Yang-Baxter deformations of supercoset sigma models with $\mathbb{Z}_{4m}$ grading*

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Abstract: We have studied Yang-Baxter deformations of supercoset sigma models with $\mathbb{Z}_{4m}$ grading. The deformations are specified by a skew-symmetric classical r-matrix satisfying the classical Yang-Baxter equations. The deformed action is constructed and the Lax pair is also presented. When $m=1$, our results reduce to those of the type IIB Green-Schwarz superstring on $AdS_5{\times}S^5$ background recently given by Kawaguchi, Matsumoto and Yoshida.

Keywords: sigma model, supercoset target, Yang-Baxter deformation, Lax pair

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1 Introduction

Non-linear sigma models with supermanifolds as targets have attracted much interest due to their applications to string theory and condensed matter physics. String theory on $AdS_d{\times}S^d$ $(d{=}2,3,5)$, $AdS_p(p{=}2,4,6)$, $AdS_5\times S^1$ and $AdS_4\times CP^3$ backgrounds are described by superspace sigma models with $\mathbb{Z}_4$ grading [1–9]. The $\mathbb{Z}_4$-grading property of the supercoset ensures its classical integrability. Bena, Polchinski and Roiban [10] found that string theory on $AdS_5{\times}S^5$ has an infinite number of non-local classically conserved charges. Thus the model is classically integrable. Subsequently Vallilo showed [11] that such charges also exist in the pure spinor formalism for the superstring. A complete proof of classical integrability of the superstring on $AdS_5{\times}S^5$ was achieved in a 'weak' sense in the pure spinor formalism by Schafer-Nameki and Mikhailov [12] and in a 'strong' sense in both the Green-Schwarz and pure spinor formalisms by Magro [13].

The study of integrable deformations of integrable non-linear sigma models is an interesting topic. The Yang-Baxter sigma-model description, which was originally introduced by Klimcik [14], is a systematic way to consider integrable deformations of 2D non-linear sigma models. By following this approach, the deformations are specified by skew-symmetric classical r-matrices satisfying the modified classical Yang-Baxter equations (mCYBE). Delduc, Magro and Vicedo extended this to sigma models defined on bosonic symmetric cosets [15] and succeeded in constructing a q-deformed action of the $AdS_5{\times}S^5$ superstring [16, 17]. The deformed metric and the B-field of the deformed $AdS_5{\times}S^5$ superstring were determined in Ref. [18] and the full background has been discussed in Ref. [19]. Some special cases of the background were examined in Ref. [20] and a mirror TBA was proposed in Ref. [21]. Giant magnon solutions were constructed in Ref. [22]. The deformed Neumann models were obtained in Ref. [23].

As a generalization of Ref. [16], one may consider the integrable deformations with classical r-matrices satisfying the classical Yang-Baxter equation (CYBE), rather than mCYBE. Deformations of the $AdS_5{\times}S^5$ superstring by the CYBE were proposed in Ref. [24] and the supergravity solution was constructed in Ref. [25]. The supercoset construction of Yang-Baxter deformed $AdS_5\times S^5$ backgrounds was performed in Ref. [26]. The generic formulation for group manifolds and bosonic cosets was

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introduced in Ref. [27]. In a series of works [24, 28–33], the associated classical r-matrices were identified with the well-known type IIB supergravity solutions including the γ deformations of S^5, gravity duals for noncommutative gauge theories, and Schrödinger spacetimes. The Yang-Baxter deformations have been further generalized to 4D Minkowski spacetime [34, 35].

Young [36] generalized Bena, Polchinski and Roiban's results [10] to all coset (super-)spaces G/H in which, at the level of the Lie algebras, $\mathfrak{h}$ is the grade-zero subspace of a $\mathbb{Z}_m$-grading of $\mathfrak{g}$. The quantum behaviour and the Hamiltonian analysis of supercoset sigma models with $\mathbb{Z}_{2n}$ grading were discussed in Refs. [37, 38]. The supercoset sigma models with the $\mathbb{Z}_{4m}$ grading are integrable non-linear sigma models which include the Green-Schwarz sigma models with the $\mathbb{Z}_4$ grading as a special case $m=1$. The action and the flat currents of supercoset sigma models with $\mathbb{Z}_{4m}$ grading was investigated in Ref. [39]. The classical exchange algebra of the model was studied in Ref. [40]. This type of sigma model may have applications in condensed matter physics, string theory and other domains in physics. In this paper, we investigate Yang-Baxter deformations of supercoset sigma models with $\mathbb{Z}_{4m}$ grading.

2 Yang-Baxter deformations of supercoset sigma models with $\mathbb{Z}_{4m}$ grading

Let $\mathfrak{g}$ be a Lie superalgebra admitting a $\mathbb{Z}_{4m}$ grading. That is, $\mathfrak{g}$ may be written as a direct sum $\mathfrak{g}=\sum\limits_{k=0}^{4m-1}\mathfrak{g}^{(k)}$ of vector subspaces where $\mathfrak{g}^{(0)}=\mathfrak{h}$, and this decomposition satisfies the algebra $\left[\mathfrak{g}^{(r)},\mathfrak{g}^{(s)}\right]\subset\mathfrak{g}^{(p)}$ with $p=r+s$ mod $4m$. Let G denote the corresponding supergroup (for examples, the supergroup $PSL(2r|2r)$, $SU(2m,2m|4m)$ etc). The supertrace is compatible with the $\mathbb{Z}_{4m}$ grading, which means that $StrX^{(i)}Y^{(j)}=0$ for $X^{(i)}\in\mathfrak{g}^{(i)}$, $Y^{(j)}\in\mathfrak{g}^{(j)}$ and $i+j\neq 0$ mod $4m$. Let $g(x^\mu)$ be a two-dimensional field valued in an even faithful matrix representation of G, where $x^0=\tau,x^1=\sigma$ are time and spatial coordinates of the string world-sheet. The left-invariant one-form is defined as $A=g^{-1}dg\in\mathfrak{g}$ and can be decomposed as $A=\sum\limits_{k=0}^{4m-1}A^{(k)}$, here $A^{(k)}\in\mathfrak{g}^{(k)},k=0,1,2,...,4m-1$.

The deformed action of the supercoset sigma models with $\mathbb{Z}_{4m}$ grading is given by

$$S=-\frac{1}{4}(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta})\int_{-\infty}^{\infty}\mathrm{d}\tau\int_0^{2\pi}\mathrm{d}\sigma Str\left(A_\alpha d\circ\frac{1}{1-\eta R_g\circ d}A_\beta\right). \tag{1}$$

Here $\gamma^{\alpha\beta}=h^{\alpha\beta}\sqrt{-h}$ is the Weyl-invariant combination of the world-sheet metric $h^{\alpha\beta}$ with $\det\gamma=-1$. In the conformal gauge $\gamma^{\alpha\beta}=\mathrm{diag}(-1,+1)$. The anti-symmetric tensor $\epsilon^{\alpha\beta}$ is normalized as $\epsilon^{\tau\sigma}=1$. The real constant $\eta\in[0,1)$ measures the associated deformation. In the $\eta\to 0$ limit, the action (1) reduces to the undeformed one [39]. The operator R_g is defined as

$$R_g(X)\equiv Ad_g^{-1}\circ R\circ Ad_g(X)=g^{-1}R(gXg^{-1})g,\quad X\in\mathfrak{g}, \tag{2}$$

where the linear operator R is antisymmetric

$$Str(R(X)Y)=-Str(XR(Y)), \tag{3}$$

and satisfies the classical Yang-Baxter equation (CYBE)

$$[R(X),R(Y)]-R([R(X),Y]+[X,R(Y)])=0. \tag{4}$$

The R-operator is related to a classical r-matrix in the tensorial notation through

$$R(X)=Str_2[r(1\otimes X)]=\sum_i(a_iStr(b_iX)-b_iStr(a_iX)), \tag{5}$$

with

$$r=\sum_i a_i\wedge b_i\equiv\sum_i(a_i\otimes b_i-b_i\otimes a_i), \tag{6}$$

where the generators a_i,b_i are some elements of $\mathfrak{g}$. The r-matrix satisfies the CYBE in the tensorial notation

$$[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0. \tag{7}$$

The operators d and $\tilde{d}$ are defined as linear combinations of operators $P_k,k=1,2,3,\ldots,4m-1$,

$$d=\sum_{r=1}^{2m-1}q_rP_r+2P_{2m}-\sum_{r=1}^{2m-1}q_rP_{4m-r}, \tag{8}$$

$$\tilde{d}=-\sum_{r=1}^{2m-1}q_rP_r+2P_{2m}+\sum_{r=1}^{2m-1}q_rP_{4m-r}, \tag{9}$$

where P_i $(i=0,1,2,\ldots,4m-1)$ are the projections to the $\mathbb{Z}_{4m}$-graded components of $\mathfrak{g}$, $P_i(X)=X^{(i)},X^{(i)}\in\mathfrak{g}^{(i)},q_r=\frac{r}{m}$. The operator $\tilde{d}$ is the transpose of d and satisfies

$$Str(X\tilde{d}(Y))=Str(d(X)Y). \tag{10}$$

We introduce the light-cone expression of A_α like

$$A_\pm\equiv A_\tau\pm A_\sigma. \tag{11}$$

With these notations, the Lagrangian of the action can be written as;

$$L=\frac{1}{4}Str(A_-d(J_+))=\frac{1}{4}Str(A_+\tilde{d}(J_-)), \tag{12}$$

where $J_\pm$ is a deformed current defined as

$$J_-=\frac{1}{1+\eta R_g\circ\tilde{d}}A_-,\quad J_+=\frac{1}{1-\eta R_g\circ d}A_+. \tag{13}$$

To find the equation of motion, we define a variation of $g\in G$ as $\delta g=g\epsilon$ with an infinitesimal parameter ϵ . Then the following relations are derived:

$$\delta A_\pm=\partial_\pm\epsilon+[A_\pm,\epsilon], \tag{14}$$

$$\delta(R_g\circ d)(X)=(R_g\circ d)(\delta X)+[(R_g\circ d)(X),\epsilon]-R_g([d(X),\epsilon]. \tag{15}$$

By use of Eqs. (14), (15) and (10), after some calculation, we get the equation of motion

$$\mathcal{E}\equiv\partial_+\tilde{d}(J_-)+\partial_-d(J_+)+[J_+,\tilde{d}(J_-)]+[J_-,d(J_+)]=0. \quad (16)$$

By definition of the undeformed current $A_\pm$, the zero-curvature condition is

$$\mathcal{Z}\equiv\partial_+A_--\partial_-A_++[A_+,A_-]=0, \quad (17)$$

which can be rewritten in terms of the deformed current $J_\pm$ as

$$\begin{aligned}\mathcal{Z} \equiv\ & \partial_+J_--\partial_-J_++[J_+,J_-]+\eta R_g(\mathcal{E})\\ &+\eta^2\mathrm{CYBE}_g(d(J_+),\tilde{d}(J_-))=0,\end{aligned} \quad (18)$$

where

$$\begin{aligned}\mathrm{CYBE}_g(X,Y) \equiv\ & [R_g(X),R_g(Y)]-R_g([R_g(X),Y]\\ &+[X,R_g(Y)]).\end{aligned} \quad (19)$$

Note that $\mathrm{CYBE}_g(X,Y)$ vanishes if the R-operator satisfies the CYBE in Eq.(4). The relations in Eq.(18) mean that $J_\pm$ also satisfies the flatness condition with the equations of motion $\mathcal{E}=0$.

With the help of the projection operators P_i ($i=0,1,2,\ldots,4m-1$), it is convenient to write the equations of motion (16) and the flatness condition (17) as

$$C^{(0)}=\partial_+J_-^{(0)}-\partial_-J_+^{(0)}+[J_+^{(0)},J_-^{(0)}]+\sum_{i=1}^{4m-1}[J_+^{(i)},J_-^{(4m-i)}]=0, \quad (20)$$

$$\begin{aligned}C^{(r)} =\ & \partial_+J_-^{(r)}-\partial_-J_+^{(r)}+\sum_{i=0}^{r}[J_+^{(i)},J_-^{(r-i)}]\\ &+\sum_{i=r+1}^{4m-1}[J_+^{(i)},J_-^{(4m+r-i)}]=0,\end{aligned} \quad (21)$$

$$C^{(2m)}=\partial_-J_+^{(2m)}-\sum_{i=1}^{2m}[J_+^{(i)},J_-^{(2m-i)}]=0, \quad (22)$$

$$C^{(4m-r)}=\sum_{i=2m-r+1}^{2m}[J_+^{(i)},J_-^{(4m-i-r)}]=0, r=1,2,\ldots 2m-1, \quad (23)$$

$$\begin{aligned}D^{(r)} =\ & \partial_+J_-^{(4m-r)}-\partial_-J_+^{(4m-r)}+\sum_{i=0}^{4m-r}[J_+^{(i)},J_-^{(4m-i-r)}]\\ &+\sum_{i=4m-r+1}^{4m-1}[J_+^{(i)},J_-^{(8m-i-r)}]=0, r=1,2,\ldots 2m-1,\end{aligned} \quad (24)$$

$$\begin{aligned}D^{(2m)} =\ & \partial_+J_-^{(2m)}+[J_+^{(0)},J_-^{(2m)}]\\ &+\sum_{i=2m+1}^{4m-1}[J_+^{(i)},J_-^{(6m-i)}]=0, r=1,2,\ldots 2m-1,\end{aligned} \quad (25)$$

$$D^{(4m-r)}=\sum_{i=2m+1}^{2m+r}[J_+^{(i)},J_-^{(4m+r-i)}]=0, r=1,2,\ldots 2m-1. \quad (26)$$

A Lax pair for the deformed action is given by

$$L_+=J_+^{(0)}+\sum_{i=1}^{2m-1}\lambda^iJ_+^{(i)}+\lambda^{2m}J_+^{(2m)}+\sum_{i=1}^{2m-1}\lambda^{-i}J_+^{(4m-i)}, \quad (27)$$

$$L_-=J_-^{(0)}+\sum_{i=1}^{2m-1}\lambda^iJ_-^{(i)}+\lambda^{-2m}J_-^{(2m)}+\sum_{i=1}^{2m-1}\lambda^{-i}J_-^{(4m-i)}, \quad (28)$$

where λ is the spectral parameter. After some calculation, one can obtain the curvature of L in terms of $\mathcal{C}^{(i)}$ and $\mathcal{D}^{(i)}$ as follows:

$$\begin{aligned}&\partial_+L_--\partial_-L_++[L_+,L_-]\\ =\ & C^{(0)}+\sum_{r=1}^{2m-1}\lambda^r(C^{(r)}-D^{(4m-r)})-\lambda^{2m}C^{(2m)}\\ &+\sum_{r=1}^{2m-1}\lambda^{(4m-r)}C^{(4m-r)}+\sum_{r=1}^{2m-1}\lambda^{-r}(D^{(r)}-C^{(4m-r)})\\ &+\lambda^{-2m}D^{(2m)}+\sum_{r=1}^{2m-1}\lambda^{-(4m-r)}D^{(4m-r)}.\end{aligned} \quad (29)$$

Therefore by Eqs. (20)-(26), the current L is flat. The equation of motion $\mathcal{E}=0$ and the zero curvature condition $\mathcal{Z}=0$ are equivalent to the flatness condition of the Lax pair $L_\pm(\lambda)$

$$\partial_+L_--\partial_-L_++[L_+,L_-]=0. \quad (30)$$

Thus the models defined in Eq. (1) are classically integrable.

3 Conclusion

In this paper, we have studied the Yang-Baxter deformations of supercoset sigma models with $\mathbb{Z}_{4m}$ grading by adopting a prescription invented by Deldue, Magro and Vicedo. The deformations are specified by the skew-symmetric classical r-matrix satisfying the classical Yang-Baxter equations. The deformed action has been constructed and the Lax pair has also been presented. The existence of the flat currents is an indication of the integrability for the model. The integrability of the model is interesting and deserves further study. From the calculation, we find that our results are algebraic and do not rely on a specific choice of the supercoset. When $m=1$ and the supercoset is $PSU(2,2|4)/[SO(4,1)\times SO(5)]$, our results reduce to those of the type IIB Green-Schwarz superstring on $AdS_5\times S^5$ background recently given by Kawaguchi, Matsumoto and Yoshida in Ref. [24].

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