Naumov- and Toshev-like relations in the renormalization-group evolution of quarks and Dirac neutrinos[*]

Zhi-zhong Xing(邢志忠)[1,2,3;1]   Shun Zhou(周顺)[1,2;2]

[1] Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China
[2] School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
[3] Center for High Energy Physics, Peking University, Beijing 100871, China

Abstract: In an analytical way of studying matter effects on neutrino oscillations, the Naumov and Toshev relations have been derived to respectively link the Jarlskog invariant of CP violation and the Dirac phase in the standard parametrization of the 3×3 flavor mixing matrix to their matter-corrected counterparts. Here we show that there exist similar relations for Dirac neutrinos and charged leptons evolving with energy scales via the one-loop renormalization-group (RG) equations in the tau-dominance approximation, and for the running behaviors of up- and down-type quarks in the top-dominance approximation, provided a different parametrization is taken into account.

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1 Introduction

In particle physics the evolution of fermion masses and flavor mixing parameters from a superhigh energy scale to low scales (or vice versa) is described by the renormalization-group (RG) equations [1]. Such a tool is extremely important because it helps bridge the gap between a predictive high-scale flavor model responsible for fermion mass generation and the relevant experimental data at low energies. Recently the authors of Refs. [2] and [3] have borrowed the general idea of the RG language and applied it to the description of neutrino oscillations in matter. An interesting outcome of this successful application is the rediscovery of the Naumov relation [4]

$$\widetilde{\mathcal{J}}_\ell \widetilde{\Delta}_{21} \widetilde{\Delta}_{31} \widetilde{\Delta}_{32} = \mathcal{J}_\ell \Delta_{21} \Delta_{31} \Delta_{32} \tag{1}$$

and the Toshev relation [5]

$$\sin\widetilde{\delta}\sin 2\widetilde{\theta}_{23} = \sin\delta\sin 2\theta_{23}, \tag{2}$$

which link the Jarlskog rephasing invariant $\mathcal{J}_\ell$ of the 3×3 Pontecorvo-Maki-Nakagawa-Sakata (PMNS) flavor mixing matrix U and the Dirac CP-violating phase δ in

the standard parametrization of U [6] to their matter-corrected counterparts $\widetilde{\mathcal{J}}_\ell$ and $\widetilde{\delta}$, respectively. In the above equations $\Delta_{ij} \equiv m_i^2 - m_j^2$ (for $ij = 21, 31, 32$) denote the neutrino mass-squared differences in vacuum, θ_{23} is one of the flavor mixing angles, and $\widetilde{\Delta}_{ij}$ and $\widetilde{\theta}_{23}$ are the corresponding effective quantities in matter. Note that the original derivations of Eq. (1) [4] and Eq. (2) [5] follow an *integral* approach, starting directly from the effective Hamiltonian responsible for neutrino oscillations in a medium[3]. In comparison, the new derivations done in Refs. [2, 3] follow the *differential* approach with the help of the RG-like equations.

The analogy between the RG evolution of neutrino masses and flavor mixing parameters with the energy scale and that with the matter parameter is so suggestive that we are wondering whether some interesting relations like Eqs. (1) and (2) in the latter case can also show up in the former case. Namely, we are eager to know whether there exist the Naumov- and Toshev-like relations for massive neutrino running from an arbitrary high energy scale μ down to the electroweak scale $\Lambda_{\mathrm{EW}} \sim M_Z = 91.2\,\mathrm{GeV}$. At the one-loop level, we find that such relations *do* hold for Dirac neutrinos and charged

leptons in the tau-dominance approximation, provided a different parametrization of the PMNS matrix U advocated by Fritzsch and one of us [10, 11] is taken into account [*]. Similar results are also obtainable for up- and down-type quarks in the top-dominance approximation. The main purpose of the present paper is just to report our findings and discuss their phenomenological implications.

2   RG equations

In the minimal extension of the standard model (SM) with three massive Dirac neutrinos, the RG evolution of the Yukawa coupling matrices for the up-type quarks Y_u, the down-type quarks Y_d, the Dirac neutrinos Y_ν and the charged leptons Y_l at the one-loop level are governed by the following equations [12, 13] [†]

$$16\pi^2 \frac{dY_u}{dt} = \left[\alpha_u + \frac{3}{2}\left(Y_u Y_u^\dagger\right) - \frac{3}{2}\left(Y_d Y_d^\dagger\right)\right] Y_u,$$

$$16\pi^2 \frac{dY_d}{dt} = \left[\alpha_d - \frac{3}{2}\left(Y_u Y_u^\dagger\right) + \frac{3}{2}\left(Y_d Y_d^\dagger\right)\right] Y_d,$$

$$16\pi^2 \frac{dY_\nu}{dt} = \left[\alpha_\nu + \frac{3}{2}\left(Y_\nu Y_\nu^\dagger\right) - \frac{3}{2}\left(Y_l Y_l^\dagger\right)\right] Y_\nu,$$

$$16\pi^2 \frac{dY_l}{dt} = \left[\alpha_l - \frac{3}{2}\left(Y_\nu Y_\nu^\dagger\right) + \frac{3}{2}\left(Y_l Y_l^\dagger\right)\right] Y_l, \quad (3)$$

where $t \equiv \ln(\mu/\Lambda_{EW})$ with μ being the renormalization energy scale and Λ_{EW} being the electroweak energy scale, and

$$\alpha_u = -\frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + \chi,$$

$$\alpha_d = -\frac{1}{4}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + \chi,$$

$$\alpha_\nu = -\frac{9}{20}g_1^2 - \frac{9}{4}g_2^2 + \chi,$$

$$\alpha_l = -\frac{9}{4}g_1^2 - \frac{9}{4}g_2^2 + \chi, \quad (4)$$

with $\chi \equiv \text{Tr}\left[3\left(Y_u Y_u^\dagger\right) + 3\left(Y_d Y_d^\dagger\right) + \left(Y_\nu Y_\nu^\dagger\right) + \left(Y_l Y_l^\dagger\right)\right]$, and the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge couplings g_3, g_2 and g_1 evolving via the RG equations $16\pi^2 (dg_i/dt) = b_i g_i^3$ with the coefficients $\{b_3, b_2, b_1\} = \{-7, -19/6, 41/10\}$.

Without loss of generality, we shall work in the flavor basis where $Y_u = \text{diag}\{y_u, y_c, y_t\} \equiv D_u$ and $Y_l = \text{diag}\{y_e, y_\mu, y_\tau\} \equiv D_l$ are chosen. Given the fact that the top-quark mass is far above the bottom-quark mass and the other quark masses, the $Y_d Y_d^\dagger$ terms on the right-

hand side of Eq. (3) are safely negligible. It is certainly much safer to neglect the $Y_\nu Y_\nu^\dagger$ terms, because the neutrino masses are far below the charged lepton masses. In these good approximations Eq. (3) can be simplified to

$$16\pi^2 \frac{dD_u}{dt} = \left[\alpha_u + \frac{3}{2}D_u^2\right] D_u,$$

$$16\pi^2 \frac{dY_d}{dt} = \left[\alpha_d - \frac{3}{2}D_u^2\right] Y_d, \quad (5)$$

for quarks; and

$$16\pi^2 \frac{dY_\nu}{dt} = \left[\alpha_\nu - \frac{3}{2}D_l^2\right] Y_\nu,$$

$$16\pi^2 \frac{dD_l}{dt} = \left[\alpha_l + \frac{3}{2}D_l^2\right] D_l, \quad (6)$$

for leptons, together with $\chi \simeq 3y_t^2$. Then it is straightforward to formally integrate Eqs. (5) and (6) from $t = t_0 \equiv \ln(\Lambda/\Lambda_{EW})$ (e.g., a superhigh energy scale Λ where there may exist a kind of underlying flavor symmetry [17] [‡]) down to $t = 0$ (i.e., the electroweak scale Λ_{EW}). In the chosen basis the Yukawa coupling matrices of up-type quarks (D_u) and charged leptons (D_l) keep diagonal during the RG running. To be more explicit, let us introduce the functions

$$I_f = \exp\left[-\frac{1}{16\pi^2}\int_0^{t_0} \alpha_f(t)dt\right], \quad (7)$$

for $f = u, d, \nu, l$;

$$\xi_q = \exp\left[+\frac{3}{32\pi^2}\int_0^{t_0} y_q^2(t)dt\right], \quad (8)$$

for $q = u, c, t$; and

$$\zeta_\alpha = \exp\left[+\frac{3}{32\pi^2}\int_0^{t_0} y_\alpha^2(t)dt\right], \quad (9)$$

for $\alpha = e, \mu, \tau$. As a consequence, the fermion Yukawa coupling matrices at Λ_{EW}, which are denoted respectively as D_u', Y_d', Y_ν' and D_l', can be expressed in an integrated way as

$$D_u' = I_u T_u D_u, \quad Y_d' = I_d T_d Y_d, \quad (10)$$

and

$$D_l' = I_l T_l D_l, \quad Y_\nu' = I_\nu T_\nu Y_\nu, \quad (11)$$

where $T_d = T_u^{-1} \equiv \text{diag}\{\xi_u, \xi_c, \xi_t\}$ and $T_\nu = T_l^{-1} \equiv \text{diag}\{\zeta_e, \zeta_\mu, \zeta_\tau\}$. Note that the relationship between Y_ν' and Y_ν has previously been obtained in the minimal

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[*] Of course, only the Toshev-like relation is parametrization-dependent. A parametrization of the PMNS matrix U which can make the tau-related elements much simpler will be favored in this regard. The matter effects on neutrino oscillations in an ordinary medium are only associated with the electron flavor, and hence the Toshev relation in Eq. (2) can be derived when the standard parametrization of U, which makes the electron-related elements much simpler, is adopted.

[†] Here we do not consider the massive Majorana neutrinos, because their three CP-violating phases are entangled with one another during the RG evolution [12, 14–16], making it impossible to obtain the concise Naumov- and Toshev-like relations which are only associated with the Dirac CP-violating phase δ.

[‡] If massive neutrinos are Majorana particles, the superhigh energy scale Λ can just be where the seesaw mechanism [18–22] works.

supersymmetric standard model (MSSM) and the two-Higgs-doublet model (2HDM) [23]. Here we deal with all the four fermion sectors together in the SM framework.
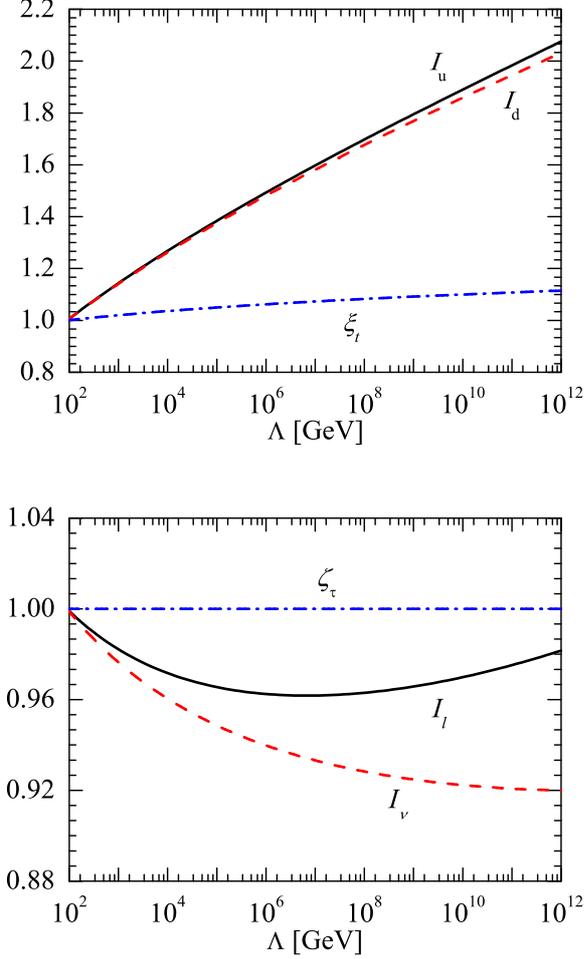

Fig. 1.   (color online) Illustration for the evolution functions I_u, I_d and ξ_t in the quark sector (left panel) and I_l, I_ν and ζ_τ in the lepton sector (right panel) for different values of Λ, where the running quark and charged-lepton masses and other SM parameters renormalized to the energy scale $\mu = M_Z$ [24] have been input.

In fact, $\xi_u \simeq \xi_c \simeq \zeta_e \simeq \zeta_\mu \simeq 1$ is an excellent approximation due to the smallness of m_u, m_c, m_e and m_μ, as compared with the vacuum expectation value of the Higgs field $v \simeq 174$ GeV. It is also reasonable to take $\zeta_\tau \simeq 1$ in the SM, but we do not do so in the present paper. Fig. 1 illustrates the magnitudes of I_u, I_d, I_l, I_ν, ξ_t and ζ_τ for different values of Λ, where the running quark and charged-lepton masses and other SM parameters renormalized to the energy scale $\mu = M_Z$ [24] have been input. It is clear that ξ_t increases by 10% when changing from

$\Lambda_{\mathrm{EW}} \sim M_Z$ to $\Lambda = 10^{12}$ GeV, whereas ζ_τ is almost unchanged. On the other hand, the values of I_u and I_d at the high-energy scale Λ can be twice larger, while those of I_ν and I_l can change by no more than 10%. Although the variation of ζ_τ is negligible in the SM, we should keep in mind that the Yukawa couplings of charged leptons in the MSSM are given by $y_\alpha^2 = (1+\tan^2\beta) m_\alpha^2 / v^2$, which can be significantly enhanced for a relatively large value of $\tan\beta$. For this reason, we retain both ξ_t and ζ_τ in the following discussions, but take the approximation $\xi_u \simeq \xi_c \simeq \zeta_e \simeq \zeta_\mu \simeq 1$ whenever it is necessary.

3   Naumov- and Toshev-like relations

Given the relations between the fermion Yukawa coupling matrices at Λ_{EW} and those at Λ as established in Eqs. (10) and (11), we now investigate their phenomenological implications and derive the Naumov-like relations for the RG evolution of quarks and Dirac neutrinos. In the chosen basis the Cabibbo-Kobayashi-Maskawa (CKM) quark flavor mixing matrix V' at $\mu = \Lambda_{\mathrm{EW}}$ can be determined from the diagonalization of

$$H_{\mathrm{d}}' \equiv Y_{\mathrm{d}}' Y_{\mathrm{d}}'^\dagger = V' D_{\mathrm{d}}'^2 V'^\dagger; \tag{12}$$

while the PMNS matrix U' at Λ_{EW} is from diagonalizing

$$H_\nu' \equiv Y_\nu' Y_\nu'^\dagger = U' D_\nu'^2 U'^\dagger, \tag{13}$$

in which $D_{\mathrm{d}}' \equiv \mathrm{diag}\{y_d', y_s', y_b'\} = \mathrm{diag}\{m_d', m_s', m_b'\}/v$ with m_q' being the down-type quark mass (for $q = d, s, b$), and $D_\nu' \equiv \mathrm{diag}\{y_1', y_2', y_3'\} = \mathrm{diag}\{m_1', m_2', m_3'\}/v$ with m_i' being the neutrino mass (for $i = 1, 2, 3$). At a superhigh-energy scale $\mu = \Lambda$ the CKM matrix V and the PMNS matrix U can be obtained in the same way as in Eqs. (12) and (13). The point is that both D_{u} and D_l keep diagonal during the RG evolution, as we have remarked before. So the eigenvalues of $H_{\mathrm{d}} \equiv Y_{\mathrm{d}} Y_{\mathrm{d}}^\dagger$ and $H_\nu \equiv Y_\nu Y_\nu^\dagger$ are given by $V^\dagger H_{\mathrm{d}} V = D_{\mathrm{d}}^2 \equiv \mathrm{diag}\{y_d^2, y_s^2, y_b^2\}$ and $U^\dagger H_\nu U = D_\nu^2 \equiv \mathrm{diag}\{y_1^2, y_2^2, y_3^2\}$, respectively.

Let us first focus on the commutators of up-type and down-type quark mass matrices at both low- and high-energy scales. More explicitly, we have

$$[D_{\mathrm{u}}'^2, H_{\mathrm{d}}'] \equiv \mathrm{i} X_{\mathrm{q}}', \quad [D_{\mathrm{u}}^2, H_{\mathrm{d}}] \equiv \mathrm{i} X_{\mathrm{q}}, \tag{14}$$

where X_{q}' and X_{q} are obviously two Hermitian matrices and their expressions can be figured out by directly calculating the commutators, i.e.,

$$X_{\mathrm{q}}' = \mathrm{i} \begin{pmatrix} 0 & \Delta_{cu}' Z_{uc}' & \Delta_{tu}' Z_{ut}' \\ \Delta_{uc}' Z_{cu}' & 0 & \Delta_{tc}' Z_{ct}' \\ \Delta_{ut}' Z_{tu}' & \Delta_{ct}' Z_{tc}' & 0 \end{pmatrix},$$

$$X_{\mathrm{q}} = \mathrm{i} \begin{pmatrix} 0 & \Delta_{cu} Z_{uc} & \Delta_{tu} Z_{ut} \\ \Delta_{uc} Z_{cu} & 0 & \Delta_{tc} Z_{ct} \\ \Delta_{ut} Z_{tu} & \Delta_{ct} Z_{tc} & 0 \end{pmatrix}, \tag{15}$$

with the notations $\Delta'_{pq} \equiv y'^2_p - y'^2_q$, $\Delta_{pq} \equiv y^2_p - y^2_q$, $Z'_{pq} \equiv y'^2_d V'_{pd} V'^*_{qd} + y'^2_s V'_{ps} V'^*_{qs} + y'^2_b V'_{pb} V'^*_{qb}$ and $Z_{pq} \equiv y^2_d V_{pd} V^*_{qd} + y^2_s V_{ps} V^*_{qs} + y^2_b V_{pb} V^*_{qb}$ (for the subscripts p and q running over the flavors u, c and t). It is straightforward to verify that the determinant of the above commutator leads us to a rephasing-invariant measure of CP violation [25]:

$$\det([D'^2_u, H'_d]) = 2i\Delta'_{cu}\Delta'_{tc}\Delta'_{tu}\text{Im}[Z'_{cu}Z'_{tc}Z'_{ut}]$$
$$= 2i\Delta'_{cu}\Delta'_{tc}\Delta'_{tu}\Delta'_{sd}\Delta'_{bd}\Delta'_{bs}\mathcal{J}'_q, \quad (16)$$

in which $\mathcal{J}'_q$ is the Jarlskog invariant of the CKM matrix at Λ_{EW}. On the other hand, the identities in Eq. (10) lead us to $H'_d = I^2_d T_d H_d T_d$ and thus $[D'^2_u, H'_d] = I^2_d T_d[D'^2_u, H_d]T_d$, where $[D'^2_u, T_d] = 0$ should be noticed. Calculating the determinant of the commutator along this line, we immediately arrive at

$$\det([D'^2_u, H'_d]) = \det(I^2_d T_d[D'^2_u, H_d]T_d)$$
$$= 2iI^6_d \xi^2_u \xi^2_c \xi^2_t \Delta'_{cu}\Delta'_{tc}\Delta'_{tu}\Delta_{sd}\Delta_{bd}\Delta_{bs}\mathcal{J}_q. \quad (17)$$

A combination of Eqs. (16) and (17) can therefore give rise to the Naumov-like relation for the RG evolution of quarks:

$$\mathcal{J}'_q\Delta'_{sd}\Delta'_{bd}\Delta'_{bs} = I^6_d \xi^2_u \xi^2_c \xi^2_t \mathcal{J}_q \Delta_{sd}\Delta_{bd}\Delta_{bs}. \quad (18)$$

In a similar way one may derive the Naumov-like relation for the RG evolution of charged leptons and Dirac neutrinos, namely,

$$\mathcal{J}'_\ell\Delta'_{21}\Delta'_{31}\Delta'_{32} = I^6_\nu \zeta^2_e \zeta^2_\mu \zeta^2_\tau \mathcal{J}_\ell \Delta_{21}\Delta_{31}\Delta_{32}, \quad (19)$$

with $\Delta'_{ij} \equiv y^2_i - y^2_j$ and $\Delta_{ij} = y^2_i - y^2_j$ (for $ij = 21,31,32$). Note that Eq. (19) has been derived in Ref. [23] by following a different approach in the MSSM and 2HDM frameworks.

Then, we proceed to explore other possible relations between the quark flavor mixing parameters at the electroweak scale Λ_{EW} and those at a superhigh-energy scale Λ. Starting with the identity $H'_d = I^2_d T_d H_d T_d$, we further adopt the top-dominance approximation, namely, $T_d \approx \text{diag}\{1, 1, \xi_t\}$. In this approximation, it has been demonstrated that the most convenient parametrization of the CKM matrix is [10, 11]

$$V = \begin{pmatrix} c_u & s_u & 0 \\ -s_u & c_u & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{i\phi_q} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \cdot \begin{pmatrix} c_d & -s_d & 0 \\ s_d & c_d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (20)$$

where $c_u \equiv \cos\theta_u$ and $s_u \equiv \sin\theta_u$ have been defined and likewise for the other two mixing angles θ and θ_d, and ϕ_q is the CP-violating phase. The parametrization in Eq. (20) actually represents three sequential rotations in

the flavor space $V = R_{12}(\theta_u) \cdot U_{\phi_q} \cdot R_{23}(\theta) \cdot R^T_{12}(\theta_d)$, where $U_{\phi_q} \equiv \text{diag}\{e^{i\phi_q}, 1, 1\}$ and $R_{ij}(\theta)$ denotes the rotation in the i-j plane with the rotation angle θ (for $ij = 12, 23$). Since $T_d \approx \text{diag}\{1, 1, \xi_t\}$ in the top-dominance approximation commutes with the rotation matrix $R_{12}(\theta_u)U_{\phi_q}$, we can obtain

$$\widetilde{H}'_d \equiv \left[R_{12}(\theta_u)U_{\phi_q}\right]^\dagger \cdot H'_d \cdot \left[R_{12}(\theta_u)U_{\phi_q}\right]$$
$$= I^2_d T_d \left[R_{23}(\theta)R^T_{12}(\theta_d) \cdot D^2_d \cdot R_{12}(\theta_d)R^T_{23}(\theta)\right] T_d, \quad (21)$$

which is a real and symmetric matrix and can be diagonalized by the following orthogonal transformation with three rotation angles $\{\widehat{\theta}_u, \widehat{\theta}, \widehat{\theta}_d\}$, i.e.,

$$\widetilde{H}'_d = \left[R_{12}(\widehat{\theta}_u)R_{23}(\widehat{\theta})R^T_{12}(\widehat{\theta}_d)\right]$$
$$\cdot D'^2_d \cdot \left[R_{12}(\widehat{\theta}_u)R_{23}(\widehat{\theta})R^T_{12}(\widehat{\theta}_d)\right]^T. \quad (22)$$

Furthermore, H'_d itself can be diagonalized via $H'_d = V'D'^2_d V'^\dagger$, where the CKM matrix V' at the electroweak scale Λ_{EW} can be parametrized in terms of $\{\theta'_u, \theta', \theta'_d, \phi'_q\}$ in the same way as in Eq. (20). With the help of Eqs. (21) and (22), we can recognize $V' = \left[R_{12}(\theta_u)U_{\phi_q}\right] \cdot \left[R_{12}(\widehat{\theta}_u)R_{23}(\widehat{\theta})R^T_{12}(\widehat{\theta}_d)\right]$ and then obtain

$$R_{12}(\theta_u)U_{\phi_q}R_{12}(\widehat{\theta}_u) = \begin{pmatrix} e^{i\varphi_1} & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} R_{12}(\theta'_u)U_{\phi'_q}, \quad (23)$$

where φ_1 and φ_2 are the unphysical phases that can be absorbed by redefining the up-type quark fields. In addition, the other two mixing angles are given by $\theta' = \widehat{\theta}$ and $\theta'_d = \widehat{\theta}_d$. In order to extract θ'_u and ϕ'_q from Eq. (23), one has to write down the explicit forms of the matrices on both sides, and identify the corresponding matrix elements. To be more explicit, we have

$$\sin\theta'_u = \left|\sin\theta_u\cos\widehat{\theta}_u + \cos\theta_u\sin\widehat{\theta}_u e^{i\phi_q}\right|,$$
$$\cos\theta'_u = \left|\cos\theta_u\cos\widehat{\theta}_u - \sin\theta_u\sin\widehat{\theta}_u e^{i\phi_q}\right|; \quad (24)$$

and

$$\varphi_1 = \arg\left[\sin\theta_u\cos\widehat{\theta}_u + \cos\theta_u\sin\widehat{\theta}_u e^{i\phi_q}\right],$$
$$\varphi_2 = \arg\left[\cos\theta_u\cos\widehat{\theta}_u - \sin\theta_u\sin\widehat{\theta}_u e^{i\phi_q}\right], \quad (25)$$

and $\phi'_q = \phi_q - \varphi_1 - \varphi_2$. Therefore, we have established the relationship between the flavor mixing parameters $\{\theta'_u, \theta', \theta'_d, \phi'_q\}$ at Λ_{EW} and $\{\theta_u, \theta, \theta_d, \phi_q\}$ at Λ in an implicit way. Then it is straightforward to verify

$$
\begin{aligned}
\sin\phi'_q \sin 2\theta'_u &= 2\mathrm{Im}\left[e^{i\phi'_q} \sin\theta'_u \cos\theta'_u \right] \\
&= 2\mathrm{Im}\left[\left(\sin\theta_u \cos\widehat{\theta}_u e^{i\phi_q} + \cos\theta_u \sin\widehat{\theta}_u \right) \left(\cos\theta_u \cos\widehat{\theta}_u - \sin\theta_u \sin\widehat{\theta}_u e^{-i\phi_q} \right) \right] \\
&= \sin\phi_q \sin 2\theta_u,
\end{aligned}
\tag{26}
$$

which is analogous to the Toshev relation in Eq. (2) for neutrino oscillations in matter with the standard parametrization of the PMNS matrix [5].

Analogously, there is no doubt that the Toshev-like relation

$$
\sin\phi'_\ell \sin 2\theta'_l = \sin\phi_\ell \sin 2\theta_l \tag{27}
$$

also holds in the RG evolution of charged-leptons and Dirac neutrinos from Λ down to Λ_{EW}, if a similar parametrization as that in Eq. (20) is adopted for the PMNS matrix $U'(\theta'_l, \theta', \theta'_\nu, \phi'_\ell)$ at Λ_{EW} or $U(\theta_l, \theta, \theta_\nu, \phi_\ell)$ at Λ and the tau-dominance approximation $T_\nu \approx \mathrm{diag}\{1, 1, \zeta_\tau\}$ is made. Although the RG equations of quark and lepton flavor mixing parameters have been extensively studied in the literature [1], we stress that such Toshev-like relations are derived here for the first time.

Some useful remarks are in order. First, the connection between the mixing parameters $\{\theta_u, \theta, \theta_d, \phi_q\}$ in Eq. (20) and those $\{\vartheta_{12}, \vartheta_{13}, \vartheta_{23}, \delta_q\}$ in the standard parametrization of the CKM matrix [6] can be found by identifying the moduli of four independent matrix elements and the Jarlskog invariant. More explicitly, we have [26]

$$
\begin{aligned}
\sin\vartheta_{13} &= \sin\theta_u \sin\theta, \\
\sin\vartheta_{23} &= \frac{\cos\theta_u \sin\theta}{\sqrt{1 - \sin^2\theta_u \sin^2\theta}}, \\
\tan\vartheta_{12} &= \frac{\left| \sin\theta_u \cos\theta_d \cos\theta - \cos\theta_u \sin\theta_d e^{i\phi_q} \right|}{\left| \sin\theta_u \sin\theta_d \cos\theta + \cos\theta_u \cos\theta_d e^{i\phi_q} \right|}, \\
\sin\delta_q &= \frac{\sin\theta_u \cos\theta_u \sin\theta_d \cos\theta_d \sin^2\theta \cos\theta \sin\phi_q}{\sin\vartheta_{12} \cos\vartheta_{12} \sin\vartheta_{23} \cos\vartheta_{23} \sin\vartheta_{13} \cos^2\vartheta_{13}}.
\end{aligned}
\tag{28}
$$

Then, it should be noticed that the Toshev-like relation in Eq. (26) for quarks has been derived in the flavor basis where the up-type quark Yukawa coupling matrix Y_u is diagonal, so the mixing angle θ_u is just introduced to parametrize the CKM matrix and not necessarily coming from the up-type quark sector. However, in specific models of quark masses and flavor mixing [27], θ_u and θ_d do arise from the unitary matrices diagonalizing the Yukawa coupling matrices of up- and down-type quarks, respec-

tively. Therefore, after transforming into the diagonal basis of up-type quarks, the rotation angle θ_u appears in the CKM matrix. Similar arguments are also applicable to the lepton sector.

In general, the Naumov- and Toshev-like relations derived for the RG running of quark and lepton flavor mixing parameters should be useful for investigating the flavor models of quarks and leptons at a superhigh energy scale, given the experimental measurements at the low-energy scale. Although two such relations may not be sufficient to completely solve the RG equations, both of them are related to the CP-violating phase and thus lead to an interesting connection between low- and high-energy CP violation. Furthermore, they can be implemented to greatly simply the approximate analytical solutions to the RG equations.

4   Summary

The profundity and usefulness of RG equations can never be overemphasized, and this powerful tool has found many important applications in a number of different fields of modern physics [1]. Recently, the RG-like equations have been derived in Refs. [2] and [3] and implemented to the study of matter effects on neutrino oscillations, from which the previously known Naumov and Toshev relations have been rediscovered. We are therefore motivated to investigate if such kinds of relations exist for the RG evolution of fermion masses and flavor mixing parameters from a superhigh-energy scale down to the electroweak scale. It turns out that such Naumov- and Toshev-like relations *do* exist, as we have shown in this paper.

Though we work in a minimal extension of the SM with three massive Dirac neutrinos, in which the RG running effects are usually small, similar results are also obtainable in the MSSM and 2HDM cases only if the top- and tau-dominance approximations can be made. Since the Naumov- and Toshev-like relations directly connect the low- and high-energy flavor parameters, they definitely provide an interesting and transparent way to understand the RG-induced quantum corrections in this respect.

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