO(d–2)×T$^-$(d–2)×T$^+$(d–2), where the last two cross products × are Cartesian products of the subgroups [16]. Another problem of the decomposition SO(1,3)=SO(3)⊗SO(3) is that the SO(1,1) local symmetry does not manifest. The local SO(1,1) symmetry is very essential in the BF-theory approach to the statistical explanation of black hole entropy [17–20]. Therefore, it is worthwhile checking the possibility of choosing the $\mathfrak{so}(1,d-1)=\mathfrak{so}(1,1)\oplus\mathfrak{so}(d-2)\oplus\mathfrak{t}^-(d-2)\oplus\mathfrak{t}^+(d-2)$-connection as the canonical configuration variable.

The decomposition of $\mathfrak{so}(1,d-1)=\mathfrak{so}(1,1)\oplus\mathfrak{so}(d-2)\oplus\mathfrak{t}^-(d-2)\oplus\mathfrak{t}^+(d-2)$ can be easily realized in a coframe consisting of two null 1-forms (e^-,e^+) and d–2 spacelike 1-forms e^a, which is similar to the Newman-Penrose form [21]. The reason is that the coframe has four types of local transformations: boost, rotation, and two types of translations, which leave the metric invariant [22]. They belong to four subgroups of the Lorentz group SO(1,d–1), namely, SO(1,1), SO(d–2), T$^-$(d–2), and T$^+$(d–2). In particular, the SO(1,1) symmetry acts on (e^-,e^+) only and the SO($d-2$) symmetry acts on e^a only. In a Bondi-like coordinate system near an isolated horizon or a beam of a gravitational wave, the null coframe e^- is chosen to be proportional to dv, which makes the SO(1,1) symmetry more obvious.

In our previous study [23], we carried out a Hamiltonian analysis of three-dimensional gravity in Bondi-like coordinates, based on Dirac's treatments of a constrained system [24]. In the three-dimensional case, g_{01} was fixed to 1; all the three variables e_0^+, e_0^2, and e_2^2 of the coframe and the connection components ω_μ^{IJ} were treated as configuration variables; the Palatini action was used; and the cosmological constant was also included. A consistency analysis was carried out successfully, and torsion-free conditions and Gauss constraints were acquired. There were only second-class constraints. The BTZ spacetime was discussed as a test, which satisfied all the constraints.

The aim of the present study is to make a Hamiltonian analysis of four-dimensional gravity in Bondi-like coordinates, by using the same method as in Ref. [23]. For convenience, we implement some modifications in the treatment. Unlike in the treatment in the three-dimensional case [23], g_{01} is not fixed, and, therefore, the metric is more general and can be applied to more cases. The other differences are that n_0, l_0, and e_0^A are treated as Lagrange multipliers and that the cosmological constant is not included. The consistency conditions of the constraints will require the multipliers n_0, l_0, and e_0^A satisfying certain equations. These equations define the first derivative of n_0, l_0, and e_0^A with respect to different coordinates, and therefore, the multiplier should satisfy the integrability conditions. Such a situation is not met in Dirac's original literature [24]. In the new approach, the torsion-free conditions will appear as the consistency conditions of the primary constraints containing π^{μ}_{IJ}. In the coframe framework, the Gauss constraints are not independent ones, and they will emerge in the consistency conditions of $\pi^{0}_{IJ}=0$.

The rest of the paper is structured as follows. In Sec. 2, the symmetry decomposition, coframe, connection, action, and Poisson brackets are introduced. In Sec. 3, the consistency conditions for the constraints are analyzed, and the equations of motion are obtained. As a part of the consistency conditions, the integrability conditions of n_0, l_0, and e^A_0 are also presented. In Sec. 4, the classifications of constraints are considered, and the local physical degrees of freedom are discussed. The scalar, vector, and Gauss constraints in the new approach are also given in this section. In Sec. 5, the summary is made. In Appendices A and B, 2 identities are proved. In Appendices C, D, and E, the integrability conditions of n_0, l_0, and e^A_0 are shown to be equivalent to Ricci identities. The nonzero Poisson brackets among the constraints are listed in Appendix F.

2 Preliminary

2.1 Symmetry decomposition

The internal symmetry group of the four-dimensional spacetime is SO(1,3), and its Lie algebra is $\mathfrak{so}(1,3)$. The generators are denoted as $L_{IJ}, I,J=0,1,2,3$, satisfying

$$[L_{IJ},L_{KL}]=\eta_{IL}L_{JK}+\eta_{JK}L_{IL}-\eta_{IK}L_{JL}-\eta_{JL}L_{JK}, \quad (2.3)$$

where $\eta_{IJ}=\mathrm{diag}(-1,1,1,1)$ is the Minkowski metric of the local space.

The generators of $\mathfrak{so}(1,3)$ can also be redefined as [16]

$$L_{-+}:=L_{01}, \quad L_{\pm A}:=\frac{1}{\sqrt{2}}(L_{0A}\pm L_{1A}), \quad L_{AB}:=L_{AB}, \quad (2.4)$$

where $A,B=2,3$. They satisfy

$$\begin{aligned}
&[L_{-+},L_{-A}]=-L_{-A}, \quad [L_{-+},L_{+A}]=L_{+A},\\
&[L_{-+},L_{AB}]=0, \quad [L_{-A},L_{-B}]=0,\\
&[L_{-A},L_{+B}]=L_{AB}-\delta_{AB}L_{-+},\\
&[L_{-A},L_{BC}]=\delta_{AB}L_{-C}-\delta_{AC}L_{-B}, \quad [L_{+A},L_{+B}]=0,\\
&[L_{+A},L_{BC}]=\delta_{AB}L_{+C}-\delta_{AC}L_{+B},\\
&[L_{AB},L_{CD}]=\delta_{AD}L_{BC}+\delta_{BC}L_{AD}-\delta_{AC}L_{BD}-\delta_{BD}L_{AC}.
\end{aligned} \quad (2.5)$$

The above equations can also be written together as (2.3) with $I,J=-,+,2,3$ and

$$(\eta_{IJ})=\begin{pmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \quad (2.6)$$

2.2 Coframe

The spacetime line element can be written in terms of the coframe,

$$\mathrm{d}s^2=\eta_{IJ}e^I\otimes e^J. \quad (2.7)$$

Corresponding to our decomposition, the coframe is $\{e^-,e^+,e^A\}$, which contains two null 1-forms e^-,e^+ (or n,l) and two spacelike 1-forms $e^A, A=2,3$. For any coframe like this, the following four types of gauge transformations [22] leave the line element (2.7) invariant:

$$E^-=\frac{e^-}{\alpha}, \quad E^+=\alpha e^+, \quad E^A=e^A, \quad (2.8)$$

$$E^-=e^--c_Ae^A+\frac{1}{2}c_Ac^Ae^+, \quad E^+=e^+, \quad E^A=e^A-c^Ae^+, \quad (2.9)$$

$$E^-=e^-, \quad E^+=e^+-b_Ae^A+\frac{1}{2}b_Ab^Ae^-, \quad E^A=e^A-b^Ae^-, \quad (2.10)$$

$$E^-=e^-, \quad E^+=e^+, \quad E^A=e^A\cos\beta-\epsilon_{AB}e^B\sin\beta, \quad (2.11)$$

which correspond to SO(1,1), $\mathrm{T}^-(2)$, $\mathrm{T}^+(2)$, and SO(2) transformations, respectively. Here, α, b^A, c^A, and β are gauge parameters, which are arbitrary functions of the coordinates.

2.3 Connection

Both e^I and E^I should satisfy torsion-free conditions

$$\mathrm{d}e^I+\omega^{IJ}\wedge e^K\eta_{JK}=0, \quad \mathrm{d}E^I+\Omega^{IJ}\wedge E^K\eta_{JK}=0. \quad (2.12)$$

If e^I and E^I are related by gauge transformations (2.8), (2.10), (2.9), and (2.11), one can obtain the relations between ω^{IJ} and Ω^{IJ}:

$$\begin{aligned}
&\Omega^{-+}=\omega^{-+}-\mathrm{d}\ln\alpha, \quad \Omega^{-A}=\frac{1}{\alpha}\omega^{-A},\\
&\Omega^{+A}=\alpha\omega^{+A}, \quad \Omega^{AB}=\omega^{AB};
\end{aligned} \quad (2.13)$$

$$\begin{aligned}
&\Omega^{-+}=\omega^{-+}-\omega^{-A}b_A, \quad \Omega^{-A}=\omega^{-A},\\
&\Omega^{AB}=\omega^{AB}+\omega^{-A}b^B-\omega^{-B}b^A,\\
&\Omega^{+A}=\omega^{+A}+\omega^{-+}b^A-\omega^{-B}b_Bb^A\\
&\qquad +\omega^{AB}b_B+\mathrm{d}b^A+\frac{1}{2}\omega^{-A}b_Bb^B;
\end{aligned} \quad (2.14)$$

$$\Omega^{-+}=\omega^{-+}+\omega^{+A}c_A,\quad \Omega^{+A}=\omega^{+A},$$
$$\Omega^{AB}=\omega^{AB}+\omega^{+A}b^B-\omega^{+B}b^A,$$
$$\Omega^{-A}=\omega^{-A}-\omega^{-+}c^A-\omega^{+B}c_Bc^A+\omega^{AB}c_B$$
$$+\mathrm{d}c^A+\frac{1}{2}\omega^{+A}c_Bc^B; \tag{2.15}$$
$$\Omega^{-+}=\omega^{-+},\quad \Omega^{\pm A}=\omega^{\pm A}\cos\beta-\epsilon_{AB}\omega^{\pm B}\sin\beta,$$
$$\Omega^{AB}=\omega^{AB}+\mathrm{d}\beta. \tag{2.16}$$

2.4 Action

In the following analysis, a special coframe is chosen

$$n=n_0\mathrm{d}v,\quad l=l_0\mathrm{d}v+\mathrm{d}r,\quad e^A=e_0^A\mathrm{d}v+e_a^A\mathrm{d}x^a \tag{2.17}$$

or written as

$$e^-=e_0^-\mathrm{d}x^0,\quad e^+=e_0^+\mathrm{d}x^1+\mathrm{d}r,\quad e^A=e_0^A\mathrm{d}v+e_a^A\mathrm{d}x^a. \tag{2.18}$$

The four-dimensional Palatini action of gravity is

$$S=\int_M F^{IJ}\wedge\Sigma_{IJ}=\int_M\frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^{IJ}e_\rho^Ke_\sigma^L\mathrm{d}v\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$$
$$=\int_M(\epsilon_{IJKL}\epsilon^{0jkl}F_{0j}^{IJ}e_k^Ke_l^L+\epsilon_{IJKL}\epsilon^{0ijl}F_{ij}^{IJ}e_0^Ke_l^L)\mathrm{d}^4x, \tag{2.19}$$

where

$$F^{IJ}=\mathrm{d}\omega^{IJ}+\eta_{KL}\omega^{IK}\wedge\omega^{LJ}. \tag{2.20}$$

Therefore, the Lagrangian is

$$L=\int\epsilon_{IJKL}\epsilon^{0ijk}(F_{0i}^{IJ}e_j^Ke_k^L+F_{ij}^{IJ}e_0^Ke_k^L)\mathrm{d}^3x. \tag{2.21}$$

In the following analysis, e_a^A and ω_μ^{IJ} will be treated as configuration variables, and their conjugate momenta are denoted as π_A^a and π_{IJ}^μ, respectively. e_0^I will be treated as Lagrange multipliers, and therefore, there are four corresponding primary constraints:

$$\epsilon_{IJKL}\epsilon^{0jkl}F_{jk}^{JK}e_l^L=\epsilon_{IJKL}\epsilon^{jkl}F_{jk}^{JK}e_l^L\approx0, \tag{2.22}$$

where ϵ^{0jkl} is written as ϵ^{jkl} for brevity. Under coframe (2.17), the above four constraints can be written as

$$\epsilon_{AB}\epsilon^{ab}F_{1a}^{+A}e_b^B+F_{23}^{23}\approx0, \tag{2.23}$$
$$\epsilon_{AB}\epsilon^{ab}F_{1a}^{-A}e_b^B\approx0, \tag{2.24}$$
$$F_{23}^{-A}+\epsilon^{ab}e_a^AF_{1b}^{-+}\approx0, \tag{2.25}$$

corresponding to n_0, l_0, and e_0^A, respectively.

2.5 Poisson bracket

The Poisson bracket of two quantities $f(v,x)$ and $g(v,y)$ at the same time v is defined as

$$\{f(v,x),g(v,y)\}=\int\left[\frac{\delta f(v,x)}{\delta e_a^A(v,z)}\frac{\delta g(v,y)}{\delta\pi_A^a(v,z)}+\frac{1}{2}\frac{\delta f(v,x)}{\delta\omega_\mu^{IJ}(v,z)}\frac{\delta g(v,y)}{\delta\pi_{IJ}^\mu(v,z)}-\frac{\delta f(v,x)}{\delta\pi_A^a(v,z)}\frac{\delta g(v,y)}{\delta e_a^A(v,z)}-\frac{1}{2}\frac{\delta f(v,x)}{\delta\pi_{IJ}^\mu(v,z)}\frac{\delta g(v,y)}{\delta\omega_\mu^{IJ}(v,z)}\right]\mathrm{d}^3z, \tag{2.26}$$

where x, y, and z stand for three-dimensional null hypersurface coordinates. The Poisson brackets of canonical pairs are

$$\{e_a^A(v,x),\pi_B^b(v,y)\}=\delta_B^A\delta_a^b\delta^3(x-y), \tag{2.27}$$
$$\{\omega_\mu^{IJ}(v,x),\pi_{KL}^\nu(v,y)\}=(\delta_K^I\delta_L^J-\delta_L^I\delta_K^J)\delta_\mu^\nu\delta^3(x-y). \tag{2.28}$$

3 Hamiltonian analysis

3.1 Total Hamiltonian

By definition, the canonical momentum P conjugate to a configuration variable Q is

$$P:=\frac{\delta L}{\delta\dot{Q}}, \tag{3.1}$$

and when the Lagrangian contains, at most, the linear term of $\dot{Q}$, the definition of the conjugate momentum P gives a primary constraint. Because the Palatini Lagrangian (2.21) is of the first order, one can obtain 28 primary constraints

$$\pi_A^a=0,\quad \pi_{-+}^0=0,\quad \pi_{-+}^1-2\epsilon_{AB}\epsilon^{ab}e_a^Ae_b^B=0,\quad \pi_{-+}^a=0, \tag{3.2}$$

$$\pi_{-A}^0=0,\quad \pi_{-A}^1=0,\quad \pi_{-A}^a-4\epsilon_{AB}\epsilon^{ab}e_b^B=0,$$
$$\pi_{+A}^0=\pi_{+A}^1=\pi_{+A}^a=\pi_{23}^0=\pi_{23}^1=\pi_{23}^a=0. \tag{3.3}$$

Together with (2.23), (2.24), and (2.25), there are 32 primary constraints in all.

By the Legendre transformation, the canonical Hamiltonian is

$$H_c=\int_V(\pi_A^a\dot{e}_a^A+\frac{1}{2}\pi_{IJ}^\mu\dot{\omega}_\mu^{IJ})\mathrm{d}^3x-\int_V\mathcal{L}\mathrm{d}^3x$$
$$=\int_V\epsilon_{AB}\epsilon^{ab}[4(\omega_{0,a}^{-A}+\omega_0^{-+}\omega_a^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-D}\omega_a^{CA}\delta_{DC}+\omega_a^{-D}\omega_0^{CA}\delta_{DC})e_b^B$$
$$+2(\omega_{0,1}^{-+}-\omega_1^{-C}\omega_0^{+D}\delta_{CD}+\omega_0^{-C}\omega_1^{+D}\delta_{CD})e_a^Ae_b^B-4F_{1a}^{-+}e_0^Ae_b^B+4F_{1a}^{-A}l_0e_b^B$$
$$-2F_{ab}^{-A}e_0^B-4n_0F_{1a}^{+A}e_b^B-n_0F_{ab}^{AB}]\mathrm{d}^3x, \tag{3.4}$$

and therefore, the total Hamiltonian with primary constraints is

$$\begin{aligned}H_T=\int_V&[4\epsilon_{AB}\epsilon^{ab}(\omega_{0,a}^{-A}+\omega_0^{-+}\omega_a^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-D}\omega_a^{CA}\delta_{DC}+\omega_a^{-D}\omega_0^{CA}\delta_{DC})e_b^B\\&+2\epsilon_{AB}\epsilon^{ab}(\omega_{0,1}^{-+}-\omega_1^{-C}\omega_0^{+D}\delta_{CD}+\omega_0^{-C}\omega_1^{+D}\delta_{CD})e_a^Ae_b^B-n_0\epsilon_{AB}\epsilon^{ab}(4F_{1a}^{+A}e_b^B+F_{ab}^{AB})\\&+4\epsilon_{AB}\epsilon^{ab}F_{1a}^{-A}e_b^Bl_0+e_0^A\epsilon_{AB}\epsilon^{ab}(2F_{ab}^{-B}-4F_{1a}^{-+}e_b^B)+\lambda_a^A\pi_A^a+\lambda_0^{-+}\pi_{-+}^0\\&+\lambda_1^{-+}(\pi_{-+}^1-2\epsilon_{AB}\epsilon^{ab}e_a^Ae_b^B)+\lambda_a^{-+}\pi_{-+}^a+\lambda_0^{-A}\pi_{-A}^0+\lambda_1^{-A}\pi_{-A}^1+\lambda_a^{-A}(\pi_{-A}^a-4\epsilon_{AB}\epsilon^{ab}e_b^B)\\&+\lambda_0^{+A}\pi_{+A}^0+\lambda_1^{+A}\pi_{+A}^1+\lambda_a^{+A}\pi_{+A}^a+\lambda_0^{23}\pi_{23}^0+\lambda_1^{23}\pi_{23}^1+\lambda_a^{23}\pi_{23}^a]\mathrm{d}^3x.\end{aligned}\tag{3.5}$$

3.2 Consistency analysis of the primary constraints

The primary constraints should always hold in the whole evolution. It means that their Poisson brackets with the total Hamiltonian should be zero on the constraint surface in the phase space. The following is the analysis of the consistency conditions for the primary constraints in detail. First, the consistency conditions for $\pi_A^a=0$ are

$$\begin{aligned}\{H_T,\pi_A^a\}=&4\epsilon_{AB}\epsilon^{ab}(\omega_{0,b}^{-B}+\omega_0^{-+}\omega_b^{-B}-\omega_b^{-+}\omega_0^{-B}-\omega_0^{-D}\omega_b^{CB}\delta_{DC}+\omega_b^{-D}\omega_0^{CB}\delta_{DC})\\&+4\epsilon_{AB}\epsilon^{ab}e_b^B(\omega_{0,1}^{-+}+\omega_1^{-C}\omega_0^{D+}\delta_{CD}-\omega_0^{-C}\omega_1^{D+}\delta_{CD})-4\epsilon_{AB}\epsilon^{ab}e_b^B\lambda_1^{-+}\\&+4\epsilon_{AB}\epsilon^{ab}(l_0F_{1b}^{-B}-e_0^BF_{1b}^{-+}-n_0F_{1b}^{+B})-4\epsilon_{AB}\epsilon^{ab}\lambda_b^{-B}\approx0.\end{aligned}\tag{3.6}$$

They will always be valid if

$$\lambda_1^{-+}\approx\omega_{0,1}^{-+}-\omega_1^{-A}\omega_0^{+B}\delta_{AB}+\omega_0^{-A}\omega_1^{+B}\delta_{AB}+X_1^{-+},\tag{3.7}$$

$$\lambda_a^{-A}\approx\omega_{0,a}^{-A}+\omega_0^{-+}\omega_a^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-B}\omega_a^{CA}\delta_{BC}+\omega_a^{-B}\omega_0^{CA}\delta_{BC}+e_0^+F_{1a}^{-A}-e_0^-F_{1a}^{+A}-e_0^AF_{1a}^{-+}-e_a^AX_1^{-+},\tag{3.8}$$

where X_1^{-+} is a function of the canonical variables to be determined.

Next, the consistency conditions of the constraints with π_{IJ}^μ are

$$\{H_T,\pi_{-+}^0\}=-4\epsilon_{AB}\epsilon^{ab}e_b^B(e_{a,1}^A+\omega_1^{AC}e_a^D\delta_{CD}-\omega_a^{-A})\approx0,\tag{3.9}$$

$$\begin{aligned}\{H_T,\pi_{-+}^1-2\epsilon_{AB}\epsilon^{ab}e_a^Ae_b^B\}=&-4\epsilon_{AB}\epsilon^{ab}e_a^A(e_{0,b}^B+\omega_b^{+B}n_0+\omega_b^{-B}l_0+\omega_b^{BC}e_0^D\delta_{CD}-\omega_0^{BC}e_b^D\delta_{CD})\\&+4\epsilon_{AB}\epsilon^{ab}e_0^B(e_{b,a}^A+\omega_a^{AC}e_b^D\delta_{CD})+4\epsilon_{AB}\epsilon^{ab}\lambda_a^Ae_b^B\approx0,\end{aligned}\tag{3.10}$$

$$\begin{aligned}\{H_T,\pi_{-+}^a\}=&4\epsilon_{AB}\epsilon^{ab}[e_b^B(e_{0,1}^A-\omega_0^{-A}+\omega_1^{+A}n_0+\omega_1^{-A}l_0+\omega_1^{AC}e_0^D\delta_{CD})\\&-e_0^B(e_{b,1}^A-\omega_b^{-A}+\omega_1^{AC}e_b^D\delta_{CD})]\approx0,\end{aligned}\tag{3.11}$$

$$\{H_T,\pi_{-A}^0\}=4\epsilon_{AB}\epsilon^{ab}e_b^B(\omega_1^{+C}e_a^D\delta_{CD}-\omega_a^{-+})+4\epsilon_{AB}\epsilon^{ab}(e_{a,b}^B+\omega_b^{BC}e_a^D\delta_{CD})\approx0,\tag{3.12}$$

$$\begin{aligned}\{H_T,\pi_{-A}^1\}=&-4\epsilon_{AB}\epsilon^{ab}e_0^B(\omega_a^{+C}e_b^D\delta_{CD})-4\epsilon_{AB}\epsilon^{ab}l_0(e_{a,b}^B+\omega_b^{BC}e_a^D\delta_{CD})\\&+4\epsilon_{AB}\epsilon^{ab}e_b^B(l_{0,a}+\omega_a^{-+}l_0-\omega_0^{+C}e_a^D\delta_{CD}+\omega_a^{+C}e_0^D\delta_{CD})\approx0,\end{aligned}\tag{3.13}$$

$$\begin{aligned}\{H_T,\pi_{-A}^a-4\epsilon_{AB}\epsilon^{ab}e_b^B\}=&4\epsilon_{AB}\epsilon^{ab}e_b^B\omega_0^{-+}+4\epsilon_{AB}\epsilon^{ab}\omega_0^{BC}e_b^D\delta_{CD}+4\epsilon_{AB}\epsilon^{ab}e_0^B\omega_1^{+C}e_b^D\delta_{CD}\\&-4\epsilon_{AB}\epsilon^{ab}e_b^B\omega_1^{+C}e_0^D\delta_{CD}-4\epsilon_{AB}\epsilon^{ab}(l_0e_b^B)_{,1}+4\epsilon_{CA}\epsilon^{ab}l_0e_b^C\omega_1^{-+}\\&-4\epsilon_{AB}\epsilon^{ab}l_0\omega_1^{BC}e_b^D\delta_{CD}-4\epsilon_{AB}\epsilon^{ab}(e_0^B)_{,b}-4\epsilon_{AB}\epsilon^{ab}\omega_b^{BC}e_0^D\delta_{CD}\\&-4\epsilon_{AB}\epsilon^{ab}e_0^B\omega_b^{-+}-4\epsilon_{AB}\epsilon^{ab}\omega_b^{+B}n_0+4\epsilon_{AB}\epsilon^{ab}\lambda_b^B\approx0,\end{aligned}\tag{3.14}$$

$$\{H_T,\pi_{+A}^0\}=-2\epsilon_{BC}\epsilon^{ab}e_a^Be_b^C\omega_1^{-A}=-4e\omega_1^{-A}\approx0,\tag{3.15}$$

$$\begin{aligned}\{H_T,\pi_{+A}^1\}=&4\epsilon_{AB}\epsilon^{ab}n_0(e_{a,b}^B+\omega_b^{BC}e_a^D\delta_{CD})+4\epsilon_{AB}\epsilon^{ab}e_0^B\omega_a^{-C}e_b^D\delta_{CD}\\&-4\epsilon_{AB}\epsilon^{ab}e_a^B(-n_{0,b}+\omega_0^{-C}e_b^D\delta_{CD}+\omega_b^{-+}n_0-\omega_b^{-C}e_0^D\delta_{CD})\approx0,\end{aligned}\tag{3.16}$$

$$\begin{aligned}\{H_T,\pi_{+A}^a\}=&-4\epsilon_{AB}\epsilon^{ab}(-n_{0,1}e_b^B-n_0e_{b,1}^B-\omega_1^{BC}n_0e_b^D\delta_{CD}+\omega_b^{-B}n_0)\\&-4\epsilon_{AB}\epsilon^{ab}e_b^B\omega_1^{-+}n_0+4\epsilon_{BC}\epsilon^{ab}e_0^Be_b^C\omega_1^{-D}\delta_{DA}\\=&-4\epsilon_{AB}\epsilon^{ab}e_b^B\omega_1^{-+}n_0+4\epsilon_{AB}\epsilon^{ab}n_{0,1}e_b^B+4\epsilon_{BC}\epsilon^{ab}e_0^Be_b^C\omega_1^{-D}\delta_{DA}\end{aligned}$$

$$+4\epsilon_{AB}\epsilon^{ab}n_0(e^B_{b,1}+\omega_1^{BC}e^D_b\delta_{CD}-\omega_b^{-B})\approx 0, \tag{3.17}$$

$$\{H_T,\pi^0_{23}\}=-4\epsilon^{ab}\omega_a^{-C}e^D_b\delta_{CD}\approx 0, \tag{3.18}$$

$$\{H_T,\pi^1_{23}\}=4\epsilon^{ab}l_0\omega_a^{-C}e^D_b\delta_{CD}-4\epsilon^{ab}n_0\omega_a^{+C}e^D_b\delta_{CD}\approx 0, \tag{3.19}$$

$$\{H_T,\pi^a_{23}\}=4\epsilon^{ab}(n_{0,b}+\omega_0^{-C}e^D_b\delta_{CD}-\omega_b^{-C}e^D_0\delta_{CD}+\omega_b^{-+}n_0)-4\epsilon^{ab}n_0(\omega_b^{-+}-\omega_1^{+C}e^D_b\delta_{CD})\approx 0. \tag{3.20}$$

The above 24 conditions are equal to 24 torsion-free conditions

$$n_{0,1}-\omega_1^{-+}n_0\approx 0, \tag{3.21}$$

$$n_{0,a}-\omega_a^{-+}n_0-\omega_0^{-A}e^B_a\delta_{AB}+\omega_a^{-A}e^B_0\delta_{AB}\approx 0, \tag{3.22}$$

$$l_{0,1}-\omega_0^{-+}+\omega_1^{+A}e^B_0\delta_{AB}+\omega_1^{-+}l_0\approx 0, \tag{3.23}$$

$$l_{0,a}+\omega_a^{+A}e^B_0\delta_{AB}-\omega_0^{+A}e^B_a\delta_{AB}+\omega_a^{-+}l_0\approx 0, \tag{3.24}$$

$$e^A_{0,1}-\omega_0^{-A}+\omega_1^{+A}n_0+\omega_1^{AB}e^C_0\delta_{BC}\approx 0, \tag{3.25}$$

$$e^A_{0,a}-\lambda^A_a+\omega_a^{-A}l_0+\omega_a^{+A}n_0+\omega_a^{AB}e^C_0\delta_{BC}-\omega_0^{AB}e^C_a\delta_{BC}\approx 0, \tag{3.26}$$

and

$$\omega_1^{-A}\approx 0, \tag{3.27}$$

$$\epsilon^{ab}\omega_a^{-A}e^B_b\delta_{AB}\approx 0, \tag{3.28}$$

$$\omega_a^{-+}-\omega_1^{+A}e^B_a\delta_{AB}\approx 0, \tag{3.29}$$

$$\epsilon^{ab}\omega_a^{+A}e^B_b\delta_{AB}\approx 0, \tag{3.30}$$

$$e^A_{a,1}-\omega_a^{-A}+\omega_1^{AB}e^C_a\delta_{BC}\approx 0, \tag{3.31}$$

$$\epsilon^{ab}(e^A_{a,b}-\omega_a^{AB}e^C_b\delta_{BC})\approx 0. \tag{3.32}$$

Eq. (3.26) consists of four torsion-free conditions by using the equations of motion of e^A_a

$$\dot{e}^A_a=\{e^A_a,H_T\}=\lambda^A_a\approx e^A_{0,a}+\omega_a^{-A}l_0+\omega_a^{+A}n_0+\omega_a^{AB}e^C_0\delta_{BC}-\omega_0^{AB}e^C_a\delta_{BC}, \tag{3.33}$$

which result in

$$e^A_{0,a}-e^A_{a,0}+\omega_a^{-A}l_0+\omega_a^{+A}n_0+\omega_a^{AB}e^C_0\delta_{BC}-\omega_0^{AB}e^C_a\delta_{BC}\approx 0. \tag{3.34}$$

The last 12 torsion-free conditions (3.27)-(3.32) contain no multipliers, and therefore, there are 12 secondary constraints.

Finally, the consistency conditions for (2.23), (2.24), and (2.25) are as follows:

$$\begin{aligned}\{\epsilon_{AB}\epsilon^{ab}F^{-A}_{1a}e^B_b,H_T\}&=\epsilon_{AB}\epsilon^{ab}\{F^{-A}_{1a},H_T\}e^B_b+\epsilon_{AB}\epsilon^{ab}F^{-A}_{1a}\{e^B_b,H_T\}\\&=\epsilon_{AB}\epsilon^{ab}(\lambda^{-A}_{a,1}-\lambda_1^{-+}\omega_a^{-A}-\omega_1^{-+}\lambda_a^{-A}-\lambda_a^{-C}\omega_1^{DA}\delta_{CD})e^B_b+\epsilon_{AB}\epsilon^{ab}F^{-A}_{1a}\lambda^B_b\approx 0,\end{aligned} \tag{3.35}$$

which will be a trivial identity after the determination of λ_1^{-+} and λ_a^{-A} (see Appendix A).

$$\{\epsilon_{AB}\epsilon^{ab}F^{+A}_{1a}e^B_b+F^{23}_{23},H_T\}=\epsilon_{AB}\epsilon^{ab}\{F^{+A}_{1a},H_T\}e^B_b+\epsilon_{AB}\epsilon^{ab}F^{+A}_{1a}\lambda^B_b+\{F^{23}_{23},H_T\}\approx 0, \tag{3.36}$$

where

$$\begin{aligned}\{F^{+A}_{1a},H_T\}=&\lambda^{+A}_{a,1}-\lambda^{+A}_{1,a}+\lambda_1^{-+}\omega_a^{+A}+\omega_1^{-+}\lambda_a^{+A}+\lambda_1^{+B}\omega_a^{CA}\delta_{BC}+\omega_1^{+B}\lambda_a^{CA}\delta_{BC}\\&-\lambda_a^{-+}\omega_1^{+A}-\omega_a^{-+}\lambda_1^{+A}-\lambda_a^{+B}\omega_1^{CA}\delta_{BC}-\omega_a^{+B}\lambda_1^{CA}\delta_{BC},\end{aligned} \tag{3.37}$$

$$\{F^{23}_{23},H_T\}=\lambda^{23}_{3,2}-\lambda^{23}_{2,3}+\epsilon^{ab}\lambda_a^{+2}\omega_b^{-3}+\epsilon^{ab}\omega_a^{+2}\lambda_b^{-3}+\epsilon^{ab}\lambda_a^{-2}\omega_b^{+3}+\epsilon^{ab}\omega_a^{-2}\lambda_b^{+3}. \tag{3.38}$$

$$\{F^{-A}_{23}+\epsilon^{ab}e^A_aF^{-+}_{1b},H_T\}=\{F^{-A}_{23},H_T\}+\epsilon^{ab}\lambda^A_aF^{-+}_{1b}+\epsilon^{ab}e^A_a\{F^{-+}_{1b},H_T\}\approx 0, \tag{3.39}$$

where

$$\{F^{-A}_{23},H_T\}=-\epsilon^{ab}\lambda^{-A}_{a,b}-\epsilon^{ab}\lambda_a^{-+}\omega_b^{-A}-\epsilon^{ab}\omega_a^{-+}\lambda_b^{-A}+\epsilon^{ab}\lambda_a^{-B}\omega_b^{CA}\delta_{BC}+\epsilon^{ab}\omega_a^{-B}\lambda_b^{CA}\delta_{BC}, \tag{3.40}$$

$$\{F^{-+}_{1b},H_T\}=\lambda^{-+}_{b,1}-\lambda^{-+}_{1,b}+\lambda_b^{-B}\omega_1^{+C}\delta_{BC}+\omega_b^{-B}\lambda_1^{+C}\delta_{BC}. \tag{3.41}$$

Eqs. (3.36) and (3.39) set three relations among the multipliers.

3.3 Consistency analysis of the secondary constraints

The secondary constraints should also be preserved in the evolution, which requires

$$\{\omega_1^{-A},H_T\}=\lambda_1^{-A}\approx 0, \tag{3.42}$$

$$\{\epsilon^{ab}\omega_a^{-A}e_b^B\delta_{AB},H_T\}=\epsilon^{ab}\lambda_a^{-A}e_b^B\delta_{AB}+\epsilon^{ab}\omega_a^{-A}\lambda_b^B\delta_{AB}\approx 0, \tag{3.43}$$

$$\{\omega_a^{-+}-\omega_1^{+A}e_a^B\delta_{AB},H_T\}=\lambda_a^{-+}-\lambda_1^{+A}e_a^B\delta_{AB}-\omega_1^{+A}\lambda_a^B\delta_{AB}\approx 0, \tag{3.44}$$

$$\{\epsilon^{ab}\omega_a^{+A}e_b^B\delta_{AB},H_T\}=\epsilon^{ab}\lambda_a^{+A}e_b^B\delta_{AB}+\epsilon^{ab}\omega_a^{+A}\lambda_b^B\delta_{AB}\approx 0, \tag{3.45}$$

$$\begin{aligned}&\{e_{a,1}^A+\omega_1^{AB}e_a^C\delta_{BC}-\omega_a^{-A},H_T\}=\lambda_{a,1}^A+\lambda_1^{AB}e_a^C\delta_{BC}+\omega_1^{AB}\lambda_a^C\delta_{BC}-\lambda_a^{-A}\\&\approx(\lambda_1^{AB}-\omega_{0,1}^{AB}+\omega_0^{-A}\omega_1^{+B}-\omega_1^{+A}\omega_0^{-B})e_a^C\delta_{CB}+2n_0F_{1a}^{+A}+F_{1a}^{AB}e_0^C\delta_{BC}+e_0^AF_{1a}^{-+}+e_a^AX_1^{-+}\approx 0,\end{aligned} \tag{3.46}$$

$$\{\epsilon^{ab}(e_{a,b}^A-\omega_a^{AB}e_b^C\delta_{BC}),H_T\}=\epsilon^{ab}\lambda_{a,b}^A-\epsilon^{ab}\lambda_a^{AB}e_b^C\delta_{BC}-\epsilon^{ab}\omega_a^{AB}\lambda_b^C\delta_{BC}\approx 0. \tag{3.47}$$

Combining (3.8), (3.26), and (3.43), one can obtain

$$\begin{aligned}&\epsilon^{ab}(\omega_{0,a}^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-B}\omega_a^{CA}\delta_{BC}+\omega_a^{-B}\omega_0^{CA}\delta_{BC})e_b^D\delta_{AD}+\epsilon^{ab}(l_0F_{1a}^{-A}-e_0^AF_{1a}^{-+}-n_0F_{1a}^{+A})e_b^B\delta_{AB}\\&-\epsilon^{ab}(e_{0,a}^A+\omega_a^{-A}l_0+\omega_a^{+A}n_0+\omega_a^{AB}e_0^C\delta_{BC}-\omega_0^{AB}e_a^C\delta_{BC})\omega_b^{-D}\delta_{AD}\approx 0,\end{aligned} \tag{3.48}$$

which will be automatically satisfied after the determination of X_1^{-+} (see Appendix B).

Eq. (3.46) leads to four expressions of λ_1^{23}:

$$\lambda_{1(1)}^{23}\approx\omega_{0,1}^{23}-\omega_0^{-2}\omega_1^{+3}+\omega_0^{-3}\omega_1^{+2}-2n_0F_{12}^{+2}(e_2^3)^{-1}-F_{12}^{23}e_0^3(e_2^3)^{-1}-e_0^2F_{12}^{-+}(e_2^3)^{-1}-e_2^2(e_2^3)^{-1}X_1^{-+}, \tag{3.49}$$

$$\lambda_{1(2)}^{23}\approx\omega_{0,1}^{23}-\omega_0^{-2}\omega_1^{+3}+\omega_1^{+2}\omega_0^{-3}-2n_0F_{13}^{+2}(e_3^3)^{-1}-F_{13}^{23}e_0^3(e_3^3)^{-1}-e_0^2F_{13}^{-+}(e_3^3)^{-1}-e_3^2(e_3^3)^{-1}X_1^{-+}, \tag{3.50}$$

$$\lambda_{1(3)}^{23}\approx\omega_{0,1}^{23}-\omega_0^{-2}\omega_1^{+3}+\omega_1^{+2}\omega_0^{-3}+2n_0F_{12}^{+3}(e_2^2)^{-1}-F_{12}^{23}e_0^2(e_2^2)^{-1}+e_0^3F_{12}^{-+}(e_2^2)^{-1}+e_2^3(e_2^2)^{-1}X_1^{-+}, \tag{3.51}$$

$$\lambda_{1(4)}^{23}\approx\omega_{0,1}^{23}-\omega_0^{-2}\omega_1^{+3}+\omega_1^{+2}\omega_0^{-3}+2n_0F_{13}^{+3}(e_3^2)^{-1}-F_{13}^{23}e_0^2(e_3^2)^{-1}+e_0^3F_{13}^{-+}(e_3^2)^{-1}+e_3^3(e_3^2)^{-1}X_1^{-+}. \tag{3.52}$$

They should be equal to each other. From them, one can obtain two new secondary constraints:

$$\epsilon^{ab}F_{1a}^{+2}e_b^3+\epsilon^{ab}F_{1a}^{+3}e_b^2\approx 0, \tag{3.53}$$

$$\epsilon^{ab}F_{1a}^{+2}e_b^2-\epsilon^{ab}F_{1a}^{+3}e_b^3\approx 0, \tag{3.54}$$

and determine X_1^{-+} as

$$X_1^{-+}\approx n_0F_{23}^{23}e^{-1}-\epsilon_{AB}\epsilon^{ab}e_0^AF_{1a}^{-+}e_b^Be^{-1}. \tag{3.55}$$

Therefore,

$$\lambda_1^{-+}\approx\omega_{0,1}^{-+}+\omega_0^{-A}\omega_1^{+B}\delta_{AB}+n_0F_{23}^{23}e^{-1}-\epsilon_{AB}\epsilon^{ab}e_0^AF_{1a}^{-+}e_b^Be^{-1}=:\Lambda_1^{-+}, \tag{3.56}$$

$$\begin{aligned}\lambda_a^{-A}\approx{}&\omega_{0,a}^{-A}+\omega_0^{-+}\omega_a^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-B}\omega_a^{CA}\delta_{BC}+\omega_a^{-B}\omega_0^{CA}\delta_{BC}+e_0^+F_{1a}^{-A}\\&-e_0^-F_{1a}^{+A}-e_0^AF_{1a}^{-+}-n_0e_a^AF_{23}^{23}e^{-1}+e_a^A\epsilon_{BC}\epsilon^{bc}e_0^BF_{1b}^{-+}e_c^Ce^{-1}=:\Lambda_a^{-A}.\end{aligned} \tag{3.57}$$

From (3.47), the multipliers λ_a^{23} can be determined:

$$\lambda_a^{23}\approx\omega_{0,a}^{23}-\omega_0^{-2}\omega_a^{+3}+\omega_0^{-3}\omega_a^{+2}-\omega_0^{+2}\omega_a^{-3}+\omega_0^{+3}\omega_a^{-2}-n_0e_a^AF_{23}^{+B}\delta_{AB}e^{-1}-l_0e_a^AF_{23}^{-B}\delta_{AB}e^{-1}-\epsilon_{AB}e_a^Ae_0^BF_{23}^{23}e^{-1}=:\Lambda_a^{23}. \tag{3.58}$$

3.4 Consistency analysis of the further secondary constraints

The consistency conditions of the further secondary constraints (3.53) and (3.54) are

$$\{\epsilon^{ab}F_{1a}^{+2}e_b^3+\epsilon^{ab}F_{1a}^{+3}e_b^2,H_T\}=\epsilon^{ab}\{F_{1a}^{+2},H_T\}e_b^3+\epsilon^{ab}F_{1a}^{+2}\lambda_b^3+\epsilon^{ab}\{F_{1a}^{+3},H_T\}e_b^2+\epsilon^{ab}F_{1a}^{+3}\lambda_b^2\approx 0, \tag{3.59}$$

$$\{\epsilon^{ab}F_{1a}^{+2}e_b^2-\epsilon^{ab}F_{1a}^{+3}e_b^3,H_T\}=\epsilon^{ab}\{F_{1a}^{+2},H_T\}e_b^2+\epsilon^{ab}F_{1a}^{+2}\lambda_b^2-\epsilon^{ab}\{F_{1a}^{+3},H_T\}e_b^3-\epsilon^{ab}F_{1a}^{+3}\lambda_b^3\approx 0, \tag{3.60}$$

where

$$\begin{aligned}\{F_{1a}^{+2},H_T\}={}&\lambda_{a,1}^{+2}-\lambda_{1,a}^{+2}+\lambda_1^{-+}\omega_a^{+2}+\omega_1^{-+}\lambda_a^{+2}+\lambda_1^{+3}\omega_a^{32}+\omega_1^{+3}\lambda_a^{32}-\lambda_a^{-+}\omega_1^{+2}-\omega_a^{-+}\lambda_1^{+2}-\lambda_a^{+3}\omega_1^{32}-\omega_a^{+3}\lambda_1^{32}\\\approx{}&\lambda_{a,1}^{+2}-\lambda_{1,a}^{+2}+\Lambda_1^{-+}\omega_a^{+2}+\omega_1^{-+}\lambda_a^{+2}+\lambda_1^{+3}\omega_a^{32}+\omega_1^{+3}\Lambda_a^{32}-\lambda_a^{-+}\omega_1^{+2}-\omega_a^{-+}\lambda_1^{+2}-\lambda_a^{+3}\omega_1^{32}-\omega_a^{+3}\Lambda_1^{32},\end{aligned} \tag{3.61}$$

$$\{F^{+3}_{1a},H_T\}=\lambda^{+3}_{a,1}-\lambda^{+3}_{1,a}+\lambda^{-+}_1\omega^{+3}_a+\omega^{-+}_1\lambda^{+3}_a+\lambda^{+2}_1\omega^{23}_a+\omega^{+2}_1\lambda^{23}_a-\lambda^{-+}_a\omega^{+3}_1-\omega^{-+}_a\lambda^{+3}_1-\lambda^{+2}_a\omega^{23}_1-\omega^{+2}_a\lambda^{23}_1$$
$$\approx\lambda^{+3}_{a,1}-\lambda^{+3}_{1,a}+\Lambda^{-+}_1\omega^{+3}_a+\omega^{-+}_1\lambda^{+3}_a+\lambda^{+2}_1\omega^{23}_a+\omega^{+2}_1\Lambda^{23}_a-\lambda^{-+}_a\omega^{+3}_1-\omega^{-+}_a\lambda^{+3}_1-\lambda^{+2}_a\omega^{23}_1-\omega^{+2}_a\Lambda^{23}_1. \quad (3.62)$$

They are relations among the multipliers.

3.5 Integrability

Eqs. (3.21)-(3.26) define the first derivatives of n_0, l_0, and e^A_0 with respect to their spatial coordinates x^1 and x^a. As a self-consistent system, these multipliers (n_0, l_0, and e^A_0) should satisfy the integrability conditions. Therefore, we should check whether the integrability conditions will result in new constraints. The direct calculations show that all the integrability conditions result in the Ricci identities. The detailed calculation will be discussed in Appendices C, D, and E, respectively.

3.6 Equations of motion

The equations of motion of the configuration variables are

$$\dot{e}^A_a=\{e^A_a,H_T\}=\lambda^A_a\approx e^A_{0,a}+\omega^{-A}_a l_0+\omega^{+A}_a n_0+\omega^{AB}_a e^C_0\delta_{BC}-\omega^{AB}_0 e^C_a\delta_{BC}, \quad (3.63)$$

$$\dot{\omega}^{-+}_0=\{\omega^{-+}_0,H_T\}=\lambda^{-+}_0, \quad \dot{\omega}^{-+}_1=\{\omega^{-+}_1,H_T\}=\lambda^{-+}_1,$$
$$\dot{\omega}^{-+}_a=\{\omega^{-+}_a,H_T\}=\lambda^{-+}_a, \quad (3.64)$$

$$\dot{\omega}^{-A}_0=\{\omega^{-A}_0,H_T\}=\lambda^{-A}_0, \quad \dot{\omega}^{-A}_1=\{\omega^{-A}_1,H_T\}=\lambda^{-A}_1\approx 0,$$
$$\dot{\omega}^{-A}_a=\{\omega^{-A}_a,H_T\}=\lambda^{-A}_a, \quad (3.65)$$

$$\dot{\omega}^{+A}_0=\{\omega^{+A}_0,H_T\}=\lambda^{+A}_0, \quad \dot{\omega}^{+A}_1=\{\omega^{+A}_1,H_T\}=\lambda^{+A}_1,$$
$$\dot{\omega}^{+A}_a=\{\omega^{+A}_a,H_T\}=\lambda^{+A}_a, \quad (3.66)$$

$$\dot{\omega}^{23}_0=\{\omega^{23}_0,H_T\}=\lambda^{23}_0, \quad \dot{\omega}^{23}_1=\{\omega^{23}_1,H_T\}=\lambda^{23}_1,$$
$$\dot{\omega}^{23}_a=\{\omega^{23}_a,H_T\}=\lambda^{23}_a. \quad (3.67)$$

The equations of motion of the non-vanishing conjugate momenta are

$$\dot{\pi}^1_{-+}=\{\pi^1_{-+},H_T\}\approx 4\epsilon_{AB}\epsilon^{ab}\lambda^A_a e^B_b,$$
$$\dot{\pi}^a_{-A}=\{\pi^a_{-A},H_T\}\approx 4\epsilon_{AB}\epsilon^{ab}\lambda^B_b. \quad (3.68)$$

4 Classifications of constraints

4.1 First- and second-class constraints

One can see that there are six first-class constraints

$$\pi^0_{-+}=0, \quad \pi^0_{-A}=0, \quad \pi^0_{+A}=0, \quad \pi^0_{23}=0, \quad (4.1)$$

because their corresponding configuration variables ω^{IJ}_0 do not exist in the constraints. The remaining 40 constraints are of the second class. The Poisson brackets of the constraints can be found in Appendix F.

4.2 Degrees of freedom

There are 4 + 24 = 28 configuration variables and 28 conjugate momenta in this system, which span a 56-dimensional phase space. There are 46 constraints, including 32 primary constraints and 14 secondary constraints. Among the 46 constraints, there are 6 first-class constraints and 40 second-class constraints, which, altogether, reduce 52 degrees of freedom in the phase space. Therefore, there are four degrees of freedom left in the phase space, which means that there are two local physical degrees of freedom. They correspond to two independent polarization modes of the gravitational wave.

4.3 Scalar and vector constraints

In $\mathfrak{su}(2)$-connection dynamics [9], the constraints are classified as the spatial scalar, spatial vector, and $\mathfrak{su}(2)$ gauge constraints. In comparison, $\epsilon_{AB}\epsilon^{ab}F^{+A}_{1a}e^B_b+F^{23}_{23}\approx 0$ and $\epsilon_{AB}\epsilon^{ab}F^{-A}_{1a}e^B_b\approx 0$ are two scalar constraints and $F^{-A}_{23}+\epsilon^{ab}e^A_aF^{-+}_{1b}\approx 0$ is a two-dimensional vector constraint. The vector constraint reduces 2 degrees of freedom in the phase space because it is actually composed of two second-class constraints.

4.4 Gauss constraints

In the new approach, the Gauss constraints are not independent ones. They can be read from the above analysis in the following way.

The SO(1,3) Gauss constraints can be written as [15]

$$D_j\pi^j_{IJ}:=\partial_j\pi^j_{IJ}+\eta_{IK}\omega^{KL}_j\pi^j_{LJ}-\eta_{JK}\omega^{KL}_j\pi^j_{LI}\approx 0. \quad (4.2)$$

By using primary constraints (3.3) to replace coframe e^A_a with non-zero conjugate momenta, one can see that the above constraints (4.2) are actually the consistency conditions of the six primary constraints $\pi^0_{IJ}=0$.

The SO(1,1) gauge constraint comes from the consistency condition of $\pi^0_{-+}=0$:

$$\begin{aligned}\{H_T,\pi^0_{-+}\} &= -4\epsilon_{AB}\epsilon^{ab}e^B_b(e^A_{a,1}+\omega^{AC}_1e^D_a\delta_{CD}-\omega^{-A}_a)\\ &\approx -\partial_j\pi^j_{-+}-\eta_{-K}\omega^{KL}_j\pi^j_{L+}+\eta_{+K}\omega^{KL}_j\pi^j_{L-}\\ &= -D_j\pi^j_{-+}\approx 0. \end{aligned} \quad (4.3)$$

The $T^-(2)$ gauge constraints come from the consistency conditions of $\pi^0_{-A}=0$:

$$\begin{aligned}\{H_T,\pi^0_{-A}\} &= 4\epsilon_{AB}\epsilon^{ab}(e^B_{a,b}+\omega^{BC}_be^D_a\delta_{CD})\\ &\quad +4\epsilon_{AB}\epsilon^{ab}e^B_b(\omega^{+C}_1e^D_a\delta_{CD}-\omega^{-+}_a)\\ &\approx -\partial_j\pi^j_{-A}-\eta_{-K}\omega^{KL}_j\pi^j_{LA}+\eta_{AK}\omega^{KL}_j\pi^j_{L-}\\ &= -D_j\pi^j_{-A}\approx 0. \end{aligned} \quad (4.4)$$

The $T^+(2)$ gauge constraints come from the consistency conditions of $\pi^0_{+A}=0$:

$$\begin{aligned}\{H_T,\pi^0_{+A}\} &= -2\epsilon_{CB}\epsilon^{ab}e^C_a e^B_b\omega_1^{-A}=-4e\omega_1^{-A}\\ &\approx -\partial_j\pi^j_{+A}-\eta_{+K}\omega_j^{KL}\pi^j_{LA}+\eta_{AK}\omega_j^{KL}\pi^j_{L+}\\ &= -D_j\pi^j_{+A}\approx 0. \end{aligned} \tag{4.5}$$

The SO(2) gauge constraint comes from the consistency condition of $\pi^0_{23}=0$:

$$\begin{aligned}\{H_T,\pi^0_{23}\} &= -4\epsilon^{ab}\omega_a^{-C}e^D_b\delta_{CD}\\ &\approx -\partial_j\pi^j_{23}-\eta_{2K}\omega_j^{KL}\pi^j_{L3}+\eta_{3K}\omega_j^{KL}\pi^j_{L2}\\ &= -D_j\pi^j_{23}\approx 0. \end{aligned} \tag{4.6}$$

5 Summary

A self-consistent Hamiltonian formalism for a four-dimensional connection dynamics has been set up in a Bondi-like coordinate system $\{v,r,x^a\}$. The advanced null coordinate v is used as the time coordinate instead of u in the Bondi-Sachs coordinates. Three components of the metric are fixed in the Bondi-like metric, and therefore, there are only 7 non-zero components in the metric. The three Bondi-like conditions can be translated into 3 conditions on the coframe and can also be treated as three primary constraints, which will be preserved in the evolution. The three-dimensional hypersurfaces labeled by v have a degenerate metric, and therefore, they are null hypersurfaces.

The internal symmetry SO(1,3) is decomposed into SO(1,1), SO(2), and $T^\pm(2)$, and the Lie algebra $\mathfrak{so}(1,3)$ is spanned by $\{L_{-+},L_{23},L_{-A},L_{+A}\}$. The coframe consists of two null 1-forms and two spacelike 1-forms. A simple coframe has been chosen to make a Hamiltonian analysis. The $\mathfrak{so}(1,3)$ connection has 24 components, which are treated as 24 independent configuration variables. They, together with four coframe coefficients e^A_a and their conjugate momenta, span a 56-dimensional phase space. There are 32 primary constraints and 14 secondary constraints. Among all the 46 constraints, there are 6 first-class constraints $\pi^0_{IJ}=0$ and 40 second-class constraints. Therefore, the two local physical degrees of freedom remain. All 24 torsion-free conditions appear as the consistency conditions for the constraints. Among the constraints, there are two scalar constraints ((2.23) and (2.24)) and one two-dimensional vector constraint (2.25). The six Gauss constraints, (4.3), (4.4), (4.5), and (4.6), are not independent.

The four Lagrange multipliers n_0, l_0, and e^A_0 satisfy eight differential equations (3.21)–(3.26). The integrability conditions of n_0, l_0, and e^A_0 are Ricci identities. The Lagrange multipliers, namely, λ^A_a, λ_1^{-A}, λ_1^{-+}, λ_a^{-A}, λ_1^{23}, and λ_a^{23} are completely solved (expressed by coframes and connections). The Lagrange multipliers λ_a^{-+} and λ_1^{+A} satisfy two algebraic equations and two differential equations. The Lagrange multipliers λ_a^{+A} satisfy one algebraic equation and three differential equations.

From the analysis, one can see that ω_0^{IJ} can also be treated as Lagrange multipliers because they are multiplied by the Gauss constraints. In this treatment, the Gauss constraints become the primary constraints. The consistency conditions containing ω_0^{IJ} are not treated as constraints but as equations of multipliers. The final degrees of freedom in the phase space will be the same.

Using (3.3), one can also replace e^A_a with π^b_{-B}, and therefore, all the canonical variables in the Hamiltonian are ω_μ^{IJ} and their conjugate momenta π^μ_{IJ}. In this way, the dynamics of gravity is recovered as the pure connection dynamics. However, the Hamiltonian analysis under this formalism will become more complicated.

The usual 1+3 spacelike foliation can be used in the initial-value analysis of the whole spacetime, whereas our foliation can only be used in a small part of the whole spacetime within a short period of time. During this short time period, we can think that there is just gravitational wave from one direction passing through a certain point in the spacetime. In the 1+3 foliation, there is 1 scalar constraint and a three-dimensional vector constraint, whereas in our decomposition there are two scalar constraints and a two-dimensional vector constraint. In the $\mathfrak{su}(2)$-connection dynamics, there are three Gauss constraints corresponding to three generators of the $\mathfrak{su}(2)$ connection as independent constraints; however, in our analysis, there are six Gauss constraints corresponding to six generators of the $\mathfrak{so}(1,3)(=\mathfrak{so}(1,1)\oplus\mathfrak{so}(2)\oplus\mathfrak{t}^-(2)\oplus\mathfrak{t}^+(2))$ connection, which are not independent constraints. Moreover, in the $\mathfrak{su}(2)$-connection dynamics, the frame rather than the coframe is used, and therefore, the torsion-free conditions do not appear, whereas in our approach the coframe is used, and therefore, the torsion-free conditions will appear as the requirements of consistency. However, in all the formalisms, there are two local physical degrees of freedom. The decomposition of symmetry and connection in the usual 1+3 way cannot be postulated to a higher dimensional spacetime, whereas our decomposition can be applied to higher dimensions in principle.

The success of the Hamiltonian analysis of gravity in three- and four-dimensional spacetimes shows that there will probably be no conceptual difficulty for the Hamiltonian analysis of gravity in a higher dimensional spacetime; however, the analysis will become much more difficult technically.

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Appendix A: Proof of (3.35)

$$\epsilon_{AB}\epsilon^{ab}(\Lambda^{-A}_{a,1}-\Lambda^{-+}_1\omega^{-A}_a-\omega^{-+}_1\Lambda^{-A}_a-\Lambda^{-C}_a\omega^{DA}_1\delta_{CD})e^B_b+\epsilon_{AB}\epsilon^{ab}F^{-A}_{1a}\Lambda^B_b$$

$$=(\epsilon_{AB}\epsilon^{ab}\Lambda^{-A}_a e^B_b)_{,1}-\epsilon_{AB}\epsilon^{ab}\Lambda^{-A}_a e^B_{b,1}-\epsilon_{AB}\epsilon^{ab}\Lambda^{-C}_a\omega^{DA}_1\delta_{CD}e^B_b-\epsilon_{AB}\epsilon^{ab}\omega^{-A}_a e^B_b\Lambda^{-+}_1$$

$$-\epsilon_{AB}\epsilon^{ab}\Lambda^{-A}_a e^B_b\omega^{-+}_1+\epsilon_{AB}\epsilon^{ab}F^{-A}_{1a}\Lambda^B_b$$

$$\approx(\epsilon_{AB}\epsilon^{ab}\Lambda^{-A}_a e^B_b)_{,1}-\epsilon_{AB}\epsilon^{ab}\Lambda^{-A}_a\omega^{-B}_b-\epsilon_{AB}\epsilon^{ab}\omega^{-A}_a e^B_b\Lambda^{-+}_1-\epsilon_{AB}\epsilon^{ab}\Lambda^{-A}_a e^B_b\omega^{-+}_1+\epsilon_{AB}\epsilon^{ab}F^{-A}_{1a}\Lambda^B_b$$

$$=(\epsilon_{AB}\epsilon^{ab}\Lambda^{-A}_a e^B_b)_{,1}-\epsilon_{AB}\epsilon^{ab}\Lambda^{-A}_a e^B_b\omega^{-+}_1-(\epsilon_{AB}\epsilon^{ab}\Lambda^{-B}_b+\epsilon_{AB}\epsilon^{ab}e^B_b\Lambda^{-+}_1)\omega^{-A}_a+\epsilon_{AB}\epsilon^{ab}F^{-A}_{1a}\Lambda^B_b$$

$$\approx[\epsilon_{AB}\epsilon^{ab}(\omega^{-A}_{0,a}+\omega^{-+}_0\omega^{-A}_a-\omega^{-+}_a\omega^{-A}_0-\omega^{-D}_0\omega^{CA}_a\delta_{DC}+\omega^{-D}_a\omega^{CA}_0\delta_{DC})e^B_b-\epsilon_{AB}F^{-A}_{23}e^B_0-n_0F^{23}_{23}]_{,1}$$

$$-\epsilon_{AB}\epsilon^{ab}(\omega^{-A}_{0,a}+\omega^{-+}_0\omega^{-A}_a-\omega^{-+}_a\omega^{-A}_0-\omega^{-D}_0\omega^{CA}_a\delta_{DC}+\omega^{-D}_a\omega^{CA}_0\delta_{DC})e^B_b\omega^{-+}_1+\epsilon_{AB}F^{-A}_{23}e^B_0\omega^{-+}_1$$

$$+n_0F^{23}_{23}\omega^{-+}_1-[\epsilon_{AB}\epsilon^{ab}e^B_b(\omega^{-+}_{0,1}+\omega^{-C}_1\omega^{D+}_0\delta_{CD}-\omega^{-C}_0\omega^{D+}_1\delta_{CD})+\epsilon_{AB}\epsilon^{ab}(l_0F^{-B}_{1b}-e^B_0F^{-+}_{1b}-n_0F^{+B}_{1b})$$

$$+\epsilon_{AB}\epsilon^{ab}(\omega^{-B}_{0,b}+\omega^{-+}_0\omega^{-B}_b-\omega^{-+}_b\omega^{-B}_0-\omega^{-D}_0\omega^{CB}_b\delta_{DC}+\omega^{-D}_b\omega^{CB}_0\delta_{DC})]\omega^{-A}_a$$

$$+\epsilon_{AB}\epsilon^{ab}(e^A_{0,a}+\omega^{-A}_a l_0+\omega^{+A}_a n_0+\omega^{AC}_a e^D_0\delta_{CD}-\omega^{AC}_0 e^D_a\delta_{CD})F^{-B}_{1b}$$

$$=\epsilon_{AB}\epsilon^{ab}(\omega^{-A}_{0,a}+\omega^{-+}_0\omega^{-A}_a-\omega^{-+}_a\omega^{-A}_0-\omega^{-D}_0\omega^{CA}_a\delta_{DC}+\omega^{-D}_a\omega^{CA}_0\delta_{DC})_{,1}e^B_b$$

$$+\epsilon_{AB}\epsilon^{ab}(\omega^{-A}_{0,a}+\omega^{-+}_0\omega^{-A}_a-\omega^{-+}_a\omega^{-A}_0-\omega^{-D}_0\omega^{CA}_a\delta_{DC}+\omega^{-D}_a\omega^{CA}_0\delta_{DC})e^B_{b,1}$$

$$-\epsilon_{AB}\epsilon^{ab}(\omega^{-A}_{0,a}+\omega^{-+}_0\omega^{-A}_a-\omega^{-+}_a\omega^{-A}_0-\omega^{-D}_0\omega^{CA}_a\delta_{DC}+\omega^{-D}_a\omega^{CA}_0\delta_{DC})e^B_b\omega^{-+}_1$$

$$-\epsilon_{AB}\epsilon^{ab}(\omega^{-B}_{0,b}+\omega^{-+}_0\omega^{-B}_b-\omega^{-+}_b\omega^{-B}_0-\omega^{-D}_0\omega^{CB}_b\delta_{DC}+\omega^{-D}_b\omega^{CB}_0\delta_{DC})\omega^{-A}_a$$

$$+\epsilon_{AB}\epsilon^{ab}e^A_{0,a}F^{-B}_{1b}+\epsilon_{AB}\epsilon^{ab}\omega^{-A}_a l_0F^{-B}_{1b}+\epsilon_{AB}\epsilon^{ab}\omega^{+A}_a n_0F^{-B}_{1b}+\epsilon_{AB}\epsilon^{ab}\omega^{AC}_a e^D_0\delta_{CD}F^{-B}_{1b}$$

$$-\epsilon_{AB}\epsilon^{ab}e^B_b(\omega^{-+}_{0,1}+\omega^{-C}_1\omega^{D+}_0\delta_{CD}-\omega^{-C}_0\omega^{D+}_1\delta_{CD})\omega^{-A}_a-\epsilon_{AB}\epsilon^{ab}\omega^{AC}_0 e^D_a\delta_{CD}F^{-B}_{1b}$$

$$-\epsilon_{AB}F^{-A}_{23,1}e^B_0-\epsilon_{AB}F^{-A}_{23}e^B_{0,1}-n_{0,1}F^{23}_{23}-n_0F^{23}_{23,1}+\epsilon_{AB}F^{-A}_{23}e^B_0\omega^{-+}_1+n_0F^{23}_{23}\omega^{-+}_1$$

$$+\epsilon_{AB}\epsilon^{ab}e^B_0F^{-+}_{1b}\omega^{-A}_a+n_0\epsilon_{AB}\epsilon^{ab}F^{+B}_{1b}\omega^{-A}_a-l_0\epsilon_{AB}\epsilon^{ab}F^{-B}_{1b}\omega^{-A}_a$$

$$\approx\epsilon_{AB}\epsilon^{ab}(\omega^{-A}_{0,1})_{,a}e^B_b+\epsilon_{AB}\epsilon^{ab}\omega^{-+}_{0,1}\omega^{-A}_a e^B_b+\epsilon_{AB}\epsilon^{ab}\omega^{-+}_0\omega^{-A}_{a,1}e^B_b-\epsilon_{AB}\epsilon^{ab}\omega^{-+}_{a,1}\omega^{-A}_0 e^B_b-\epsilon_{AB}\epsilon^{ab}\omega^{-+}_a\omega^{-A}_{0,1}e^B_b$$

$$-\epsilon_{AB}\epsilon^{ab}\omega^{-D}_{0,1}\omega^{CA}_a\delta_{DC}e^B_b-\epsilon_{AB}\epsilon^{ab}\omega^{-D}_0\omega^{CA}_{a,1}\delta_{DC}e^B_b+\epsilon_{AB}\epsilon^{ab}\omega^{-D}_{a,1}\omega^{CA}_0\delta_{DC}e^B_b+\epsilon_{AB}\epsilon^{ab}\omega^{-D}_a\omega^{CA}_{0,1}\delta_{DC}e^B_b$$

$$-\epsilon_{AB}\epsilon^{ab}(\omega^{-A}_{0,a}+\omega^{-+}_0\omega^{-A}_a-\omega^{-+}_a\omega^{-A}_0-\omega^{-D}_0\omega^{CA}_a\delta_{DC}+\omega^{-D}_a\omega^{CA}_0\delta_{DC})\omega^{BE}_1e^F_b\delta_{EF}$$

$$-\epsilon_{AB}\epsilon^{ab}(\omega^{-A}_{0,a}+\omega^{-+}_0\omega^{-A}_a-\omega^{-+}_a\omega^{-A}_0-\omega^{-D}_0\omega^{CA}_a\delta_{DC}+\omega^{-D}_a\omega^{CA}_0\delta_{DC})e^B_b\omega^{-+}_1$$

$$-\epsilon_{AB}\epsilon^{ab}e^B_b(\omega^{-+}_{0,1}-\omega^{-C}_0\omega^{D+}_1\delta_{CD})\omega^{-A}_a-n_0F^{23}_{23,1}+n_0\epsilon_{AB}\epsilon^{ab}\omega^{+A}_aF^{-B}_{1b}+n_0\epsilon_{AB}\epsilon^{ab}F^{+B}_{1b}\omega^{-A}_a$$

$$+n_0\epsilon_{AB}F^{-A}_{23}\omega^{+B}_1+\epsilon_{AB}\epsilon^{ab}\omega^{AC}_a e^D_0\delta_{CD}F^{-B}_{1b}-\epsilon_{AB}F^{-A}_{23,1}e^B_0+\epsilon_{AB}F^{-A}_{23}e^B_0\omega^{-+}_1+\epsilon_{AB}\epsilon^{ab}e^B_0F^{-+}_{1b}\omega^{-A}_a$$

$$+\epsilon_{AB}F^{-A}_{23}\omega^{BC}_1e^D_0\delta_{CD}+\omega^{-A}_0\epsilon_{AB}\epsilon^{ab}F^{-+}_{1a}e^B_b+\epsilon_{AB}\epsilon^{ab}e^A_{0,a}F^{-B}_{1b}$$

$$\approx\omega^{-+}_{1,c}\omega^{-A}_0\epsilon_{AD}\epsilon^{cd}e^D_d+\omega^{-+}_1\omega^{-A}_{0,c}\epsilon_{AD}\epsilon^{cd}e^D_d+\omega^{-B}_{0,c}\omega^{CA}_1\delta_{BC}\epsilon_{AD}\epsilon^{cd}e^D_d+\omega^{-B}_0\omega^{CA}_{1,c}\delta_{BC}\epsilon_{AD}\epsilon^{cd}e^D_d$$

$$+(n_0\epsilon^{ab}e^A_aF^{23}_{1b}e^{-1})_{,c}\epsilon_{AD}\epsilon^{cd}e^D_d+(\epsilon^{ab}e^A_aF^{-B}_{1b}e^C_0\epsilon_{BC}e^{-1})_{,c}\epsilon_{AD}\epsilon^{cd}e^D_d-\epsilon_{AB}\epsilon^{ab}\omega^{-+}_a\omega^{-A}_{0,1}e^B_b$$

$$-\epsilon_{AB}\epsilon^{ab}\omega^{-D}_{0,1}\omega^{CA}_a\delta_{DC}e^B_b-\epsilon_{AB}\epsilon^{ab}\omega^{-A}_{0,a}\omega^{BE}_1e^F_b\delta_{EF}-\epsilon_{AB}\epsilon^{ab}\omega^{-A}_{0,a}e^B_b\omega^{-+}_1+\epsilon_{AB}\epsilon^{ab}\omega^{-+}_0\omega^{-A}_{a,1}e^B_b$$

$$-\epsilon_{AB}\epsilon^{ab}\omega^{-+}_{a,1}\omega^{-A}_0e^B_b-\epsilon_{AB}\epsilon^{ab}\omega^{-D}_0\omega^{CA}_{a,1}\delta_{DC}e^B_b+\epsilon_{AB}\epsilon^{ab}\omega^{-D}_{a,1}\omega^{CA}_0\delta_{DC}e^B_b+\epsilon_{AB}\epsilon^{ab}\omega^{-+}_a\omega^{-A}_0\omega^{BE}_1e^F_b\delta_{EF}$$

$$+\epsilon_{AB}\epsilon^{ab}\omega^{-D}_0\omega^{CA}_a\delta_{DC}\omega^{BE}_1e^F_b\delta_{EF}-\epsilon_{AB}\epsilon^{ab}\omega^{-D}_a\omega^{CA}_0\delta_{DC}\omega^{BE}_1e^F_b\delta_{EF}-\epsilon_{AB}\epsilon^{ab}\omega^{-+}_0\omega^{-A}_a e^B_b\omega^{-+}_1$$

$$+\epsilon_{AB}\epsilon^{ab}\omega^{-+}_a\omega^{-A}_0e^B_b\omega^{-+}_1+\epsilon_{AB}\epsilon^{ab}\omega^{-D}_0\omega^{CA}_a\delta_{DC}e^B_b\omega^{-+}_1+\epsilon_{AB}\epsilon^{ab}e^B_b\omega^{-C}_0\omega^{D+}_1\delta_{CD}\omega^{-A}_a-n_0F^{23}_{23,1}$$

$$+n_0\epsilon_{AB}\epsilon^{ab}\omega^{+A}_aF^{-B}_{1b}+n_0\epsilon_{AB}\epsilon^{ab}F^{+B}_{1b}\omega^{-A}_a+n_0\epsilon_{AB}F^{-A}_{23}\omega^{+B}_1-\epsilon_{AB}F^{-A}_{23,1}e^B_0+\epsilon_{AB}\epsilon^{ab}\omega^{AC}_ae^D_0\delta_{CD}F^{-B}_{1b}$$

$$+\epsilon_{AB}F^{-A}_{23}e^B_0\omega^{-+}_1+\epsilon_{AB}\epsilon^{ab}e^B_0F^{-+}_{1b}\omega^{-A}_a+\epsilon_{AB}F^{-A}_{23}\omega^{BC}_1e^D_0\delta_{CD}+\omega^{-A}_0\epsilon_{AB}\epsilon^{ab}F^{-+}_{1a}e^B_b+\epsilon_{AB}\epsilon^{ab}e^A_{0,a}F^{-B}_{1b}$$

$$
\begin{aligned}
&=n_0\epsilon^{ab}e_{,a}F_{1b}^{23}e^{-1}+n_0\epsilon^{ab}F_{1b}^{23}e_{,c}^{-1}\epsilon^{cd}\epsilon_{AD}e_a^Ae_d^D+n_0\epsilon^{ab}F_{1b}^{23}e^{-1}\epsilon_{AD}\epsilon^{cd}e_a^Ae_{c,d}^D-n_0F_{23,1}^{23}\\
&\quad+n_0\epsilon^{ab}F_{1b,c}^{23}e^{-1}\epsilon^{cd}\epsilon_{AD}e_a^Ae_d^D+n_0\epsilon_{AB}\epsilon^{ab}\omega_a^{+A}F_{1b}^{-B}+n_0\epsilon_{AB}\epsilon^{ab}F_{1b}^{+B}\omega_a^{-A}+n_0\epsilon_{AB}F_{23}^{-A}\omega_1^{+B}\\
&\quad-\epsilon_{DB}\epsilon^{cd}\omega_c^{ED}\delta_{AE}e_d^Bn_0\epsilon^{ab}e_a^AF_{1b}^{23}e^{-1}+\epsilon_{AB}\epsilon^{ab}\omega_a^{AC}e_0^D\delta_{CD}F_{1b}^{-B}-\epsilon_{AB}F_{23,1}^{-A}e_0^B+\epsilon_{AB}F_{23}^{-A}e_0^B\omega_1^{-+}\\
&\quad+\epsilon_{AB}\epsilon^{ab}e_0^BF_{1b}^{-+}\omega_a^{-A}+\epsilon_{AB}F_{23}^{-A}\omega_1^{BC}e_0^D\delta_{CD}+\epsilon^{ab}e_{a,c}^AF_{1b}^{-B}e_0^C\epsilon_{BC}e^{-1}\epsilon_{AD}\epsilon^{cd}e_d^D\\
&\quad+\epsilon^{ab}e_a^AF_{1b,c}^{-B}e_0^C\epsilon_{BC}e^{-1}\epsilon_{AD}\epsilon^{cd}e_d^D-\omega_c^{-C}e_0^D\delta_{CD}\epsilon^{ab}e_a^AF_{1b}^{23}e^{-1}\epsilon_{AB}\epsilon^{cd}e_d^B+\epsilon^{ab}e_a^AF_{1b}^{-B}e_0^C\epsilon_{BC}e_{,c}^{-1}\epsilon_{AD}\epsilon^{cd}e_d^D\\
&\quad-\epsilon_{AD}\epsilon^{cd}\omega_c^{-+}e_d^D\epsilon^{ab}e_a^AF_{1b}^{-B}e_0^C\epsilon_{BC}e^{-1}-\epsilon_{DF}\epsilon^{cd}\omega_c^{ED}\delta_{AE}e_d^F\epsilon^{ab}e_a^AF_{1b}^{-B}e_0^C\epsilon_{BC}e^{-1}\\
&\approx-n_0\epsilon^{ab}F_{1a,b}^{23}-n_0F_{23,1}^{23}+n_0\epsilon_{AB}\epsilon^{ab}\omega_a^{+A}F_{1b}^{-B}+n_0\epsilon_{AB}\epsilon^{ab}F_{1b}^{+B}\omega_a^{-A}+n_0\epsilon_{AB}F_{23}^{-A}\omega_1^{+B}\\
&\quad+e_0^C\epsilon_{AB}\epsilon^{ab}F_{1b}^{-B}\omega_a^{AD}\delta_{CD}-\epsilon_{AB}F_{23,1}^{-A}e_0^B+\epsilon_{AB}F_{23}^{-A}e_0^B\omega_1^{-+}+\epsilon_{AB}F_{23}^{-A}\omega_1^{BC}e_0^D\delta_{CD}\\
&\quad+e_0^A\epsilon_{AB}\epsilon^{ab}F_{1a,b}^{-B}+e_0^B\epsilon_{AB}\epsilon^{ab}F_{1b}^{-+}\omega_a^{-A}-e_0^B\delta_{AB}\epsilon^{ab}F_{1b}^{23}\omega_a^{-A}+e_0^A\epsilon_{AB}\epsilon^{ab}\omega_a^{-+}F_{1b}^{-B}\\
&\approx n_0\epsilon_{AB}F_{23}^{-A}\omega_1^{+B}+n_0\epsilon^{ab}\epsilon_{AB}\omega_{a,b}^{-A}\omega_1^{+B}+n_0\epsilon_{AB}\epsilon^{ab}F_{1a}^{-A}\omega_b^{+B}-n_0\epsilon^{ab}\epsilon_{AB}\omega_{a,1}^{-A}\omega_b^{+B}+n_0\epsilon^{ab}\epsilon_{AB}\omega_a^{-A}\omega_{1,b}^{+B}\\
&\quad-n_0\epsilon^{ab}\epsilon_{AB}\omega_a^{-A}\omega_{b,1}^{+B}+n_0\epsilon_{AB}\epsilon^{ab}F_{1b}^{+B}\omega_a^{-A}+e_0^B\epsilon_{AB}\epsilon^{ab}F_{1b}^{-+}\omega_a^{-A}-\epsilon_{AB}e_0^B\epsilon^{ab}\omega_{b,1}^{-+}\omega_a^{-A}\\
&\quad+e_0^B\epsilon_{AB}\epsilon^{ab}\omega_{1,b}^{-+}\omega_a^{-A}+e_0^A\epsilon_{AB}\epsilon^{ab}\omega_a^{-+}F_{1b}^{-B}-\epsilon_{AB}e_0^A\epsilon^{ab}\omega_a^{-+}\omega_{b,1}^{-B}+e_0^C\epsilon_{AB}\epsilon^{ab}F_{1b}^{-B}\omega_a^{AD}\delta_{CD}\\
&\quad-\epsilon_{AC}e_0^C\epsilon^{ab}\omega_{b,1}^{-B}\omega_a^{AD}\delta_{BD}-e_0^B\delta_{AB}\epsilon^{ab}F_{1b}^{23}\omega_a^{-A}-e_0^B\epsilon^{ab}\omega_a^{-A}\omega_{1,b}^{23}\delta_{AB}+e_0^B\epsilon^{ab}\omega_a^{-A}\omega_{b,1}^{23}\delta_{AB}\\
&\quad+e_0^B\epsilon_{AB}\epsilon^{ab}\omega_1^{-+}\omega_{a,b}^{-A}+\epsilon_{AB}F_{23}^{-A}e_0^B\omega_1^{-+}-e_0^B\epsilon^{ab}\omega_{a,b}^{-A}\omega_1^{23}\delta_{AB}-e_0^BF_{23}^{-A}\omega_1^{23}\delta_{AB}\\
&\approx n_0\epsilon_{AB}F_{23}^{-A}\omega_1^{+B}-n_0\epsilon_{AB}F_{23}^{-A}\omega_1^{+B}+n_0\epsilon_{AB}\epsilon^{ab}F_{1a}^{-A}\omega_b^{+B}-n_0\epsilon^{ab}\epsilon_{AB}F_{1a}^{-A}\omega_b^{+B}-n_0\epsilon^{ab}\epsilon_{AB}\omega_a^{-A}F_{1b}^{+B}\\
&\quad+n_0\epsilon_{AB}\epsilon^{ab}F_{1b}^{+B}\omega_a^{-A}+e_0^B\epsilon_{AB}\epsilon^{ab}F_{1b}^{-+}\omega_a^{-A}-\epsilon_{AB}e_0^B\epsilon^{ab}F_{1b}^{-+}\omega_a^{-A}+e_0^A\epsilon_{AB}\epsilon^{ab}\omega_a^{-+}F_{1b}^{-B}\\
&\quad-\epsilon_{AB}e_0^A\epsilon^{ab}\omega_a^{-+}F_{1b}^{-B}+e_0^C\epsilon_{AB}\epsilon^{ab}F_{1b}^{-B}\omega_a^{AD}\delta_{CD}-\epsilon_{AC}e_0^C\epsilon^{ab}F_{1b}^{-B}\omega_a^{AD}\delta_{BD}-e_0^B\delta_{AB}\epsilon^{ab}F_{1b}^{23}\omega_a^{-A}\\
&\quad+e_0^B\epsilon^{ab}\omega_a^{-A}F_{1b}^{23}\delta_{AB}-e_0^B\epsilon_{AB}\epsilon^{ab}\omega_1^{-+}F_{ab}^{-A}+\epsilon_{AB}F_{23}^{-A}e_0^B\omega_1^{-+}+e_0^B\epsilon^{ab}F_{ab}^{-A}\omega_1^{23}\delta_{AB}-e_0^BF_{23}^{-A}\omega_1^{23}\delta_{AB}=0.
\end{aligned}
$$

In the 1st "$\approx$", (3.26) has been used, whereas in the 2nd "$\approx$", (2.23) and (2.24) have been used. The identity $\epsilon^{ab}F_{1a}^{-A}e_b^B\delta_{AB}\approx0$ and (3.24), (3.26), (3.28), and (3.30) have been used in the 3rd "$\approx$", and (2.25), (3.24), (3.25), (3.26), (3.28), (3.30), and (3.31) have been used in the 4th "$\approx$". In the 5th "$\approx$", (3.32) has been used. (3.30) has been used in the 6th and the last "$\approx$".

The proof of the additional identity is as follows:

$$
\begin{aligned}
\epsilon^{ab}F_{1a}^{-A}e_b^B\delta_{AB}&=\epsilon^{ab}(\omega_{a,1}^{-A}-\omega_{1,a}^{-A}-\omega_1^{-+}\omega_a^{-A}+\omega_1^{-C}\omega_a^{DA}\delta_{CD}-\omega_a^{-C}\omega_1^{DA}\delta_{CD}+\omega_a^{-+}\omega_1^{-A})e_b^B\delta_{AB}\\
&\approx\epsilon^{ab}(\omega_{a,1}^{-A}-\omega_a^{-C}\omega_1^{DA}\delta_{CD})e_b^B\delta_{AB}=\delta_{AB}\epsilon^{ab}\omega_{a,1}^{-A}e_b^B-\delta_{AB}\epsilon^{ab}\omega_a^{-C}\omega_1^{DA}\delta_{CD}e_b^B\\
&=(\delta_{AB}\epsilon^{ab}\omega_a^{-A}e_b^B)_{,1}-\delta_{AB}\epsilon^{ab}\omega_a^{-A}e_{b,1}^B-\delta_{AB}\epsilon^{ab}\omega_a^{-C}\omega_1^{DA}\delta_{CD}e_b^B\\
&\approx-\delta_{CD}\epsilon^{ab}\omega_a^{-C}e_{b,1}^D-\delta_{AB}\epsilon^{ab}\omega_a^{-C}\omega_1^{DA}\delta_{CD}e_b^B=-\delta_{CD}\epsilon^{ab}\omega_a^{-C}(e_{b,1}^D+\omega_1^{DA}e_b^B\delta_{AB})\\
&\approx-\delta_{CD}\epsilon^{ab}\omega_a^{-C}\omega_b^{-D}=0.
\end{aligned}
$$

In the 1st "$\approx$", (3.30) and (3.31) have been used. In the 2nd and 3rd "$\approx$", (3.28) and (3.31) have been used, respectively.

Appendix B: Proof of (3.48)

$$
\begin{aligned}
&\epsilon^{ab}(\omega_{0,a}^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-B}\omega_a^{CA}\delta_{BC}+\omega_a^{-B}\omega_0^{CA}\delta_{BC})e_b^D\delta_{AD}+\epsilon^{ab}(l_0F_{1a}^{-A}-e_0^AF_{1a}^{-+}-n_0F_{1a}^{+A})e_b^B\delta_{AB}\\
&\quad-\epsilon^{ab}(e_{0,a}^A+\omega_a^{-A}l_0+\omega_a^{+A}n_0+\omega_a^{AB}e_0^C\delta_{BC}-\omega_0^{AB}e_a^C\delta_{BC})\omega_b^{-D}\delta_{AD}\\
&\approx-\epsilon^{ab}e_0^AF_{1a}^{-+}e_b^E\delta_{AE}-\epsilon^{ab}n_0F_{1a}^{+A}e_b^E\delta_{AE}+\epsilon^{ab}(\omega_{0,a}^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-D}\omega_a^{CA}\delta_{DC})e_b^E\delta_{AE}\\
&\quad+\epsilon^{ab}\omega_a^{-A}(e_{0,b}^B+\omega_b^{+B}n_0+\omega_b^{BC}e_0^D\delta_{CD})\delta_{AB}\\
&=\epsilon^{ab}e_b^E\delta_{AE}\omega_{0,a}^{-A}-\epsilon^{ab}e_b^E\delta_{AE}\omega_a^{-+}\omega_0^{-A}-\epsilon^{ab}e_b^E\delta_{AE}\omega_0^{-D}\omega_a^{CA}\delta_{DC}+\epsilon^{ab}\omega_a^{-A}\delta_{AB}e_{0,b}^B+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{+B}n_0\\
&\quad+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{BC}e_0^D\delta_{CD}-\epsilon^{ab}e_0^AF_{1a}^{-+}e_b^E\delta_{AE}-\epsilon^{ab}n_0F_{1a}^{+A}e_b^E\delta_{AE}
\end{aligned}
$$

$$
\begin{aligned}
&=\epsilon^{ab}(\delta_{AE}\omega_0^{-A}e_b^E)_{,a}-\epsilon^{ab}\delta_{AE}\omega_0^{-A}e_{b,a}^E-\epsilon^{ab}e_b^E\delta_{AE}\omega_a^{-+}\omega_0^{-A}-\epsilon^{ab}e_b^E\delta_{AE}\omega_0^{-D}\omega_a^{CA}\delta_{DC}+\epsilon^{ab}(\omega_a^{-A}\delta_{AB}e_0^B)_{,b}\\
&\quad-\epsilon^{ab}\omega_{a,b}^{-A}\delta_{AB}e_0^B+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{+B}n_0+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{BC}e_0^D\delta_{CD}-\epsilon^{ab}e_0^AF_{1a}^{-+}e_b^E\delta_{AE}-\epsilon^{ab}n_0F_{1a}^{+A}e_b^E\delta_{AE}\\
&=-\epsilon^{ab}(\delta_{AE}\omega_0^{-A}e_a^E)_{,b}+\epsilon^{ab}(\omega_a^{-A}\delta_{AB}e_0^B)_{,b}-\epsilon^{ab}\delta_{AE}\omega_0^{-A}e_{b,a}^E-\epsilon^{ab}e_b^E\delta_{AE}\omega_a^{-+}\omega_0^{-A}-\epsilon^{ab}e_b^E\delta_{AE}\omega_0^{-D}\omega_a^{CA}\delta_{DC}\\
&\quad+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{+B}n_0-\epsilon^{ab}n_0F_{1a}^{+A}e_b^E\delta_{AE}-\epsilon^{ab}\omega_{a,b}^{-A}\delta_{AB}e_0^B+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{BC}e_0^D\delta_{CD}-\epsilon^{ab}e_0^AF_{1a}^{-+}e_b^E\delta_{AE}\\
&\approx\epsilon^{ab}(-n_{0,a}+\omega_a^{-+}n_0)_{,b}+\epsilon^{ab}\delta_{AE}\omega_0^{-A}\omega_a^{EF}e_b^G\delta_{FG}-\epsilon^{ab}e_b^E\delta_{AE}\omega_a^{-+}\omega_0^{-A}-\epsilon^{ab}e_b^E\delta_{AE}\omega_0^{-D}\omega_a^{CA}\delta_{DC}\\
&\quad+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{+B}n_0-\epsilon^{ab}n_0F_{1a}^{+A}e_b^E\delta_{AE}-\epsilon^{ab}\omega_{a,b}^{-A}\delta_{AB}e_0^B+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{BC}e_0^D\delta_{CD}-\epsilon^{ab}e_0^AF_{1a}^{-+}e_b^E\delta_{AE}\\
&=\epsilon^{ab}(\omega_{a,b}^{-+}n_0+\omega_a^{-+}n_{0,b})-\epsilon^{ab}e_b^E\delta_{AE}\omega_a^{-+}\omega_0^{-A}+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{+B}n_0-\epsilon^{ab}n_0F_{1a}^{+A}e_b^E\delta_{AE}\\
&\quad-\epsilon^{ab}\omega_{a,b}^{-A}\delta_{AB}e_0^B+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{BC}e_0^D\delta_{CD}-\epsilon^{ab}e_0^AF_{1a}^{-+}e_b^E\delta_{AE}\\
&\approx\epsilon^{ab}\omega_{a,b}^{-+}n_0-\epsilon^{ab}\omega_b^{-+}(\omega_a^{-+}n_0+\omega_0^{-A}e_a^B\delta_{AB}-\omega_a^{-A}e_0^B\delta_{AB})-\epsilon^{ab}e_b^E\delta_{AE}\omega_a^{-+}\omega_0^{-A}+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{+B}n_0\\
&\quad-\epsilon^{ab}n_0F_{1a}^{+A}e_b^E\delta_{AE}-\epsilon^{ab}\omega_{a,b}^{-A}\delta_{AB}e_0^B+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{BC}e_0^D\delta_{CD}-\epsilon^{ab}e_0^AF_{1a}^{-+}e_b^E\delta_{AE}\\
&=\epsilon^{ab}\omega_{a,b}^{-+}n_0+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{+B}n_0-\epsilon^{ab}n_0F_{1a}^{+A}e_b^E\delta_{AE}-\epsilon^{ab}\omega_{a,b}^{-A}\delta_{AB}e_0^B+\epsilon^{ab}\omega_a^{-A}\delta_{AB}\omega_b^{BC}e_0^D\delta_{CD}\\
&\quad+\epsilon^{ab}\omega_b^{-+}\omega_a^{-A}e_0^B\delta_{AB}-\epsilon^{ab}e_0^AF_{1a}^{-+}e_b^E\delta_{AE}\\
&=-n_0(F_{23}^{-+}+\epsilon^{ab}F_{1a}^{+A}e_b^E\delta_{AE})+e_0^A\delta_{AB}(F_{23}^{-B}-\epsilon^{ab}F_{1a}^{-+}e_b^B)\approx-n_0(F_{23}^{-+}+\epsilon^{ab}F_{1a}^{+A}e_b^B\delta_{AB})\\
&=-n_0[\epsilon^{ab}(\omega_{a,1}^{+A}-\omega_{1,a}^{+A}+\omega_1^{-+}\omega_a^{+A}+\omega_1^{+C}\omega_a^{DA}\delta_{CD}-\omega_a^{-+}\omega_1^{+A}-\omega_a^{+C}\omega_1^{DA}\delta_{CD})e_b^B\delta_{AB}\\
&\quad+(\omega_{3,2}^{-+}-\omega_{2,3}^{-+}-\omega_2^{-A}\omega_3^{+B}\delta_{AB}+\omega_3^{-A}\omega_2^{+B}\delta_{AB})]\\
&=-n_0[(\omega_{3,2}^{-+}-\omega_{2,3}^{-+}-\omega_2^{-A}\omega_3^{+B}\delta_{AB}+\omega_3^{-A}\omega_2^{+B}\delta_{AB}+(\epsilon^{ab}\omega_a^{+A}e_b^B\delta_{AB})_{,1}-\epsilon^{ab}\omega_a^{+A}e_{b,1}^B\delta_{AB}\\
&\quad-(\epsilon^{ab}\omega_1^{+A}e_b^B\delta_{AB})_{,a}+\epsilon^{ab}\omega_1^{+A}e_{b,a}^B\delta_{AB}+\epsilon^{ab}\omega_1^{-+}\omega_a^{+A}e_b^B\delta_{AB}+\epsilon^{ab}\omega_1^{+C}\omega_a^{DA}\delta_{CD}e_b^B\delta_{AB}\\
&\quad-\epsilon^{ab}\omega_a^{-+}\omega_1^{+A}e_b^B\delta_{AB}-\epsilon^{ab}\omega_a^{+C}\omega_1^{DA}\delta_{CD}e_b^B\delta_{AB}]\approx-n_0(\epsilon^{ab}\omega_1^{+A}\omega_b^{BC}e_a^D\delta_{AB}+\epsilon^{ab}\omega_1^{+C}\omega_a^{DA}\delta_{CD}e_b^B\delta_{AB})=0.
\end{aligned}
$$

In the 1st "≈", 1 identity $\epsilon^{ab}F_{1a}^{-A}e_b^B\delta_{AB}\approx0$ has been used, whereas in the 2nd "≈", (3.25) and (3.32) have been used. In the 3rd and 4th "≈", (3.22) and (2.25) have been used, respectively. In the last "≈", (3.22), (3.23), (3.24), and (3.25) have been used.

Appendix C: Integrability of n_0

The integrability of n_0 requires that

$$n_{0,1a}-n_{0,a1}=0, \tag{C1}$$

$$\epsilon^{ab}n_{0,ab}=0. \tag{C2}$$

From (3.21), one can obtain

$$(n_{0,1}-\omega_1^{-+}n_0)_{,a}\approx n_{0,1a}-\omega_{1,a}^{-+}n_0-\omega_1^{-+}n_{0,a}\approx n_{0,a1}-\omega_{1,a}^{-+}n_0-\omega_1^{-+}(\omega_a^{-+}n_0+\omega_0^{-A}e_a^B\delta_{AB}-\omega_a^{-A}e_0^B\delta_{AB})\approx0. \tag{C3}$$

On the other hand, from (3.22), one can obtain

$$
\begin{aligned}
&(n_{0,a}-\omega_a^{-+}n_0-\omega_0^{-A}e_a^B\delta_{AB}+\omega_a^{-A}e_0^B\delta_{AB})_{,1}\\
&\approx n_{0,a1}-\omega_{a,1}^{-+}n_0-\omega_a^{-+}n_{0,1}-\omega_{0,1}^{-A}e_a^B\delta_{AB}-\omega_0^{-A}e_{a,1}^B\delta_{AB}+\omega_{a,1}^{-A}e_0^B\delta_{AB}+\omega_a^{-A}e_{0,1}^B\delta_{AB}\\
&\approx n_{0,a1}-\omega_{a,1}^{-+}n_0-\omega_a^{-+}\omega_1^{-+}n_0-\omega_{0,1}^{-A}e_a^B\delta_{AB}-\omega_0^{-B}(\omega_a^{-A}-\omega_1^{AC}e_a^D\delta_{CD})\delta_{AB}+\omega_{a,1}^{-A}e_0^B\delta_{AB}\\
&\quad+\omega_a^{-A}\delta_{AB}(\omega_0^{-B}-\omega_1^{+B}n_0-\omega_1^{BC}e_0^D\delta_{CD})\\
&\approx n_{0,a1}-\omega_{a,1}^{-+}n_0-\omega_a^{-+}\omega_1^{-+}n_0-\omega_a^{-A}\delta_{AB}\omega_1^{+B}n_0-\omega_{0,1}^{-A}e_a^B\delta_{AB}+\omega_0^{-B}\omega_1^{AC}e_a^D\delta_{CD}\delta_{AB}\\
&\quad+\omega_{a,1}^{-A}e_0^B\delta_{AB}-\omega_a^{-A}\delta_{AB}\omega_1^{BC}e_0^D\delta_{CD}\approx0.
\end{aligned} \tag{C4}
$$

Eq. (C1) requires

$$\begin{aligned}
&-\omega_{1,a}^{-+}n_0-\omega_1^{-+}\omega_a^{-+}n_0-\omega_1^{-+}\omega_0^{-A}e_a^B\delta_{AB}+\omega_1^{-+}\omega_a^{-A}e_0^B\delta_{AB}\\
\approx&-\omega_{a,1}^{-+}n_0-\omega_a^{-+}\omega_1^{-+}n_0-\omega_a^{-A}\delta_{AB}\omega_1^{+B}n_0-\omega_{0,1}^{-A}e_a^B\delta_{AB}+\omega_0^{-B}\omega_1^{AC}e_a^D\delta_{CD}\delta_{AB}\\
&+\omega_{a,1}^{-A}e_0^B\delta_{AB}-\omega_a^{-A}\delta_{AB}\omega_1^{BC}e_0^D\delta_{CD},
\end{aligned}\tag{C5}$$

which is equivalent to

$$\begin{aligned}
&\omega_{a,1}^{-+}n_0-\omega_{1,a}^{-+}n_0+\omega_a^{-A}\delta_{AB}\omega_1^{+B}n_0-\omega_1^{-+}\omega_0^{-A}e_a^B\delta_{AB}+\omega_{0,1}^{-A}e_a^B\delta_{AB}-\omega_0^{-B}\omega_1^{AC}e_a^D\delta_{CD}\delta_{AB}\\
&+\omega_1^{-+}\omega_a^{-A}e_0^B\delta_{AB}-\omega_{a,1}^{-A}e_0^B\delta_{AB}+\omega_a^{-A}\delta_{AB}\omega_1^{BC}e_0^D\delta_{CD}\\
\approx&F_{1a}^{-+}n_0+(\omega_{0,1}^{-A}-\omega_1^{-+}\omega_0^{-A}+\omega_0^{-C}\omega_1^{DA}\delta_{CD})e_a^B\delta_{AB}-F_{1a}^{-A}e_0^B\delta_{AB}\\
\approx&F_{1a}^{-+}n_0-F_{01}^{-A}e_a^B\delta_{AB}-F_{1a}^{-A}e_0^B\delta_{AB}=-(\eta_{IJ}F^{-I}\wedge e^J)_{01a}=0.
\end{aligned}\tag{C6}$$

The integrability conditions (C6) are Ricci identities.

From (3.22), one obtains

$$\epsilon^{ab}n_{0,ab}\approx\epsilon^{ab}(\omega_{a,b}^{-+}n_0+\omega_a^{-+}n_{0,b}+\omega_{0,b}^{-B}e_a^A\delta_{AB}+\omega_0^{-B}e_{a,b}^A\delta_{AB}-\omega_{a,b}^{-A}e_0^B\delta_{AB}-\omega_a^{-A}e_{0,b}^B\delta_{AB}).\tag{C7}$$

Eq. (C2) requires that

$$\begin{aligned}
&\epsilon^{ab}(\omega_{a,b}^{-+}n_0+\omega_a^{-+}n_{0,b}+\omega_{0,b}^{-B}e_a^A\delta_{AB}+\omega_0^{-B}e_{a,b}^A\delta_{AB}-\omega_{a,b}^{-A}e_0^B\delta_{AB}-\omega_a^{-A}e_{0,b}^B\delta_{AB})\\
\approx&\epsilon^{ab}[\omega_{a,b}^{-+}n_0+\omega_a^{-+}(\omega_b^{-+}n_0+\omega_0^{-A}e_b^B\delta_{AB}-\omega_b^{-A}e_0^B\delta_{AB})+\omega_{0,b}^{-B}e_a^A\delta_{AB}+\omega_0^{-B}e_{a,b}^A\delta_{AB}-\omega_{a,b}^{-A}e_0^B\delta_{AB}-\omega_a^{-A}e_{0,b}^B\delta_{AB}]\\
\approx&\epsilon^{ab}\omega_{a,b}^{-+}n_0+\epsilon^{ab}(-\omega_{a,b}^{-A}-\omega_a^{-+}\omega_b^{-A})e_0^B\delta_{AB}-\epsilon^{ab}\omega_a^{-A}e_{0,b}^B\delta_{AB}+\epsilon^{ab}(e_{a,b}^B+\omega_a^{-+}e_b^B)\omega_0^{-A}\delta_{AB}-\epsilon^{ab}\omega_{0,a}^{-A}e_b^B\delta_{AB}\\
\approx&\epsilon^{ab}\omega_{a,b}^{-+}n_0+\epsilon^{ab}(-\omega_{a,b}^{-A}-\omega_a^{-+}\omega_b^{-A})e_0^B\delta_{AB}-\epsilon^{ab}\omega_a^{-A}e_{0,b}^B\delta_{AB}+\epsilon^{ab}(\omega_a^{BC}e_b^D\delta_{CD}+\omega_a^{-+}e_b^B)\omega_0^{-A}\delta_{AB}-\epsilon^{ab}\omega_{0,a}^{-A}e_b^B\delta_{AB}\\
\approx&\epsilon^{ab}\omega_{a,b}^{-+}n_0+\epsilon^{ab}(-\omega_{a,b}^{-A}-\omega_a^{-+}\omega_b^{-A})e_0^B\delta_{AB}-\epsilon^{ab}\omega_a^{-A}e_{0,b}^B\delta_{AB}-\epsilon^{ab}(\omega_{0,a}^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-C}\omega_a^{DA}\delta_{CD})e_b^B\delta_{AB}\\
\approx&\epsilon^{ab}(\omega_{a,b}^{-+}-\omega_a^{+A}\omega_b^{-B}\delta_{AB})n_0+\epsilon^{ab}(-\omega_{a,b}^{-A}-\omega_a^{-+}\omega_b^{-A}+\omega_a^{-C}\omega_b^{DA}\delta_{CD})e_0^B\delta_{AB}\\
&-\epsilon^{ab}\omega_a^{-A}(e_{0,b}^B+\omega_b^{-B}l_0+\omega_b^{+B}n_0+\omega_b^{BC}e_0^D\delta_{CD}-\omega_0^{BC}e_b^D\delta_{CD})\delta_{AB}\\
&-\epsilon^{ab}(\omega_{0,a}^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-C}\omega_a^{DA}\delta_{CD}+\omega_a^{-C}\omega_0^{DA}\delta_{CD}+\omega_0^{-+}\omega_a^{-A})e_b^B\delta_{AB}\\
\approx&-\epsilon^{ab}(\omega_{0,a}^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-C}\omega_a^{DA}\delta_{CD}+\omega_a^{-C}\omega_0^{DA}\delta_{CD}+\omega_0^{-+}\omega_a^{-A})e_b^B\delta_{AB}-F_{23}^{-+}n_0+F_{23}^{-A}e_0^B\delta_{AB}-\epsilon^{ab}\omega_a^{-A}\lambda_b^B\delta_{AB}\\
\approx&\epsilon^{ab}\lambda_a^{-A}e_b^B\delta_{AB}-\epsilon^{ab}(\omega_{0,a}^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-C}\omega_a^{DA}\delta_{CD}+\omega_a^{-C}\omega_0^{DA}\delta_{CD}+\omega_0^{-+}\omega_a^{-A})e_b^B\delta_{AB}-F_{23}^{-+}n_0+F_{23}^{-A}e_0^B\delta_{AB}\\
\approx&\epsilon^{ab}F_{0a}^{-A}e_b^B\delta_{AB}-F_{23}^{-+}n_0+F_{23}^{-A}e_0^B\delta_{AB}=(\eta_{IJ}F^{-I}\wedge e^J)_{023}=0,
\end{aligned}\tag{C8}$$

Here, (3.22), (3.26), (3.32), and (3.43) have been used. The integrability condition (C8) is a Ricci identity.

Appendix D: Integrability of l_0

Similarly, the integrability conditions for l_0 require

$$l_{0,1a}-l_{0,a1}=0,\tag{D1}$$

$$\epsilon^{ab}l_{0,ab}=0.\tag{D2}$$

The left-hand side of (D1) is

$$\begin{aligned}
l_{0,1a}\approx&\omega_{0,a}^{-+}-\omega_{1,a}^{+A}e_0^B\delta_{AB}-\omega_1^{+A}e_{0,a}^B\delta_{AB}-\omega_{1,a}^{-+}l_0-\omega_1^{-+}l_{0,a}\\
\approx&\omega_{0,a}^{-+}-\omega_{1,a}^{+A}e_0^B\delta_{AB}-\omega_1^{+A}e_{0,a}^B\delta_{AB}-\omega_{1,a}^{-+}l_0+\omega_1^{-+}(\omega_a^{+A}e_0^B\delta_{AB}-\omega_0^{+A}e_a^B\delta_{AB}+\omega_a^{-+}l_0)\\
\approx&\omega_{0,a}^{-+}-(\omega_{1,a}^{+A}-\omega_1^{-+}\omega_a^{+A})e_0^B\delta_{AB}-\omega_1^{+A}e_{0,a}^B\delta_{AB}-(\omega_{1,a}^{-+}-\omega_1^{-+}\omega_a^{-+})l_0-\omega_1^{-+}\omega_0^{+A}e_a^B\delta_{AB}\\
\approx&\omega_{0,a}^{-+}-(\omega_{1,a}^{+A}-\omega_1^{-+}\omega_a^{+A})e_0^B\delta_{AB}+(-\lambda_a^A+\omega_a^{-A}l_0+\omega_a^{+A}n_0+\omega_a^{AC}e_0^D\delta_{CD}-\omega_0^{AC}e_a^D\delta_{CD})\omega_1^{+B}\delta_{AB}
\end{aligned}$$

$$
\begin{aligned}
&-(\omega_{1,a}^{-+}-\omega_1^{-+}\omega_a^{-+})l_0-\omega_1^{-+}\omega_0^{+A}e_a^B\delta_{AB}\\
&\approx\omega_{0,a}^{-+}-(\omega_{1,a}^{+A}-\omega_1^{-+}\omega_a^{+A}-\omega_1^{+C}\omega_a^{DA}\delta_{CD})e_0^B\delta_{AB}-\lambda_a^A\omega_1^{+B}\delta_{AB}+n_0\omega_a^{+A}\omega_1^{+B}\delta_{AB}\\
&-(\omega_{1,a}^{-+}-\omega_1^{-+}\omega_a^{-+}-\omega_a^{-A}\omega_1^{+B}\delta_{AB})l_0-(\omega_1^{+C}\omega_0^{DB}\delta_{CD}+\omega_1^{-+}\omega_0^{+B})e_a^A\delta_{AB},
\end{aligned}\tag{D3}
$$

and the right-hand side of (D1) is

$$
\begin{aligned}
l_{0,a1}&\approx\omega_{0,1}^{+A}e_a^B\delta_{AB}+\omega_0^{+B}e_{a,1}^A\delta_{AB}-\omega_{a,1}^{+A}e_0^B\delta_{AB}-\omega_a^{+A}e_{0,1}^B\delta_{AB}-\omega_{a,1}^{-+}l_0-\omega_a^{-+}l_{0,1}\\
&\approx\omega_{0,1}^{+A}e_a^B\delta_{AB}+(\omega_a^{-A}-\omega_1^{AC}e_a^D\delta_{CD})\omega_0^{+B}\delta_{AB}-\omega_{a,1}^{+A}e_0^B\delta_{AB}-\omega_a^{+A}(\omega_0^{-B}-\omega_1^{+B}n_0-\omega_1^{BC}e_0^D\delta_{CD})\delta_{AB}\\
&-\omega_{a,1}^{-+}l_0-\omega_a^{-+}(\omega_0^{-+}-\omega_1^{+A}e_0^B\delta_{AB}-\omega_1^{-+}l_0)\\
&\approx\omega_{0,1}^{+A}e_a^B\delta_{AB}+(\omega_a^{-A}-\omega_1^{AC}e_a^D\delta_{CD})\omega_0^{+B}\delta_{AB}-(\omega_{a,1}^{+A}-\omega_a^{+C}\omega_1^{DA}\delta_{CD}-\omega_a^{-+}\omega_1^{+A})e_0^B\delta_{AB}\\
&-(\omega_0^{-+}\omega_a^{-+}+\omega_0^{-A}\omega_a^{+B}\delta_{AB})+\omega_a^{+A}\omega_1^{+B}n_0\delta_{AB}-(\omega_{a,1}^{-+}-\omega_a^{-+}\omega_1^{-+})l_0.
\end{aligned}\tag{D4}
$$

Eq. (D1) requires

$$
\begin{aligned}
&\omega_{0,a}^{-+}-(\omega_{1,a}^{+A}-\omega_1^{-+}\omega_a^{+A})e_0^B\delta_{AB}-\omega_1^{+A}e_{0,a}^B\delta_{AB}-(\omega_{1,a}^{-+}-\omega_1^{-+}\omega_a^{-+})l_0-\omega_1^{-+}\omega_0^{+A}e_a^B\delta_{AB}\\
&\approx\omega_{0,1}^{+A}e_a^B\delta_{AB}+(\omega_a^{-A}-\omega_1^{AC}e_a^D\delta_{CD})\omega_0^{+B}\delta_{AB}-(\omega_{a,1}^{+A}-\omega_a^{+C}\omega_1^{DA}\delta_{CD}-\omega_a^{-+}\omega_1^{+A})e_0^B\delta_{AB}\\
&-(\omega_0^{-+}\omega_a^{-+}+\omega_0^{-A}\omega_a^{+B}\delta_{AB})+\omega_a^{+A}\omega_1^{+B}n_0\delta_{AB}-(\omega_{a,1}^{-+}-\omega_a^{-+}\omega_1^{-+})l_0,
\end{aligned}\tag{D5}
$$

which is equivalent to

$$
\begin{aligned}
&\omega_{0,a}^{-+}+(\omega_{a,1}^{+A}-\omega_a^{+C}\omega_1^{DA}\delta_{CD}-\omega_a^{-+}\omega_1^{+A}-\omega_{1,a}^{+A}+\omega_1^{-+}\omega_a^{+A})e_0^B\delta_{AB}-\omega_1^{+A}e_{0,a}^B\delta_{AB}+\omega_{a,1}^{-+}l_0-\omega_{1,a}^{-+}l_0\\
&-\omega_1^{-+}\omega_0^{+A}e_a^B\delta_{AB}-\omega_{0,1}^{+A}e_a^B\delta_{AB}-(\omega_a^{-A}-\omega_1^{AC}e_a^D\delta_{CD})\omega_0^{+B}\delta_{AB}+\omega_a^{+A}\omega_0^{-B}\delta_{AB}-\omega_a^{+A}\omega_1^{+B}n_0\delta_{AB}+\omega_a^{-+}\omega_0^{-+}\\
\approx&-(\omega_{0,1}^{+A}+\omega_1^{-+}\omega_0^{+A}-\omega_0^{+C}\omega_1^{DA}\delta_{CD})e_a^B\delta_{AB}+(\omega_{0,a}^{-+}+\omega_a^{-+}\omega_0^{-+})-\omega_1^{+A}e_{0,a}^B\delta_{AB}-\omega_a^{+A}\omega_1^{+B}n_0\delta_{AB}\\
&-\omega_a^{-A}\omega_0^{+B}\delta_{AB}+\omega_a^{+A}\omega_0^{-B}\delta_{AB}+(\omega_{a,1}^{-+}-\omega_{1,a}^{-+})l_0+(\omega_{a,1}^{+A}-\omega_{1,a}^{+A}-\omega_a^{+C}\omega_1^{DA}\delta_{CD}-\omega_a^{-+}\omega_1^{+A}+\omega_1^{-+}\omega_a^{+A})e_0^B\delta_{AB}\\
\approx&-(\omega_{0,1}^{+A}+\omega_1^{-+}\omega_0^{+A}-\omega_0^{+C}\omega_1^{DA}\delta_{CD})e_a^B\delta_{AB}+(\omega_{0,a}^{-+}+\omega_a^{-+}\omega_0^{-+})-\omega_1^{+A}e_{0,a}^B\delta_{AB}-\omega_a^{+A}\omega_1^{+B}n_0\delta_{AB}\\
&-\omega_a^{-A}\omega_0^{+B}\delta_{AB}+\omega_a^{+A}\omega_0^{-B}\delta_{AB}+F_{1a}^{-+}l_0-\omega_a^{-A}\omega_1^{+B}\delta_{AB}l_0+F_{1a}^{+A}e_0^B\delta_{AB}-\omega_1^{+C}\omega_a^{DA}\delta_{CD}e_0^B\delta_{AB}\\
\approx&-(e_{0,a}^A+\omega_a^{+A}n_0+\omega_a^{-A}l_0+\omega_a^{AC}e_0^D\delta_{CD})\omega_1^{+B}\delta_{AB}-(\omega_{0,1}^{+A}+\omega_1^{-+}\omega_0^{+A}-\omega_0^{+C}\omega_1^{DA}\delta_{CD})e_a^B\delta_{AB}\\
&+(\omega_{0,a}^{-+}+\omega_a^{-+}\omega_0^{-+}-\omega_a^{-A}\omega_0^{+B}\delta_{AB}+\omega_a^{+A}\omega_0^{-B}\delta_{AB})+F_{1a}^{-+}l_0+F_{1a}^{+A}e_0^B\delta_{AB}\\
\approx&-(e_{0,a}^A+\omega_a^{+A}n_0+\omega_a^{-A}l_0+\omega_a^{AC}e_0^D\delta_{CD}-\omega_0^{AC}e_a^D\delta_{CD})\omega_1^{+B}\delta_{AB}\\
&-(\omega_{0,1}^{+A}+\omega_1^{-+}\omega_0^{+A}-\omega_0^{+C}\omega_1^{DA}\delta_{CD}+\omega_1^{+C}\omega_0^{DA}\delta_{CD}-\omega_0^{-+}\omega_1^{+A})e_a^B\delta_{AB}\\
&+(\omega_{0,a}^{-+}-\omega_a^{-A}\omega_0^{+B}\delta_{AB}+\omega_a^{+A}\omega_0^{-B}\delta_{AB})+F_{1a}^{-+}l_0+F_{1a}^{+A}e_0^B\delta_{AB}\\
=&-\omega_1^{+A}\lambda_a^B\delta_{AB}-(\omega_{0,1}^{+A}+\omega_1^{-+}\omega_0^{+A}-\omega_0^{+C}\omega_1^{DA}\delta_{CD}+\omega_1^{+C}\omega_0^{DA}\delta_{CD}-\omega_0^{-+}\omega_1^{+A})e_a^B\delta_{AB}\\
&+(\omega_{0,a}^{-+}-\omega_a^{-A}\omega_0^{+B}\delta_{AB}+\omega_a^{+A}\omega_0^{-B}\delta_{AB})+F_{1a}^{-+}l_0+F_{1a}^{+A}e_0^B\delta_{AB}\\
\approx&\lambda_1^{+A}e_a^B\delta_{AB}-(\omega_{0,1}^{+A}+\omega_1^{-+}\omega_0^{+A}-\omega_0^{+C}\omega_1^{DA}\delta_{CD}+\omega_1^{+C}\omega_0^{DA}\delta_{CD}-\omega_0^{-+}\omega_1^{+A})e_a^B\delta_{AB}\\
&-\lambda_a^{-+}+(\omega_{0,a}^{-+}-\omega_a^{-A}\omega_0^{+B}\delta_{AB}+\omega_a^{+A}\omega_0^{-B}\delta_{AB})+F_{1a}^{-+}l_0+F_{1a}^{+A}e_0^B\delta_{AB}\\
\approx&F_{01}^{+A}e_a^B\delta_{AB}-F_{0a}^{-+}+F_{1a}^{-+}l_0+F_{1a}^{+A}e_0^B\delta_{AB}=(\eta_{IJ}F^{+I}\wedge e^J)_{01a}=0.
\end{aligned}\tag{D6}
$$

Here, (3.26), (3.44), (3.64), and (3.66) have been used. The integrability conditions (D6) are Ricci identities.

From (3.24), one obtains

$$
l_{0,ab}\approx\omega_{0,b}^{+A}e_a^B\delta_{AB}+\omega_0^{+A}e_{a,b}^B\delta_{AB}-\omega_{a,b}^{+A}e_0^B\delta_{AB}-\omega_a^{+A}e_{0,b}^B\delta_{AB}-\omega_{a,b}^{-+}l_0-\omega_a^{-+}l_{0,b}.\tag{D7}
$$

Eq. (D2) requires

$$
\begin{aligned}
\epsilon^{ab}l_{0,ab}&\approx\epsilon^{ab}(\omega_{0,b}^{+A}e_a^B\delta_{AB}+\omega_0^{+A}e_{a,b}^B\delta_{AB}-\omega_{a,b}^{+A}e_0^B\delta_{AB}-\omega_a^{+A}e_{0,b}^B\delta_{AB}-\omega_{a,b}^{-+}l_0-\omega_a^{-+}l_{0,b})\\
&\approx\epsilon^{ab}[\omega_{0,b}^{+A}e_a^B\delta_{AB}+\omega_0^{+A}e_{a,b}^B\delta_{AB}-\omega_{a,b}^{+A}e_0^B\delta_{AB}-\omega_a^{+A}e_{0,b}^B\delta_{AB}-\omega_{a,b}^{-+}l_0-\omega_a^{-+}(\omega_0^{+A}e_b^B\delta_{AB}-\omega_b^{+A}e_0^B\delta_{AB}-\omega_b^{-+}l_0)]
\end{aligned}
$$

$$
\begin{aligned}
&\approx -\epsilon^{ab}\omega_{a,b}^{-+}l_0+\epsilon^{ab}(-\omega_{a,b}^{+A}+\omega_a^{-+}\omega_b^{+A})e_0^B\delta_{AB}-\epsilon^{ab}\omega_a^{+A}e_{0,b}^B\delta_{AB}-\epsilon^{ab}\omega_{0,a}^{+A}e_b^B\delta_{AB}-\epsilon^{ab}\omega_0^{+A}(e_{b,a}^B+\omega_a^{-+}e_b^B)\delta_{AB}\\
&\approx -\epsilon^{ab}\omega_{a,b}^{-+}l_0+\epsilon^{ab}(-\omega_{a,b}^{+A}+\omega_a^{-+}\omega_b^{+A})e_0^B\delta_{AB}-\epsilon^{ab}\omega_a^{+A}e_{0,b}^B\delta_{AB}-\epsilon^{ab}\omega_{0,a}^{+A}e_b^B\delta_{AB}+\epsilon^{ab}\omega_a^{BC}e_b^D\delta_{CD}\omega_0^{+A}\delta_{AB}\\
&\quad -\epsilon^{ab}\omega_a^{-+}e_b^B\omega_0^{+A}\delta_{AB}\\
&\approx -\epsilon^{ab}\omega_{a,b}^{-+}l_0+\epsilon^{ab}(-\omega_{a,b}^{+A}+\omega_a^{-+}\omega_b^{+A})e_0^B\delta_{AB}-\epsilon^{ab}\omega_a^{+A}e_{0,b}^B\delta_{AB}+\epsilon^{ab}(-\omega_{0,a}^{+A}+\omega_0^{+C}\omega_a^{DA}\delta_{CD}-\omega_a^{-+}\omega_0^{+A})e_b^B\delta_{AB}\\
&\approx F_{23}^{-+}l_0+F_{23}^{+A}e_0^B\delta_{AB}-\epsilon^{ab}\omega_a^{+A}\lambda_b^B\delta_{AB}+\epsilon^{ab}(-\omega_{0,a}^{+A}+\omega_0^{+C}\omega_a^{DA}\delta_{CD}-\omega_a^{-+}\omega_0^{+A})e_b^B\delta_{AB}\approx 0,\\
&\approx F_{23}^{-+}l_0+F_{23}^{+A}e_0^B\delta_{AB}+\epsilon^{ab}(\lambda_a^{+A}-\omega_{0,a}^{+A}+\omega_0^{+C}\omega_a^{DA}\delta_{CD}-\omega_a^{-+}\omega_0^{+A})e_b^B\delta_{AB}\\
&\approx F_{23}^{-+}l_0+F_{23}^{+A}e_0^B\delta_{AB}+\epsilon^{ab}F_{0a}^{+A}e_b^B\delta_{AB}=(\eta_{IJ}F^{+I}\wedge e^J)_{023}=0,
\end{aligned}
\tag{D8}
$$

where (3.66) has been used. The integrability condition (D8) is a Ricci identity.

Appendix E: Integrability of e_0^A

Finally, the integrability conditions for e_0^A require

$$e_{0,1a}^A-e_{0,a1}^A=0, \tag{E1}$$

$$\epsilon^{ab}e_{0,ab}^A=0. \tag{E2}$$

From (3.25) and (3.26), one has

$$
\begin{aligned}
e_{0,1a}^A&\approx\omega_{0,a}^{-A}-\omega_{1,a}^{+A}n_0-\omega_1^{+A}n_{0,a}-\omega_{1,a}^{AB}e_0^C\delta_{BC}-\omega_1^{AB}e_{0,a}^C\delta_{BC}\\
&\approx\omega_{0,a}^{-A}-\omega_{1,a}^{+A}n_0-\omega_{1,a}^{AB}e_0^C\delta_{BC}-\omega_1^{AB}\delta_{BC}(\lambda_a^C-\omega_a^{-C}l_0-\omega_a^{+C}n_0-\omega_a^{CD}e_0^E\delta_{DE}+\omega_0^{CD}e_a^E\delta_{DE})\\
&\quad-\omega_1^{+A}(\omega_a^{-+}n_0+\omega_0^{-B}e_a^C\delta_{BC}-\omega_a^{-B}e_0^C\delta_{BC}),
\end{aligned}
\tag{E3}
$$

$$
\begin{aligned}
e_{0,a1}^A&\approx\lambda_{a,1}^A-\omega_{a,1}^{-A}l_0-\omega_a^{-A}l_{0,1}-\omega_{a,1}^{+A}n_0-\omega_a^{+A}n_{0,1}-\omega_{a,1}^{AB}e_0^C\delta_{BC}-\omega_a^{AB}e_{0,1}^C\delta_{BC}+\omega_{0,1}^{AB}e_a^C\delta_{BC}+\omega_0^{AB}e_{a,1}^C\delta_{BC}\\
&\approx\lambda_{a,1}^A-\omega_{a,1}^{-A}l_0-\omega_a^{-A}(\omega_0^{-+}-\omega_1^{+A}e_0^B\delta_{AB}-\omega_1^{-+}l_0)-\omega_{a,1}^{+A}n_0-\omega_a^{+A}\omega_1^{-+}n_0-\omega_{a,1}^{AB}e_0^C\delta_{BC}\\
&\quad-\omega_a^{AB}(\omega_0^{-C}-\omega_1^{+C}n_0-\omega_1^{CD}e_0^E\delta_{DE})\delta_{BC}+\omega_{0,1}^{AB}e_a^C\delta_{BC}+\omega_0^{AB}(\omega_a^{-C}-\omega_1^{CD}e_a^E\delta_{DE})\delta_{BC}.
\end{aligned}
\tag{E4}
$$

Eq. (E1) requires

$$
\begin{aligned}
&\omega_{0,a}^{-A}-\omega_{1,a}^{+A}n_0-\omega_{1,a}^{AB}e_0^C\delta_{BC}-\omega_1^{AB}\delta_{BC}(\lambda_a^C-\omega_a^{-C}l_0-\omega_a^{+C}n_0-\omega_a^{CD}e_0^E\delta_{DE}+\omega_0^{CD}e_a^E\delta_{DE})\\
&-\omega_1^{+A}(\omega_a^{-+}n_0+\omega_0^{-B}e_a^C\delta_{BC}-\omega_a^{-B}e_0^C\delta_{BC})-\lambda_{a,1}^A+\omega_{a,1}^{-A}l_0+\omega_a^{-A}(\omega_0^{-+}-\omega_1^{+B}e_0^C\delta_{BC}-\omega_1^{-+}l_0)\\
&+\omega_{a,1}^{+A}n_0+\omega_a^{+A}\omega_1^{-+}n_0+\omega_{a,1}^{AB}e_0^C\delta_{BC}+\omega_a^{AB}(\omega_0^{-C}-\omega_1^{+C}n_0-\omega_1^{CD}e_0^E\delta_{DE})\delta_{BC}-\omega_{0,1}^{AB}e_a^C\delta_{BC}\\
&-\omega_0^{AB}(\omega_a^{-C}-\omega_1^{CD}e_a^E\delta_{DE})\delta_{BC}\\
\approx&\,\omega_{0,a}^{-A}-\omega_{1,a}^{+A}n_0-\omega_{1,a}^{AB}e_0^C\delta_{BC}-\omega_1^{AB}\delta_{BC}\lambda_a^C+\omega_1^{AB}\delta_{BC}\omega_a^{-C}l_0+\omega_1^{AB}\delta_{BC}\omega_a^{+C}n_0+\omega_1^{AB}\delta_{BC}\omega_a^{CD}e_0^E\delta_{DE}\\
&-\omega_1^{AB}\delta_{BC}\omega_0^{CD}e_a^E\delta_{DE}-\omega_1^{+A}\omega_a^{-+}n_0-\omega_1^{+A}\omega_0^{-B}e_a^C\delta_{BC}+\omega_1^{+A}\omega_a^{-B}e_0^C\delta_{BC}-\lambda_{a,1}^A+\omega_{a,1}^{-A}l_0+\omega_a^{-A}\omega_0^{-+}\\
&-\omega_a^{-A}\omega_1^{+B}e_0^C\delta_{BC}-\omega_a^{-A}\omega_1^{-+}l_0+\omega_{a,1}^{+A}n_0+\omega_a^{+A}\omega_1^{-+}n_0+\omega_{a,1}^{AB}e_0^C\delta_{BC}+\omega_a^{AB}\omega_0^{-C}\delta_{BC}-\omega_a^{AB}\omega_1^{+C}n_0\delta_{BC}\\
&-\omega_a^{AB}\omega_1^{CD}e_0^E\delta_{DE}\delta_{BC}-\omega_{0,1}^{AB}e_a^C\delta_{BC}-\omega_0^{AB}\omega_a^{-C}\delta_{BC}+\omega_0^{AB}\omega_1^{CD}e_a^E\delta_{DE}\delta_{BC}\\
\approx&\,(\omega_{0,a}^{-A}+\omega_a^{-A}\omega_0^{-+}+\omega_a^{AB}\omega_0^{-C}\delta_{BC}-\omega_0^{AB}\omega_a^{-C}\delta_{BC}-\lambda_a^{-A})+(\omega_{a,1}^{-A}-\omega_a^{-A}\omega_1^{-+}+\omega_1^{AB}\omega_a^{-C}\delta_{BC})l_0\\
&+(\omega_{a,1}^{+A}-\omega_{1,a}^{+A}+\omega_1^{AB}\omega_a^{+C}\delta_{BC}-\omega_1^{+A}\omega_a^{-+}+\omega_a^{+A}\omega_1^{-+}-\omega_a^{AB}\omega_1^{+C}\delta_{BC})n_0\\
&+(\omega_{a,1}^{AB}-\omega_{1,a}^{AB}+\omega_1^{AD}\delta_{DE}\omega_a^{EB}+\omega_1^{+A}\omega_a^{-B}-\omega_a^{-A}\omega_1^{+B}-\omega_a^{AD}\omega_1^{EB}\delta_{DE})e_0^C\delta_{BC}\\
&+(\lambda_1^{AB}-\omega_{0,1}^{AB}-\omega_1^{AD}\omega_0^{EB}\delta_{DE}-\omega_1^{+A}\omega_0^{-B}+\omega_0^{AD}\omega_1^{EB}\delta_{DE})e_a^C\delta_{BC}\\
\approx&\,(\omega_{0,a}^{-A}+\omega_0^{-+}\omega_a^{-A}-\omega_a^{-+}\omega_0^{-A}-\omega_0^{-B}\omega_a^{CA}\delta_{BC}+\omega_a^{-B}\omega_0^{CA}\delta_{BC}-\lambda_a^{-A})+F_{1a}^{-A}l_0+F_{1a}^{+A}n_0+F_{1a}^{AB}e_0^C\delta_{BC}\\
&+(\lambda_1^{AB}-\omega_{0,1}^{AB}+\omega_0^{-A}\omega_1^{+B}-\omega_1^{AD}\omega_0^{EB}\delta_{DE}-\omega_1^{+A}\omega_0^{-B}+\omega_0^{AD}\omega_1^{EB}\delta_{DE})e_a^C\delta_{BC}\\
\approx&-F_{0a}^{-A}+F_{1a}^{-A}l_0+F_{1a}^{+A}n_0+F_{1a}^{AB}e_0^C\delta_{BC}+F_{01}^{AB}e_a^C\delta_{BC}=(\eta_{IJ}F^{AI}\wedge e^J)_{01a}=0,
\end{aligned}
\tag{E5}
$$

where (3.46) has been used. The integrability conditions (E5) are Ricci identities.

From (3.26), one obtains

$$e^A_{0,ab}\approx\lambda^A_{a,b}-\omega^{-A}_{a,b}l_0-\omega^{-A}_a l_{0,b}-\omega^{+A}_{a,b}n_0-\omega^{+A}_a n_{0,b}-\omega^{AB}_{a,b}e^C_0\delta_{BC}-\omega^{AB}_a e^C_{0,b}\delta_{BC}+\omega^{AB}_{0,b}e^C_a\delta_{BC}+\omega^{AB}_0 e^C_{a,b}\delta_{BC}. \tag{E6}$$

Eq. (E2) requires

$$\begin{aligned}\epsilon^{ab}e^A_{0,ab}\approx&\epsilon^{ab}\lambda^A_{a,b}-\epsilon^{ab}\omega^{-A}_{a,b}l_0-\epsilon^{ab}\omega^{-A}_a l_{0,b}-\epsilon^{ab}\omega^{+A}_{a,b}n_0-\epsilon^{ab}\omega^{+A}_a n_{0,b}-\epsilon^{ab}\omega^{AB}_{a,b}e^C_0\delta_{BC}-\epsilon^{ab}\omega^{AB}_a e^C_{0,b}\delta_{BC}\\&+\epsilon^{ab}\omega^{AB}_{0,b}e^C_a\delta_{BC}+\epsilon^{ab}\omega^{AB}_0 e^C_{a,b}\delta_{BC}\\&\approx\lambda^A_{a,b}-\omega^{-A}_{a,b}l_0+\omega^{-A}_a(\omega^{+B}_b e^C_0\delta_{BC}-\omega^{+B}_0 e^C_b\delta_{BC}+\omega^{-+}_b l_0)-\omega^{+A}_{a,b}n_0-\omega^{+A}_a(\omega^{-+}_b n_0+\omega^{-B}_0 e^C_b\delta_{BC}-\omega^{-B}_b e^C_0\delta_{BC})\\&-\omega^{AB}_{a,b}e^C_0\delta_{BC}-\omega^{AB}_a(\lambda^C_b-\omega^{-C}_b l_0-\omega^{+C}_b n_0-\omega^{CD}_b e^E_0\delta_{DE}-\omega^{CD}_0 e^E_b\delta_{DE})\delta_{BC}+\omega^{AB}_{0,b}e^C_a\delta_{BC}+\omega^{AB}_0 e^C_{a,b}\delta_{BC}\\&\approx\epsilon^{ab}\lambda^A_{a,b}-\epsilon^{ab}\omega^{-A}_{a,b}l_0+\epsilon^{ab}\omega^{-A}_a\omega^{+B}_b e^C_0\delta_{BC}-\epsilon^{ab}\omega^{-A}_a\omega^{+B}_0 e^C_b\delta_{BC}+\epsilon^{ab}\omega^{-A}_a\omega^{-+}_b l_0-\epsilon^{ab}\omega^{+A}_{a,b}n_0\\&-\epsilon^{ab}\omega^{+A}_a\omega^{-+}_b n_0-\epsilon^{ab}\omega^{+A}_a\omega^{-B}_0 e^C_b\delta_{BC}+\epsilon^{ab}\omega^{+A}_a\omega^{-B}_b e^C_0\delta_{BC}-\epsilon^{ab}\omega^{AB}_{a,b}e^C_0\delta_{BC}-\epsilon^{ab}\omega^{AB}_a\lambda^C_b\delta_{BC}\\&+\epsilon^{ab}\omega^{AB}_a\omega^{-C}_b l_0\delta_{BC}+\epsilon^{ab}\omega^{AB}_a\omega^{+C}_b n_0\delta_{BC}+\epsilon^{ab}\omega^{AB}_a\omega^{CD}_b e^E_0\delta_{DE}\delta_{BC}+\epsilon^{ab}\omega^{AB}_a\omega^{CD}_0 e^E_b\delta_{DE}\delta_{BC}\\&+\epsilon^{ab}\omega^{AB}_{0,b}e^C_a\delta_{BC}+\epsilon^{ab}\omega^{AB}_0\omega^{CD}_a e^E_b\delta_{DE}\delta_{BC}\\&\approx(-\epsilon^{ab}\omega^{+A}_{a,b}-\epsilon^{ab}\omega^{+C}_a\omega^{AB}_b\delta_{BC}+\epsilon^{ab}\omega^{-+}_a\omega^{+A}_b)n_0+(-\epsilon^{ab}\omega^{-A}_{a,b}-\epsilon^{ab}\omega^{-+}_a\omega^{-A}_b+\epsilon^{ab}\omega^{-B}_a\omega^{CA}_b\delta_{BC})l_0\\&+(\epsilon^{ab}\omega^{-A}_a\omega^{+B}_b+\epsilon^{ab}\omega^{+A}_a\omega^{-B}_b-\epsilon^{ab}\omega^{AB}_{a,b}+\epsilon^{ab}\omega^{AD}_a\omega^{EB}_b\delta_{DE})e^C_0\delta_{BC}\\&+(\epsilon^{ab}\lambda^{AB}_a-\epsilon^{ab}\omega^{-A}_a\omega^{+B}_0-\epsilon^{ab}\omega^{+A}_a\omega^{-B}_0+\epsilon^{ab}\omega^{AD}_a\omega^{EB}_0\delta_{DE}-\epsilon^{ab}\omega^{AB}_{0,a}+\epsilon^{ab}\omega^{AD}_0\omega^{EB}_a\delta_{DE})e^C_b\delta_{BC}\\&\approx\epsilon^{ab}(\lambda^{AB}_a-\omega^{AB}_{0,a}-\omega^{-A}_a\omega^{+B}_0-\omega^{+A}_a\omega^{-B}_0+\omega^{AD}_a\omega^{EB}_0\delta_{DE}+\omega^{AD}_0\omega^{EB}_a\delta_{DE})e^C_b\delta_{BC}+F^{+A}_{23}n_0+F^{-A}_{23}l_0+F^{AB}_{23}e^C_0\delta_{BC}\\&\approx\epsilon^{ab}F^{AB}_{0a}e^C_b\delta_{BC}+F^{+A}_{23}n_0+F^{-A}_{23}l_0+F^{AB}_{23}e^C_0\delta_{BC}=(\eta_{IJ}F^{AI}\wedge e^J)_{023}=0,\end{aligned} \tag{E7}$$

where (3.22), (3.24), (3.26), (3.32), and (3.47) have been used. The integrability conditions (E7) are Ricci identities.

Appendix F: Poisson brackets among the constraints

All non-zero Poisson brackets among the constraints are listed as follows:

$$\{\pi^a_A(x),(\pi^1_{-+}-2\epsilon_{BC}\epsilon^{bc}e^B_b e^C_c)(y)\}=4\epsilon_{AB}\epsilon^{ab}e^B_b(y)\delta(x-y), \tag{F1}$$

$$\{\pi^a_A(x),(\pi^b_{-B}-4\epsilon_{BC}\epsilon^{bc}e^C_c)(y)\}=4\epsilon_{AB}\epsilon^{ab}\delta(x-y), \tag{F2}$$

$$\{\pi^a_A(x),(\epsilon^{bc}\omega^{-B}_b e^C_c\delta_{BC})(y)\}=\epsilon^{ab}\omega^{-A}_b(y)\delta(x-y), \tag{F3}$$

$$\{\pi^a_A(x),(\omega^{-+}_b-\omega^{+B}_1 e^C_b\delta_{BC})(y)\}=\omega^{+A}_1(y)\delta(x-y)\delta^a_b, \tag{F4}$$

$$\{\pi^a_A(x),(\epsilon^{bc}\omega^{+B}_b e^C_c\delta_{BC})(y)\}=\epsilon^{ab}\omega^{+A}_b(y)\delta(x-y), \tag{F5}$$

$$\{\pi^a_A(x),(e^B_{b,1}-\omega^{-B}_b+\omega^{BC}_1 e^D_b\delta_{CD})(y)\}=\omega^{AB}_1(y)\delta(x-y)\delta^a_b-\delta(x-y)_{,y^1}\delta^B_A\delta^a_b, \tag{F6}$$

$$\{\pi^a_A(x),\epsilon^{bc}(e^B_{b,c}-\omega^{BC}_b e^D_c\delta_{CD})(y)\}=\epsilon^{ab}\omega^{+A}_b(y)\delta(x-y)-\epsilon^{ab}\delta(x-y)_{,y^b}\delta^B_A, \tag{F7}$$

$$\{\pi^a_A(x),(\epsilon_{BC}\epsilon^{bc}F^{-B}_{1b}e^C_c)(y)\}=\epsilon_{AB}\epsilon^{ab}F^{-B}_{1b}(y)\delta(x-y), \tag{F8}$$

$$\{\pi^a_A(x),(\epsilon_{BC}\epsilon^{bc}F^{+B}_{1b}e^C_c+F^{23}_{23})(y)\}=\epsilon_{AB}\epsilon^{ab}F^{+B}_{1b}(y)\delta(x-y), \tag{F9}$$

$$\{\pi^a_A(x),(F^{-B}_{23}+\epsilon^{bc}e^B_b F^{-+}_{1c})(y)\}=\epsilon^{ab}F^{-+}_{1b}(y)\delta(x-y)\delta^B_A, \tag{F10}$$

$$\{\pi^a_A(x),(\epsilon^{bc}F^{+2}_{1b}e^3_c+\epsilon^{bc}F^{+3}_{1b}e^2_c)(y)\}=-\epsilon^{ab}F^{+2}_{1b}(y)\delta(x-y)\delta^3_A-\epsilon^{ab}F^{+3}_{1b}(y)\delta(x-y)\delta^2_A, \tag{F11}$$

$$\{\pi^a_A(x),(\epsilon^{bc}F^{+2}_{1b}e^2_c-\epsilon^{bc}F^{+3}_{1b}e^3_c)(y)\}=-\epsilon^{ab}F^{+2}_{1b}(y)\delta(x-y)\delta^2_A+\epsilon^{ab}F^{+3}_{1b}(y)\delta(x-y)\delta^3_A, \tag{F12}$$

$$\{(\pi^1_{-+}-2\epsilon_{AB}\epsilon^{ab}e^A_a e^B_b)(x),(\epsilon_{CD}\epsilon^{cd}F^{-C}_{1c}e^D_d)(y)\}=\epsilon_{AB}\epsilon^{ab}\omega^{-A}_a(y)e^B_b(y)\delta(x-y), \tag{F13}$$

$$\{(\pi^1_{-+}-2\epsilon_{AB}\epsilon^{ab}e^A_a e^B_b)(x),(\epsilon_{CD}\epsilon^{cd}F^{+C}_{1c}e^D_d+F^{23}_{23})(y)\}=\epsilon_{AB}\epsilon^{ab}\omega^{+A}_a(y)e^B_b(y)\delta(x-y), \tag{F14}$$

$$\{(\pi^1_{-+}-2\epsilon_{AB}\epsilon^{ab}e^A_ae^B_b)(x),(F^{-C}_{23}+\epsilon^{cd}e^C_cF^{-+}_{1d})(y)\}=\epsilon^{ab}e^C_a(y)\delta(x-y)_{,y^b}, \tag{F15}$$

$$\{(\pi^1_{-+}-2\epsilon_{AB}\epsilon^{ab}e^A_ae^B_b)(x),(\epsilon^{cd}F^{+2}_{1c}e^3_d+\epsilon^{cd}F^{+3}_{1c}e^2_d)(y)\}=-[\epsilon^{ab}\omega^{+2}_a(y)e^3_b(y)+\epsilon^{ab}\omega^{+3}_a(y)e^2_b(y)]\delta(x-y), \tag{F16}$$

$$\{(\pi^1_{-+}-2\epsilon_{AB}\epsilon^{ab}e^A_ae^B_b)(x),(\epsilon^{cd}F^{+2}_{1c}e^2_d-\epsilon^{cd}F^{+3}_{1c}e^3_d)(y)\}=\epsilon^{ab}\omega^{+3}_a(y)e^3_b(y)\delta(x-y)-\epsilon^{ab}\omega^{+2}_a(y)e^2_b(y)\delta(x-y), \tag{F17}$$

$$\{\pi^a_{-+}(x),(\omega^{-+}_b-\omega^{+A}_1e^B_b\delta_{AB})(y)\}=-\delta(x-y)\delta^a_b, \tag{F18}$$

$$\{\pi^a_{-+}(x),(\epsilon_{AB}\epsilon^{bc}F^{+A}_{1b}e^B_c+F^{23}_{23})(y)\}=\epsilon_{AB}\epsilon^{ab}\omega^{+A}_1(y)e^B_b(y)\delta(x-y), \tag{F19}$$

$$\{\pi^a_{-+}(x),(F^{-A}_{23}+\epsilon^{bc}e^A_bF^{-+}_{1c})(y)\}=\epsilon^{ab}e^A_b(y)\delta(x-y)_{,y^1}, \tag{F20}$$

$$\{\pi^a_{-+}(x),(\epsilon^{bc}F^{+2}_{1b}e^3_c+\epsilon^{bc}F^{+3}_{1b}e^2_c)(y)\}=\epsilon^{ab}\omega^{+2}_1(y)e^3_b(y)\delta(x-y)+\epsilon^{ab}\omega^{+3}_1(y)e^2_b(y)\delta(x-y), \tag{F21}$$

$$\{\pi^a_{-+}(x),(\epsilon^{bc}F^{+2}_{1b}e^2_c-\epsilon^{bc}F^{+3}_{1b}e^3_c)(y)\}=\epsilon^{ab}\omega^{+2}_1(y)e^2_b(y)\delta(x-y)-\epsilon^{ab}\omega^{+3}_1(y)e^3_b(y)\delta(x-y), \tag{F22}$$

$$\{\pi^1_{-A}(x),\omega^{-B}_1(y)\}=-\delta(x-y)\delta^B_A, \tag{F23}$$

$$\{(\pi^a_{-A}-4\epsilon_{AB}\epsilon^{ab}e^B_b)(x),(\epsilon^{cd}\omega^{-C}_ce^D_d\delta_{CD})(y)\}=-\epsilon^{ab}e^A_b(y)\delta(x-y), \tag{F24}$$

$$\{(\pi^a_{-A}-4\epsilon_{AB}\epsilon^{ab}e^B_b)(x),(e^C_{c,1}-\omega^{-C}_c+\omega^{CD}_1e^E_c\delta_{DE})(y)\}=\delta(x-y)\delta^C_A\delta^a_c, \tag{F25}$$

$$\{(\pi^a_{-A}-4\epsilon_{AB}\epsilon^{ab}e^B_b)(x),(\epsilon_{CD}\epsilon^{cd}F^{-C}_{1c}e^D_d)(y)\}=\epsilon_{AB}\epsilon^{ab}[\omega^{-+}_1(y)e^B_b(y)\delta(x-y)-e^B_b(y)\delta(x-y)_{,y^1}], \tag{F26}$$

$$\{(\pi^a_{-A}-4\epsilon_{AB}\epsilon^{ab}e^B_b)(x),(\epsilon_{CD}\epsilon^{cd}F^{+C}_{1c}e^D_d+F^{23}_{23})(y)\}=\epsilon^{ab}\omega^{+2}_b(y)\delta^3_A\delta(x-y)-\epsilon^{ab}\omega^{+3}_b(y)\delta^2_A\delta(x-y), \tag{F27}$$

$$\{(\pi^a_{-A}-4\epsilon_{AB}\epsilon^{ab}e^B_b)(x),(F^{-C}_{23}+\epsilon^{cd}e^C_cF^{-+}_{1d})(y)\}=\epsilon^{ab}\omega^{+A}_1(y)e^C_b(y)\delta(x-y)-\epsilon^{bc}\delta(x-y)_{,y^b}\delta^C_A\delta^a_c, \tag{F28}$$

$$\{\pi^1_{+A}(x),(\omega^{-+}_a-\omega^{+B}_1e^C_a\delta_{BC})(y)\}=e^A_a(y)\delta(x-y), \tag{F29}$$

$$\{\pi^1_{+A}(x),(\epsilon_{BC}\epsilon^{bc}F^{+B}_{1b}e^C_c+F^{23}_{23})(y)\}=[\epsilon^{ab}\omega^{23}_a(y)e^A_b(y)+\epsilon_{AB}\epsilon^{ab}\omega^{-+}_a(y)e^B_b(y)]\delta(x-y)$$
$$-\epsilon^{ab}e^A_a(y)\delta(x-y)_{,y^b}, \tag{F30}$$

$$\{\pi^1_{+A}(x),(F^{-B}_{23}+\epsilon^{bc}e^B_bF^{-+}_{1c})(y)\}=\epsilon^{ab}\omega^{-A}_a(y)e^B_b(y)\delta(x-y), \tag{F31}$$

$$\{\pi^1_{+A}(x),(\epsilon^{ab}F^{+2}_{1a}e^3_b+\epsilon^{ab}F^{+3}_{1a}e^2_b)(y)\}=\epsilon^{ab}\omega^{23}_a(y)e^3_b(y)\delta(x-y)\delta^3_A+\epsilon^{ab}\omega^{-+}_a(y)e^3_b(y)\delta(x-y)\delta^2_A$$
$$+\epsilon^{ab}e^2_b(y)\delta(x-y)_{,y^a}\delta^3_A+\epsilon^{ab}\omega^{23}_a(y)e^2_b(y)\delta(x-y)\delta^2_A-\epsilon^{ab}\omega^{-+}_a(y)e^2_b(y)\delta(x-y)\delta^3_A+\epsilon^{ab}e^3_b(y)\delta(x-y)_{,y^a}\delta^2_A, \tag{F32}$$

$$\{\pi^1_{+A}(x),(\epsilon^{ab}F^{+2}_{1a}e^2_b-\epsilon^{ab}F^{+3}_{1a}e^3_b)(y)\}=\epsilon^{ab}\omega^{23}_a(y)e^2_b(y)\delta(x-y)\delta^3_A+\epsilon^{ab}\omega^{-+}_a(y)e^2_b(y)\delta(x-y)\delta^2_A$$
$$-\epsilon^{ab}e^3_b(y)\delta(x-y)_{,y^a}\delta^3_A-\epsilon^{ab}\omega^{23}_a(y)e^3_b(y)\delta(x-y)\delta^2_A+\epsilon^{ab}\omega^{-+}_a(y)e^3_b(y)\delta(x-y)\delta^3_A+\epsilon^{ab}e^2_b(y)\delta(x-y)_{,y^a}\delta^2_A, \tag{F33}$$

$$\{\pi^a_{+A}(x),\epsilon^{bc}(\omega^{+B}_be^C_c\delta_{BC})(y)\}=-\epsilon^{ab}e^A_b(y)\delta(x-y), \tag{F34}$$

$$\{\pi^a_{+A}(x),(\epsilon_{BC}\epsilon^{bc}F^{+B}_{1b}e^C_c+F^{23}_{23})(y)\}=\epsilon^{ab}\omega^{-2}_b(y)\delta^3_A\delta(x-y)-\epsilon^{ab}\omega^{-3}_b(y)\delta^2_A\delta(x-y)$$
$$-\epsilon_{AB}\epsilon^{ab}\omega^{-+}_1(y)e^B_b(y)\delta(x-y)-\epsilon^{ab}\omega^{23}_1(y)e^A_b(y)\delta(x-y)-\epsilon_{AB}\epsilon^{ab}e^B_b(y)\delta(x-y)_{,y^1}, \tag{F35}$$

$$\{\pi^a_{+A}(x),(\epsilon^{bc}F^{+2}_{1b}e^3_c+\epsilon^{bc}F^{+3}_{1b}e^2_c)(y)\}=-\epsilon^{ab}\omega^{-+}_1(y)e^3_b(y)\delta(x-y)\delta^2_A-\epsilon^{ab}\omega^{23}_1(y)e^3_b(y)\delta(x-y)\delta^3_A$$
$$-\epsilon^{ab}e^2_b(y)\delta(x-y)_{,y^1}\delta^3_A-\epsilon^{ab}\omega^{-+}_1(y)e^2_b(y)\delta(x-y)\delta^3_A+\epsilon^{ab}\omega^{23}_1(y)e^2_b(y)\delta(x-y)\delta^2_A-\epsilon^{ab}e^3_b(y)\delta(x-y)_{,y^1}\delta^2_A, \tag{F36}$$

$$\{\pi^a_{+A}(x),(\epsilon^{bc}F^{+2}_{1b}e^2_c-\epsilon^{bc}F^{+3}_{1b}e^3_c)(y)\}=-\epsilon^{ab}\omega^{-+}_1(y)e^2_b(y)\delta(x-y)\delta^2_A-\epsilon^{ab}\omega^{23}_1(y)e^2_b(y)\delta(x-y)\delta^3_A$$
$$+\epsilon^{ab}e^3_b(y)\delta(x-y)_{,y^1}\delta^3_A+\epsilon^{ab}\omega^{-+}_1(y)e^3_b(y)\delta(x-y)\delta^3_A-\epsilon^{ab}\omega^{23}_1(y)e^3_b(y)\delta(x-y)\delta^2_A-\epsilon^{ab}e^2_b(y)\delta(x-y)_{,y^1}\delta^2_A, \tag{F37}$$

$$\{\pi^1_{23}(x),(e^A_{a,1}-\omega^{-A}_a+\omega^{AB}_1e^C_a\delta_{BC})(y)\}=-\epsilon_{AB}e^B_a(y)\delta(x-y), \tag{F38}$$

$$\{\pi^1_{23}(x),(\epsilon^{ab}F^{+2}_{1a}e^3_b+\epsilon^{ab}F^{+3}_{1a}e^2_b)(y)\}=-\epsilon^{ab}\omega^{+3}_a(y)e^3_b(y)\delta(x-y)+\epsilon^{ab}\omega^{+2}_a(y)e^2_b(y)\delta(x-y), \tag{F39}$$

$$\{\pi^1_{23}(x),(\epsilon^{ab}F^{+2}_{1a}e^2_b-\epsilon^{ab}F^{+3}_{1a}e^3_b)(y)\}=-\epsilon^{ab}\omega^{+3}_a(y)e^2_b(y)\delta(x-y)-\epsilon^{ab}\omega^{+2}_a(y)e^3_b(y)\delta(x-y), \tag{F40}$$

$$\{\pi^a_{23}(x),\epsilon^{bc}(e^A_{b,c}-\omega^{AB}_be^C_c\delta_{BC})(y)\}=\epsilon_{AB}\epsilon^{ab}e^B_b(y)\delta(x-y), \tag{F41}$$

$$\{\pi^a_{23}(x),(\epsilon_{AB}\epsilon^{bc}F^{+A}_{1b}e^B_c+F^{23}_{23})(y)\}=\epsilon^{ab}\omega^{+A}_1(y)e^B_b(y)\delta_{AB}\delta(x-y)+\epsilon^{ab}\delta(x-y)_{,y^b}, \tag{F42}$$

$$\{\pi^a_{23}(x),(F^{-A}_{23}+\epsilon^{bc}e^A_bF^{-+}_{1c})(y)\}=-\epsilon_{AB}\epsilon^{ab}\omega^{-B}_b(y)\delta(x-y), \tag{F43}$$

$$\{\pi^a_{23}(x),(\epsilon^{bc}F^{+2}_{1b}e^3_c+\epsilon^{bc}F^{+3}_{1b}e^2_c)(y)\}=\epsilon^{ab}\omega^{+3}_1(y)e^3_b(y)\delta(x-y)-\epsilon^{ab}\omega^{+2}_1(y)e^2_b(y)\delta(x-y), \tag{F44}$$

$$\{\pi^a_{23}(x),(\epsilon^{bc}F^{+2}_{1b}e^2_c-\epsilon^{bc}F^{+3}_{1b}e^3_c)(y)\}=\epsilon^{ab}\omega^{+3}_1(y)e^2_b(y)\delta(x-y)+\epsilon^{ab}\omega^{+2}_1(y)e^3_b(y)\delta(x-y). \tag{F45}$$

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