Possible supersymmetric kinematics [*]

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Abstract: The contraction method in different limits to obtain 22 different realizations of kinematical algebras is applied to study the supersymmetric extension of AdS algebra and its contractions. It is shown that $\mathbf{p}_2$ $\mathbf{h}_-$, $\mathbf{p}'$, $\mathbf{c}_2$ and $\mathbf{g}'$ algebras, in addition to $\mathbf{d}_-$, $\mathbf{p}$, $\mathbf{n}_-$, $\mathbf{g}$ and $\mathbf{c}$ algebras, have supersymmetric extension, while $\mathbf{n}_{-2}$, $\mathbf{g}_2$ and $\mathbf{g}'_2$ algebras have no supersymmetric extension. The connections among the superalgebras are established.

Key words: possible supersymmetric kinematics, contraction, superalgebras

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1 Introduction

The contraction is a useful method in mathematical physics. It reveals the relations among groups and algebras. It may also be used to establish the relation among geometries. By the Inönü-Wigner (IW) contraction method [1], Bacry and Lévy-Leblond established the connection among the 11 kinematical algebras of 8 types [2]. All these algebras satisfy the assumption that a kinematical group should possess (i) an $SO(3)$ isotropy generated by $\boldsymbol{J}$, (ii) automorphism of parity

$$\Pi{:}H{\to}H, \boldsymbol{P}{\to}{-}\boldsymbol{P}, \boldsymbol{K}{\to}{-}\boldsymbol{K}, \boldsymbol{J}{\to}\boldsymbol{J}, \tag{1}$$

and time-reversal

$$\Theta{:}H{\to}{-}H, \boldsymbol{P}{\to}\boldsymbol{P}, \boldsymbol{K}{\to}{-}\boldsymbol{K}, \boldsymbol{J}{\to}\boldsymbol{J}, \tag{2}$$

and (iii) non-compact one-dimensional subgroup generated by each boost K_i [2], where K_i is the components of $\boldsymbol{K}$. (The same convention will be used for $\boldsymbol{J}$ and $\boldsymbol{P}$, etc.) The 11 kinematical algebras are the Poincaré ($\mathbf{p}$), de Sitter (dS or $\mathbf{d}_+$), anti-de Sitter (AdS or $\mathbf{d}_-$), inhomogenous $SO(4)$ ($\mathbf{e}'$ or $\mathbf{p}'_+$ in literature), para-Poincaré ($\mathbf{p}'$), Galilei ($\mathbf{g}$), Newton-Hooke ($\mathbf{n}_+$), anti-Newton-Hooke ($\mathbf{n}_-$), para-Galilei ($\mathbf{g}'$), Carroll ($\mathbf{c}$), and static ($\mathbf{s}$) algebras. Relaxing the third condition, three geometrically kinematical groups — Euclid ($\mathbf{e}$), Riemann ($\mathbf{r}$), and Lobachevsky ($\mathbf{l}$) algebras — should be added.

The contraction of a kinematical algebra can be studied in two different ways. One is as in Ref. [2]. An algebra is defined by the commutation relations of a set of abstract generators. Then, a dimensionless parameter ε is introduced and multiplied to some generators, which will not alter the algebraic structure. Finally, the limit

of $\varepsilon{\to}0$ is taken and the contracted algebra is attained. In the manipulation, the realization of the generators are not used. Hence, it is an abstract way of contraction. In the other way, in contrast, an algebra is realized by a special realization of a set of generators and their commutators, and the infinity or zero limit of the parameter(s) in the realization is taken to obtain the contracted algebra. For example, $\mathbf{d}_\pm$, realized by a set of partial differential operators in a given coordinate system, contract to $\mathbf{p}$ when the invariant length l tends to ∞ [3]. Similarly, $\mathbf{p}$ tends to $\mathbf{g}$ or $\mathbf{c}$ when the speed of light $c{\to}\infty$ or $c{\to}0$, respectively. This is referred to as the concrete approach of IW contraction.

In the concrete approach, one may ask: what is the $l{\to}0$ limit of $\mathbf{d}_\pm$, is it the same as $\mathbf{p}$, and does it have the same physical significance as the ordinary Poincaré algebra? These problems have been studied in Ref. [4–6]. It has been shown that when $\mathbf{d}_\pm$ are realized in terms of the Beltrami coordinates [7, 8], in the $l{\to}0$ limit $\mathbf{d}_\pm$ also contract to an $\mathbf{iso}(1,3)$ algebra but the generators and thus the contracted algebra has very different physical significance. The new realization does not generate the translation invariance of the Minkowski spacetime. In fact, the geometry invariant under the transformations generated by the new realization is a degenerate one. Therefore, it is called the second realization of Poincaré algebra and denoted by $\mathbf{p}_2$. The systematic studies on the contractions of the Beltrami realization of $\mathbf{so}(\mathbf{p}, 5{-}\mathbf{p})$ $(0{\leqslant}p{\leqslant}2)$ kinematical algebras with two invariant parameters c and l are made in Ref. [9]. It has been shown that there are 22 different realizations of possible kinematical algebras in all, whose generators are all expressed in terms of coordinate partial differential operators. (The

static algebra in Ref. [2] has been excluded because its time-translation generator is expressed in terms of central charge.) All these realizations of possible kinematics are re-classified and their underlying geometries are presented. It is worth mentioning that the 22 realizations of possible kinematical algebras are first obtained by the combinatorial method in the vector space spanned by the projective general linear algebra **pgl**(5,$\mathbb{R}$) [10].

The possible kinematical superalgebras have also been studied by the IW contraction method [11–14]. It has been shown that the Galilei superalgebra can be contracted from the Poincaré superalgebra [11–14] and that the Poincaré superalgebra can be obtained form AdS superalgebra **osp**(1|4) [14]. In Ref. [15], the supersymmetric extension of all possible kinematics in Bacry-Lévy-Leblond [2] are presented in an abstract way. In Ref. [16], the superalgebras such as supersymmetric extensions of **p**, **g**, **c**, **n**$_-$, and **s**, are re-obtained in the abstract way and are arranged in a figure to show their relations[1]. When the concrete realizations of superalgebras are taken into account, many realizations will be added. But, in comparison with the contraction scheme in Ref. [9], not all the realizations have their supersymmetric extensions. The purpose of the present paper is to study the IW contraction of the Beltrami realization of AdS superalgebra and present more realizations of kinematical superalgebras for completion.

The paper is organized in the following way. In the next section, we shall briefly review the IW contraction by taking the IW contraction of Beltrami realization of **d**$_-$ in two opposite limits, as an example, and list the Beltrami realizations of all possible kinematical algebras contracting from **d**$_-$. In Section 3, the supersymmetric extension of **d**$_-$ (i.e. **osp**(1|4)) will be reviewed. In Section 4, we will use the contraction method to study the supersymmetric extension of the kinematical algebras in Section 2. The concluding remarks will be given in the final section.

2   Beltrami realization of AdS algebra and its contractions

2.1   Beltrami realization of AdS algebra

It is well known that a 4d AdS space-time embedded in $\mathbb{R}^{2,3}$ with the metric $\eta_{AB}=\mathrm{diag}(1,-1,-1,-1,1)$,

$$\eta_{AB}\xi^A\xi^B=\eta_{\mu\nu}\xi^\mu\xi^\nu+(\xi^4)^2=l^2, \qquad (3)$$

is mapped onto itself by the group $SO(2,3)$, where A, B, $\cdots$ run from 0 to 4 and the lowercase Greek letters μ, ν, $\cdots$ run from 0 to 3, $\eta_{\mu\nu}=\mathrm{diag}(1, -1, -1, -1)$, and l is the AdS radius. The generators for $SO(2,3)$ and the commutation relations are

$$J_{AB} = \xi_A\partial_{\xi^B}-\xi_B\partial_{\xi^A} \qquad (4)$$

$$[J_{AB},J_{CD}] = J_{AD}\eta_{BC}-J_{AC}\eta_{BD}+$$
$$J_{BC}\eta_{AD}-J_{BD}\eta_{AC}. \qquad (5)$$

In terms of the 4d Beltrami coordinates

$$x^\mu=l\frac{\xi^\mu}{\xi^4}, \qquad (6)$$

the generators read

$$\begin{cases} (lP_\mu^-):=J_{4\mu}=l(\partial_{x^\mu}+l^{-2}x_\mu x^\nu\partial_{x^\nu}), \\ J_{\mu\nu}=x_\mu\partial_{x^\nu}-x_\nu\partial_{x^\mu}. \end{cases} \qquad (7)$$

In $3d$ realization, they are

$$\begin{cases} H^-=\partial_t+\nu^2tx^\mu\partial_{x^\mu}, & P_i^-=\partial_i+l^{-2}x_ix^\mu\partial_{x^\mu}, \\ K_i=t\partial_i-c^{-2}x_i\partial_t, & J_i=\dfrac{1}{2}\epsilon_i^{\ jk}(x_j\partial_k-x_k\partial_j), \end{cases} \qquad (8)$$

where $H^- = cP_0^-$, $t = x^0/c$, $\nu = c/l$, $\partial_t = \dfrac{\partial}{\partial t}$, $\partial_i = \dfrac{\partial}{\partial x^i}$, $x_i = -\delta_{ij}x^j$. The dimensions of H^-, P_i^-, and K_i are T^{-1}, L^{-1}, and c^{-1}, respectively.

2.2   IW contraction of Beltrami realization of AdS algebra in two opposite limits

Suppose the generators T_I $(I=1,\cdots n)$ span the Lie algebra of a Lie group

$$[T_I,T_J]=C_{IJ}^K T_K. \qquad (9)$$

Under a linear homogeneous non-singular transformation,

$$S_I=U_I^{\ J}T_J, \qquad (10)$$

the structure of the algebra will not change though the structure constants C_{IJ}^K will be replaced by other constants. If the first $m<n$ generators span a subalgebra of the algebra and the matrix U in Eq.(10) takes the form

$$(U_I^{\ J})=\begin{pmatrix} I_m & 0 \\ 0 & \varepsilon I_{n-m} \end{pmatrix}, \qquad (11)$$

where I_m and I_{n-m} are unit matrices, then the transformation (10) becomes singular when $\epsilon\to 0$ and will lead to a new algebra [1]. The operation is known as the IW contraction with respect to the subalgebra.

In order to study the explicit contraction of the Beltrami realization of **d**$_-$, the invariant parameters l and/or c in Eqs. (6)–(8) are replaced by running parameter l_r and/or c_r, respectively, which are still invariant under the AdS transformations. As a result, all quantities in Eqs. (6)–(8) will be replaced by the running ones. Then, multiplying a suitable parameter in terms of l_r

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1) The Konopel'chenko algebra [17] in the figure should be excluded because there exists no Hermite representations of generators for the fermionic part in it [16].

and/or c_r and taking the limit, one may obtain the contracted algebras.

For example, the Beltrami realization of $\mathbf{d}_-$ can be contracted explicitly in the following two ways to obtain the completely different realizations of the Poincaré algebra. At any point in the neighborhood of the "north pole" of Eq. (3), we have

$$\xi^\mu \approx 0, \qquad \xi^4 \approx l_r. \tag{12}$$

Following Eqs. (10) and (11), the generators of $SO(2,3)$ can be written as [18]

$$\begin{pmatrix} J_{\mu\nu} \\ P_\mu^- \end{pmatrix} = \begin{pmatrix} I_6 & 0 \\ 0 & \varepsilon I_4 \end{pmatrix} \begin{pmatrix} J_{\mu\nu} \\ J_{4\mu} \end{pmatrix} \quad \text{with } \varepsilon = l_r^{-1}. \tag{13}$$

In the limit $l_r \to \infty$ or $\varepsilon \to 0$, we have

$$J_{\mu\nu} = x_\mu \partial_{x^\nu} - x_\nu \partial_{x^\mu} \to J_{\mu\nu}, \tag{14}$$

$$\begin{aligned} P_\mu^- &= \frac{1}{l_r} J_{4\mu} = \frac{1}{l_r}(\xi_4 \partial_{\xi^\mu} - \xi_\mu \partial_{\xi^4}) \\ &= (\partial_{x^\mu} + l_r^{-2} x_\mu x^\nu \partial_{x^\nu}) \to \partial_{x^\mu} =: P_\mu. \end{aligned} \tag{15}$$

They are the generators of the Poincaré group.

On the other hand, one may use an alternative set of generators

$$\begin{pmatrix} J_{\mu\nu} \\ \Pi_\mu^- \end{pmatrix} = \begin{pmatrix} I_6 & 0 \\ 0 & \varepsilon' I_4 \end{pmatrix} \begin{pmatrix} J_{\mu\nu} \\ J_{4\mu} \end{pmatrix} \quad \text{with } \varepsilon' = l_r/l^2. \tag{16}$$

In the limit $l_r \to 0$,

$$J_{\mu\nu} = x_\mu \partial_{x^\nu} - x_\nu \partial_{x^\mu} \to J_{\mu\nu}, \tag{17}$$

$$\begin{aligned} \Pi_\mu^- &= \frac{l_r}{l^2} J_{4\mu} = \frac{l_r}{l^2}(\xi_4 \partial_{\xi^\mu} - \xi_\mu \partial_{\xi^4}) \\ &= \frac{l_r^2}{l^2}(\partial_{x^\mu} + l_r^{-2} x_\mu x^\nu \partial_{x^\nu}) \to l^{-2} x_\mu x^\nu \partial_{x^\nu} =: P_\mu'. \end{aligned} \tag{18}$$

They generate a group isomorphic to the Poincaré group. However, P_μ' are obviously different from P_μ. The new realization of $\mathbf{iso}(1,3)$ does not generate the translations on a Minkowski space-time, and thus P_μ' are called the pseudo-translation generators. (It should be noted that the definitions of the pseudo-translation generators here are different by a minus from the ones in Refs. [4–6, 9, 10], which do not affect the algebraic structure.) For brevity, the realization (2.12) and (2.13) is referred as to the ordinary Poincaré algebra, while the new realization (2.15) and (2.16) is referred as to the second Poincaré algebra.

Obviously, the ordinary and the second Poincaré algebras are the contraction of the same AdS algebra in two opposite limits. Algebraically, the ordinary and the second Poincaré algebras are identical to each other. If the abstract generators are dealt with just like the treatment in Ref. [2], the second Poincaré algebra cannot be distinguished from the ordinary one. However, if the realization is taken into account, the ordinary and second Poincaré algebras are dramatically different from each other in physics and geometry.

For example, the underlying geometries for the second Poincaré algebra contracting from AdS algebra is described [4–6] by a type-(2,0) degenerate symmetric tensor field

$$\boldsymbol{g} = -\frac{l^2}{(x \cdot x)^2}(\eta_{\mu\rho}\eta_{\nu\tau} - \eta_{\mu\nu}\eta_{\rho\tau})x^\rho x^\tau \, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu,$$

$$x \cdot x = \eta_{\mu\nu} x^\mu x^\nu > 0, \tag{19}$$

a type-(0,2) degenerate symmetric tensor field

$$\boldsymbol{h} = \frac{x \cdot x}{l^4} x^\mu x^\nu \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}, \tag{20}$$

and a connection ∇ compatible to $\boldsymbol{g}$ and $\boldsymbol{h}$, i.e.

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma_{\lambda\nu}^\kappa g_{\mu\kappa} - \Gamma_{\mu\lambda}^\kappa g_{\kappa\nu} = 0, \tag{21}$$

$$\nabla_\lambda h^{\mu\nu} = \partial_\lambda h^{\mu\nu} + \Gamma_{\lambda\kappa}^\nu h^{\mu\kappa} + \Gamma_{\lambda\kappa}^\mu h^{\kappa\nu} = 0, \tag{22}$$

with connection coefficients in Beltrami coordinate system,

$$\Gamma_{\nu\lambda}^\mu = -\frac{x_\nu \delta_\lambda^\mu + x_\lambda \delta_\nu^\mu}{x \cdot x}. \tag{23}$$

$(M^{\mathbf{P}_2}, \boldsymbol{g}, \boldsymbol{h}, \nabla)$ is invariant under P_2 transformations, namely, $\forall \xi \in \mathbf{p}_2 \subset TM^{\mathbf{P}_2}$, equations

$$\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{g} = (\xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda}\partial_\nu \xi^\lambda + g_{\lambda\nu}\partial_\mu \xi^\lambda)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = 0,$$

$$\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{h} = (\xi^\lambda \partial_\lambda h^{\mu\nu} - h^{\mu\lambda}\partial_\nu \xi^\lambda - h^{\lambda\nu}\partial_\mu \xi^\lambda)\partial_\mu \otimes \partial_\nu = 0,$$

$$[\mathcal{L}_{\boldsymbol{\xi}}, \nabla] = 0. \tag{24}$$

are valid simultaneously. The curvature tensor and Ricci curvature tensor are

$$R_{\mu\nu\rho}^\sigma = -l^{-2}(\delta_\rho^\sigma g_{\mu\nu} - \delta_\nu^\sigma g_{\mu\rho}), \quad R_{\mu\nu} = -3l^{-2}g_{\mu\nu}, \tag{25}$$

respectively. The structure of the manifold will become more transparent when the following coordinate transformation is made

$$x^0 = l^2 \eta^{-1} \cosh(r/l), \tag{26}$$

$$x^1 = l^2 \eta^{-1} \sinh(r/l)\sin(\theta)\cos(\phi), \tag{27}$$

$$x^2 = l^2 \eta^{-1} \sinh(r/l)\sin(\theta)\sin(\phi), \tag{28}$$

$$x^3 = l^2 \eta^{-1} \sinh(r/l)\cos(\theta). \tag{29}$$

Under the transformation, (19), (20) and (25) read

$$\boldsymbol{g} = \bar{g}_{\mu\nu}\mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu = -\mathrm{d}r^2 - l^2 \sinh^2(r/l)\mathrm{d}\Omega_2^2, \tag{30}$$

$$\boldsymbol{h} = \bar{h}^{\mu\nu}\partial_\mu \otimes \partial_\nu = \frac{\partial}{\partial\eta} \otimes \frac{\partial}{\partial\eta}, \tag{31}$$

$$\bar{R}_{\mu\nu} = -3l^2 \mathrm{diag}(0, -1, -\sinh^2(r/l), -\sinh^2(r/l)\sin^2\theta), \tag{32}$$

Table 1.   *AdS* algebra and its contractions.

algebra	symbol	generator set	$[\mathcal{H},\mathcal{P}]$	$[\mathcal{H},\mathcal{K}]$	$[\mathcal{P},\mathcal{P}]$	$[\mathcal{K},\mathcal{K}]$	$[\mathcal{P},\mathcal{K}]$	limit
AdS	$\mathbf{d}_-$	$(H^-,\boldsymbol{P}^-,\boldsymbol{K},\boldsymbol{J})$	$-\nu^2\boldsymbol{K}$	$\boldsymbol{P}^-$	$-l^{-2}\boldsymbol{J}$	$-c^{-2}\boldsymbol{J}$	$c^{-2}H^-$	
Poincaré	$\mathbf{p}$	$(H,\boldsymbol{P},\boldsymbol{K},\boldsymbol{J})$	0	$\boldsymbol{P}$	0	$-c^{-2}\boldsymbol{J}$	$c^{-2}H$	$l_r\to\infty$
	$\mathbf{p}_2$	$(H',\boldsymbol{P}',\boldsymbol{K},\boldsymbol{J})$		$\boldsymbol{P}'$			$c^{-2}H'$	$l_r\to0$
Galilei	$\mathbf{g}$	$(H,\boldsymbol{P},\boldsymbol{K}^{\mathbf{g}},\boldsymbol{J})$	0	$\boldsymbol{P}$	0	0	0	$l_r,c_r\to\infty,\nu_r\to0$
	$\mathbf{g}_2$	$(H',\boldsymbol{P}',\boldsymbol{K}^{\mathbf{c}},\boldsymbol{J})$		$\boldsymbol{P}'$				$l_r,c_r\to0,\nu_r\to\infty$
Carroll	$\mathbf{c}$	$(H,\boldsymbol{P},\boldsymbol{K}^{\mathbf{c}},\boldsymbol{J})$	0	0	0	0	$c^{-2}H$	$l_r\to\infty,c_r\to0$
	$\mathbf{c}_2$	$(H',\boldsymbol{P}',\boldsymbol{K}^{\mathbf{g}},\boldsymbol{J})$					$c^{-2}H'$	$l_r\to0,c_r\to\infty$
NH_-	$\mathbf{n}_-$	$(H^-,\boldsymbol{P},\boldsymbol{K}^{\mathbf{g}},\boldsymbol{J})$	$-\nu^2\boldsymbol{K}^{\mathbf{g}}$	$\boldsymbol{P}$	0	0	0	$l_r,c_r\to\infty,\nu_r=\nu$
	$\mathbf{n}_{-2}$	$(H^-,\boldsymbol{P}',\boldsymbol{K}^{\mathbf{c}},\boldsymbol{J})$	$-\nu^2\boldsymbol{K}^{\mathbf{c}}$	$\boldsymbol{P}'$				$l_r,c_r\to0,\nu_r=\nu$
para-Galilei	$\mathbf{g}'$	$(H',\boldsymbol{P},\boldsymbol{K}^{\mathbf{g}},\boldsymbol{J})$	$-\nu^2\boldsymbol{K}^{\mathbf{g}}$	0	0	0	0	$l_r,c_r,\nu_r\to\infty$
	$\mathbf{g}'_2$	$(H,\boldsymbol{P}',\boldsymbol{K}^{\mathbf{c}},\boldsymbol{J})$	$-\nu^2\boldsymbol{K}^{\mathbf{c}}$					$l_r,c_r,\nu_r\to0$
HN_-	$\mathbf{h}_-$	$(H,\boldsymbol{P}^-,\boldsymbol{K}^{\mathbf{c}},\boldsymbol{J})$	$-\nu^2\boldsymbol{K}^{\mathbf{c}}$	0	$-l^{-2}\boldsymbol{J}$	0	$c^{-2}H$	$c_r\to0$
	$\mathbf{p}'$	$(H',\boldsymbol{P}^-,\boldsymbol{K}^{\mathbf{g}},\boldsymbol{J})$	$-\nu^2\boldsymbol{K}^{\mathbf{g}}$				$c^{-2}H'$	$c_r\to\infty$

respectively. The geometry can be extended through $\eta=0$ and is globally $\mathbb{R}\times\mathbb{H}_3$.

With the same technique, one may obtain another 10 realizations of contracted algebras, which have the same **so**(3) spanned by $\boldsymbol{J}$ [9, 10]. Table 1 lists the generators and commutation relations of $\mathbf{d}_-$ and its contractions, except the common commutation relations involving $\boldsymbol{J}$. In Table 1, the generators are defined by

$$\begin{cases} H=\partial_t, & H'=\nu^2 tx^\mu\partial_{x^\mu}, \\ P_i=\partial_i, & P'_i=l^{-2}x_i x^\mu\partial_{x^\mu}, \\ K_i^{\mathbf{g}}=t\partial_i, & K_i^{\mathbf{c}}=-c^{-2}x_i\partial_t, \end{cases} \quad (33)$$

in addition to the definition of Eq. (8), and $\mathcal{H}$, $\mathcal{P}$, or $\mathcal{K}$ stands for the suitable one in $\{H^-, H, H'\}$, $\{\boldsymbol{P}^-, \boldsymbol{P}, \boldsymbol{P}'\}$, or $\{\boldsymbol{K}, \boldsymbol{K}^{\mathbf{g}}, \boldsymbol{K}^{\mathbf{c}}\}$, respectively. Their underlying geometries are presented in Ref. [9]. The name of anti-Hooke-Newton algebra HN_- comes from the fact that it is different from the anti-Newton-Hooke algebra NH_- by the replacement $H^- \leftrightarrow H$, $P_i \leftrightarrow P_i^-$, $K_i^{\mathbf{g}} \leftrightarrow K_i^{\mathbf{c}}$ [10]. It is also called the para-Poincaré algebra [2]. From the geometrical point of view, the first version of HN_- algebra is referred to as the anti-Hooke-Newton algebra ($\mathbf{h}_-$), while its second version is para-Poincaré algebra ($\mathbf{p}'$) [9]. In Table 1, the static algebra has been excluded because its time-translation generator $H^{\mathbf{s}}$ is meaningful only when the central extension is considered. It should be mentioned that in comparison with the table in Ref. [10] and [9] the minus in the definitions of the pseudo-translation generators H' and $\boldsymbol{P}'$ have been removed, thus the sets of generators for $\mathbf{n}_{-2}$ and $\mathbf{p}'$ are modified correspondingly, and the structure constants of $\mathbf{g}'$ have an overall minus, which do not affect the algebraic structure.

3   Supersymmetric extension of *AdS* algebra

The supersymmetric extension of $\mathbf{d}_-$ can be establi-

shed on a superspace spanned by coordinates ξ^A of dimension L subject to Eq. (3), a Majorana fermionic coordinate θ of dimension $L^{1/2}$. Concretely, we may choose the Weyl basis of the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

$$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \bar{\sigma}^0 \end{pmatrix} =: \gamma^4, \quad (34)$$

and take the charge conjugation matrix

$$C=\mathrm{i}\gamma^2\gamma^0=\mathrm{i}\begin{pmatrix} \sigma^2\bar{\sigma}^0 & 0 \\ 0 & \bar{\sigma}^2\sigma^0 \end{pmatrix}, \quad (35)$$

satisfying $C\gamma^\mu C^{-1} = -(\gamma^\mu)^{\mathrm{T}}$, where the superscript T denotes the transpose,

$$\begin{aligned} \sigma^\mu &= (\sigma^0,\sigma^{\mathrm{i}})=(I_2,\tau^{\mathrm{i}}), \\ \bar{\sigma}^\mu &= (\bar{\sigma}^0,\bar{\sigma}^{\mathrm{i}})=(I_2,-\tau^{\mathrm{i}}). \end{aligned} \quad (36)$$

in which I_2 is the 2×2 unit matrix and τ^{i} are 3 Pauli matrices. Acting on a scalar superfield, the bosonic generators are extended as

$$J_{\mu\nu}=x_\mu\partial_\nu-x_\nu\partial_\mu+\bar{\theta}\Sigma_{\mu\nu}\frac{\partial}{\partial\bar{\theta}}, \quad (37)$$

and

$$P_\mu^-=\frac{\partial}{\partial x^\mu}+\frac{1}{l^2}x_\mu x^\nu\partial_\nu+\frac{1}{l^2}\frac{x^\nu}{1+\sqrt{\sigma}}\bar{\theta}\Sigma_{\nu\mu}\frac{\partial}{\partial\bar{\theta}}, \quad (38)$$

where $\bar{\theta}$ is the Dirac conjugate of θ,

$$\Sigma^{\mu\nu}=\frac{1}{4}[\gamma^\mu,\gamma^\nu], \quad (39)$$

and $\sigma(x) := 1+l^{-2}\eta_{\mu\nu}x^\mu x^\nu > 0$ defines the domain of Beltrami-*AdS* spacetime. The fermionic generator Q and its conjugate $\bar{Q}$, which obey the Majorana condition $Q=C\bar{Q}^{\mathrm{T}}$ and are of dimension $L^{-1/2}$, may be chosen as

[19]

$$Q = \left(1-\frac{1}{4l}\bar{\theta}\theta\right)\Lambda\left\{\text{i}(1+\frac{1}{4l}\bar{\theta}\theta)\frac{\partial}{\partial\bar{\theta}}+\frac{\text{i}}{l}\theta\bar{\theta}\frac{\partial}{\partial\bar{\theta}}\right.$$
$$-\frac{\text{i}}{4l}(\gamma^\nu\theta)\bar{\theta}\left(\gamma_\nu-\frac{2\text{i}}{l}\frac{x^\mu}{1+\sqrt{\sigma}}\Sigma_{\mu\nu}\right)\frac{\partial}{\partial\bar{\theta}}$$
$$\left.-\frac{1}{2}\sqrt{\sigma}\gamma^\nu\theta\left(\frac{\partial}{\partial x^\nu}+\frac{l^{-2}x_\nu x^\mu}{1+\sqrt{\sigma}}\partial_\mu\right)\right\}, \quad (40)$$

$$\bar{Q} = -\left(1-\frac{1}{4l}\bar{\theta}\theta\right)\left\{\text{i}\left(1+\frac{1}{4l}\bar{\theta}\theta\right)\frac{\partial}{\partial\theta}\Lambda^{-1}-\frac{\text{i}}{l}(\bar{\theta}\Lambda^{-1})\bar{\theta}\frac{\partial}{\partial\bar{\theta}}\right.$$
$$-\frac{\text{i}}{4l}(\bar{\theta}\gamma^\nu\Lambda^{-1})\bar{\theta}\left(\gamma_\nu-\frac{2\text{i}}{l}\frac{x^\mu}{1+\sqrt{\sigma}}\Sigma_{\mu\nu}\right)\frac{\partial}{\partial\bar{\theta}}$$
$$\left.-\frac{1}{2}\sqrt{\sigma}(\bar{\theta}\gamma^\nu\Lambda^{-1})\left(\frac{\partial}{\partial x^\nu}+\frac{l^{-2}x_\nu x^\mu}{1+\sqrt{\sigma}}\partial_\mu\right)\right\}, \quad (41)$$

where

$$\Lambda=\left(\frac{1+\sqrt{\sigma}}{2\sqrt{\sigma}}\right)^{\frac{1}{2}}\left(I+\frac{\text{i}}{l}\frac{x^\mu}{1+\sqrt{\sigma}}\gamma_\mu\right), \quad (42)$$

I is a 4×4 unit matrix.

The commutators among the pure bosonic generators $J_{\mu\nu}$ and $P_\mu^-=l^{-1}J_{4\mu}$ with the supersymmetric extensions (37) and (38) are still given by Eq. (5). The structure relations of $\mathbf{osp}(1|4)$ involving fermionic parts are

$$\{Q,\bar{Q}\}=\frac{\text{i}}{c}\gamma^0 H^-+\text{i}\gamma^i P_i^--\frac{1}{l}\Sigma^{ij}J_{ij}-\frac{2c}{l}\Sigma^{0i}K_i. \quad (43)$$

$$[J_{ij},Q]=-\Sigma_{ij}Q, \qquad [K_i,Q]=-\frac{1}{c}\Sigma_{0i}Q, \quad (44)$$

$$[H^-,Q]=\frac{\text{i}\nu}{2}\gamma_0 Q, \qquad [P_i^-,Q]=\frac{\text{i}}{2l}\gamma_i Q, \quad (45)$$

When the invariant parameters l and/or c vary finitely and are labeled by l_r and/or c_r, respectively, the AdS superalgebra spanned by $(H_r^-, \boldsymbol{P}_r^-, \boldsymbol{K}_r, \boldsymbol{J}, Q_r, \bar{Q}_r)$ will remain the form of Eqs. (5), (44), (45), and (43) with l and/or c replaced by l_r and/or c_r. That is,

$$[J_i,J_j] = -\epsilon_{ij}^k J_k, \quad [J_i,(K_r)_j]=-\epsilon_{ij}^k (K_r)_k,$$
$$[J_i,H_r^-]=0, \quad [J_i,(P_r^-)_j]=-\epsilon_{ij}^k (P_r^-)_k,$$
$$[J_i,Q_r] = -\frac{1}{2}\epsilon_i^{jk}\Sigma_{jk}Q_r, [(K_r)_i,Q_r]=-\frac{1}{c_r}\Sigma_{0i}Q_r,$$
$$[H_r^-,Q_r] = \frac{\text{i}c_r}{2l_r}\gamma_0 Q_r, \quad [(P_r^-)_i,Q_r]=\frac{\text{i}}{2l_r}\gamma_i Q_r, \quad (46)$$

and

$$\{Q_r,\bar{Q}_r\} = \frac{\text{i}}{c_r}\gamma^0 H_r^-+i\gamma^i (P_r^-)_i-\frac{1}{l}\,_r\Sigma^{ij}J_{ij}$$
$$-\frac{2c_r}{l_r}\Sigma^{0i}(K_r)_i. \quad (47)$$

Further, the AdS superalgebra still remains if θ and $\bar{\theta}$ undergo a scale transformation

$$\theta\to\epsilon\theta, \qquad \bar{\theta}\to\epsilon\bar{\theta}, \quad (48)$$

in addition to the replacement of l and/or c by l_r and/or c_r, respectively.

4   Possible kinematical superalgebras

The contraction of the Beltrami realization of AdS superalgebra in different limits can be attained in the same way as the contraction of the Beltrami realization of AdS algebra. In all contractions, the supersymmetric extension of $J_i=\frac{1}{2}\epsilon_i^{jk}J_{jk}$ remains unchanged:

$$J_{ij}=x_i\partial_j-x_j\partial_i+\bar{\theta}\Sigma_{ij}\frac{\partial}{\partial\bar{\theta}}. \quad (49)$$

In the following, we shall consider these contractions one by one.

4.1   Two realizations of the Poincaré superalgebra

Recall that the ordinary and second realization of the generators of Poincaré algebra can be contracted from the generators of AdS algebra in the following way,

$$\mathbf{p:} \quad H=\lim_{l_r\to\infty}H_r^-, \qquad \boldsymbol{P}=\lim_{l_r\to\infty}\boldsymbol{P}_r^-,$$
$$\boldsymbol{K}, \qquad\qquad \boldsymbol{J}, \quad (50)$$

$$\mathbf{p_2:} \quad H'=\lim_{l_r\to0}\frac{l_r^2}{l^2}H_r^-, \qquad \boldsymbol{P}'=\lim_{l_r\to0}\frac{l_r^2}{l^2}\boldsymbol{P}_r^-,$$
$$\boldsymbol{K}, \qquad\qquad \boldsymbol{J}. \quad (51)$$

After supersymmetric extension and in the same limits,

$$\mathbf{p:} \quad H=\partial_t, \qquad P_i=\partial_{x^i},$$
$$K_i=t\partial_{x^i}-\frac{1}{c^2}x_i\partial_t+\frac{1}{c}\bar{\theta}\Sigma_{0i}\frac{\partial}{\partial\bar{\theta}}, \quad (52)$$

$$\mathbf{p_2:} \quad H'=\frac{c^2}{l^2}tx^\mu\partial_{x^\mu}, \quad P_i'=l^{-2}x_ix^\mu\partial_{x^\mu},$$
$$K_i=t\partial_{x^i}-\frac{1}{c^2}x_i\partial_t+\frac{1}{c}\bar{\theta}\Sigma_{0i}\frac{\partial}{\partial\bar{\theta}}, \quad (53)$$

plus Eq. (49). Their commutators are pure bosonic parts for the corresponding contracted superalgebras.

In the limit of $l_r\to\infty$, the fermionic generator reduces to that of Poincaré superalgebra, as expected,

$$Q^{\mathbf{p}}:=\lim_{l_r\to\infty}Q_r=\text{i}\frac{\partial}{\partial\bar{\theta}}-\frac{1}{2}\gamma^\mu\theta\frac{\partial}{\partial x^\mu}, \quad (54)$$

and the (anti-)commutators of super AdS algebra, (44), (45) and (43), contract to those for Poincaré superalgebra:

$$\{Q^{\mathbf{p}},\bar{Q}^{\mathbf{p}}\}=\text{i}\gamma^\mu P_\mu, \quad (55)$$

$$[J_{ij},Q^{\mathbf{p}}]=-\Sigma_{ij}Q^{\mathbf{p}},[K_i,Q^{\mathbf{p}}]=-\frac{1}{c}\Sigma_{0i}Q^{\mathbf{p}}, \quad (56)$$

$$[H,Q^{\mathbf{p}}]=0, \qquad [P_i,Q^{\mathbf{p}}]=0, \quad (57)$$

On the other hand, as $l_r \to 0$ the AdS superalgebra with scaled θ also contracts to the Poincaré superalgebra if $\epsilon \to 0$ according to l_r/l, denoted by ε_m^2, and the fermionic generator is chosen as

$$Q^{\mathbf{P2}} := \lim_{l_r \to 0} \varepsilon_m^2 Q_r$$

$$= \Lambda^{\mathbf{P2}} \left(\mathrm{i} \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2l^2} (\gamma^\nu \theta) x_\nu x^\mu \partial_\mu \right), \quad (58)$$

with $\quad \Lambda^{\mathbf{P2}} := \lim_{l_r \to 0} \Lambda_r = \frac{1}{\sqrt{2}} \left(I + \mathrm{i} \frac{x^\mu \gamma_\mu}{\sqrt{x \cdot x}} \right), \quad (59)$

and $\quad x \cdot x = \eta_{\mu\nu} x^\mu x^\nu.$

The fermionic parts of algebraic relations read

$$\{Q^{\mathbf{P2}}, \bar{Q}^{\mathbf{P2}}\} = \mathrm{i} \gamma^\mu P'_\mu, \quad (60)$$

$$[J_{ij}, Q^{\mathbf{P2}}] = -\Sigma_{ij} Q^{\mathbf{P2}}, \quad [K_i, Q^{\mathbf{P2}}] = -\frac{1}{c} \Sigma_{0i} Q^{\mathbf{P2}}, \quad (61)$$

$$[H', Q^{\mathbf{P2}}] = 0, \quad [P'_i, Q^{\mathbf{P2}}] = 0, \quad (62)$$

which can also be obtained from the algebraic relations for $\mathbf{osp}(1|4)$ in the limit $l_r \to 0$. These relations together with the pure bosonic ones present the supersymmetric extension of the second realization of Poincaré algebra. For brevity, we call it the second Poincaré superalgebra. Obviously, the invariant length parameter l appears in the fermionic operators as well as in the bosonic operators in the second Poincaré superalgebra. The second Poincaré superalgebra, as the supersymmetric extension of the kinematical algebra on the geometry Eqs. (30), (31) and (32) has a dramatically different meaning from the ordinary Poincaré superalgebra, though their algebraic structures are the same.

4.2   Supersymmetric extension of anti-Newton-Hooke algebra

It has been shown [9] that the algebra $\mathbf{n}_-$ and $\mathbf{n}_{-2}$ can be acquired by the contraction from $\mathbf{d}_-$,

$$\mathbf{n}_-: \quad H^- = \lim_{\substack{c_r, l_r \to \infty \\ \nu_r = \nu \, \text{fixed}}} H_r^-, \quad \boldsymbol{P} = \lim_{\substack{c_r, l_r \to \infty \\ \nu_r = \nu}} \boldsymbol{P}_r^-,$$

$$\boldsymbol{K}^{\mathbf{g}} = \lim_{\substack{c_r, l_r \to \infty \\ \nu_r = \nu}} \boldsymbol{K}_r, \quad \boldsymbol{J}, \quad (63)$$

$$\mathbf{n}_{-2}: \quad H^- = \lim_{\substack{c_r, l_r \to 0 \\ \nu_r = \nu}} H_r^-, \quad \boldsymbol{P}' = \lim_{\substack{c_r, l_r \to 0 \\ \nu_r = \nu}} \frac{l_r^2}{l^2} \boldsymbol{P}_r^-,$$

$$\boldsymbol{K}^{\mathbf{c}} = \lim_{\substack{c_r, l_r \to 0 \\ \nu_r = \nu}} \frac{c_r^2}{c^2} \boldsymbol{K}_r, \quad \boldsymbol{J}. \quad (64)$$

The supersymmetric extensions of the generators for $\mathbf{n}_-$ are

$$H^- = \partial_t + \frac{c^2}{l^2} t x^\mu \partial_{x^\mu}, \quad P_i = \partial_{x^i}, \quad K_i^{\mathbf{g}} = t \partial_{x^i}, \quad (65)$$

together with Eq. (49). The fermionic generator for the anti-Newton-Hooke superalgebra is attained in the limit

of $c_r, l_r \to \infty$ but $\nu_r = \nu$,

$$Q^{\mathbf{n}-} := \lim_{\substack{l_r, c_r \to \infty \\ \nu_r = \text{fixed}}} Q_r$$

$$= \Lambda^{\mathbf{n}-} \left(\mathrm{i} \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \sqrt{\sigma_{\mathbf{n}}} (\gamma^i \theta) \frac{\partial}{\partial x^i} \right), \quad (66)$$

$$\Lambda^{\mathbf{n}-} = \left(\frac{1 + \sqrt{\sigma_{\mathbf{n}}}}{2\sqrt{\sigma_{\mathbf{n}}}} \right)^{\frac{1}{2}} \left(I + \frac{\mathrm{i}\nu t \gamma_0}{1 + \sqrt{\sigma_{\mathbf{n}}}} \right), \quad (67)$$

where $\sigma_{\mathbf{n}} = 1 + \nu^2 t^2$. With these generators, the AdS superalgebra contracts to the $\mathbf{n}_-$ superalgebra,

$$\{Q^{\mathbf{n}-}, \bar{Q}^{\mathbf{n}-}\} = \mathrm{i} \gamma^i P_i - 2\nu \Sigma^{0i} K_i^{\mathbf{g}}, \quad (68)$$

$$[J_{ij}, Q^{\mathbf{n}-}] = -\Sigma_{ij} Q^{\mathbf{n}-}, \quad [K_i^{\mathbf{g}}, Q^{\mathbf{n}-}] = 0, \quad (69)$$

$$[H^-, Q^{\mathbf{n}-}] = \frac{\mathrm{i}\nu}{2} \gamma_0 Q^{\mathbf{n}-}, \quad [P_i, Q^{\mathbf{n}-}] = 0, \quad (70)$$

together the commutators for the bosonic generators as shown in Table 1.

In the process of the limit $c_r, l_r \to 0$, $\nu_r = \nu$, $\sigma \to 1 + \nu^2 t^2 - l_r^{-2} \delta_{ij} x^i x^j$ cannot preserve positive. Therefore, in the limit the fermionic generators are ill-defined even though the bosonic generators might be written down in the same way as for the supersymmetric extension of $\mathbf{n}_-$. In other words, no supersymmetric extension of the algebra $\mathbf{n}_{-2}$ can be obtained from the contraction of $\mathbf{osp}(1|4)$.

4.3   Supersymmetric extension of $\mathbf{h}_-$ and $\mathbf{p}'$

The generators of HN_- algebra ($\mathbf{h}_-$) and para-Poincaré algebra ($\mathbf{p}'$) can be contracted from AdS algebra as follows [9]:

$$\mathbf{h}_-: \quad H = \lim_{c_r \to 0} H_r^-, \quad \boldsymbol{P}^-,$$

$$\boldsymbol{K}^{\mathbf{c}} = \lim_{c_r \to 0} \frac{c_r^2}{c^2} \boldsymbol{K}_r, \quad \boldsymbol{J}, \quad (71)$$

$$\mathbf{p}': \quad H' = \lim_{c_r \to \infty} \frac{c^2}{c_r^2} H_r^-, \quad \boldsymbol{P}^-,$$

$$\boldsymbol{K}^{\mathbf{g}} = \lim_{c_r \to \infty} \boldsymbol{K}_r, \quad \boldsymbol{J}. \quad (72)$$

Their supersymmetric extensions are

$$\mathbf{h}_-: \quad H = \partial_t, \quad K_i^{\mathbf{c}} = -\frac{1}{c^2} x_i \partial_t,$$

$$P_i^{\mathbf{h}-} = \partial_{x^i} + l^{-2} x_i x^\mu \partial_{x^\mu} +$$

$$\frac{1}{l^2} \frac{x^j}{1 + \sqrt{\sigma_b}} \bar{\theta} \Sigma_{ji} \frac{\partial}{\partial \bar{\theta}}, \quad (73)$$

$$\mathbf{p}': \quad H' = \frac{c^2}{l^2} t x^\mu \partial_{x^\mu}, \quad K_i^{\mathbf{g}} = t \partial_{x^i},$$

$$P_i^{\mathbf{p}'-} = \partial_{x^i} + l^{-2} x_i x^\mu \partial_{x^\mu} \pm \frac{1}{l^2} \bar{\theta} \Sigma_{0i} \frac{\partial}{\partial \bar{\theta}}, \quad (74)$$

and Eq. (49), where $\sigma_b = \lim_{c_r \to 0} = 1 - l^{-2} \delta_{ij} x^i x^j > 0$ gives the the domain condition when $c_r \to 0$. The condition

defines the domain inside a ball of radius l and thus the σ in the limit is denoted with a subscript b. The third term in $P^{p'^-}{}_i$ is taken the same sign as t. It is remarkable that the spatial translation generators for $\mathbf{h}_-$ and $\mathbf{p}'$ are the same before supersymmetric extension, but after that they become different.

In the $c_r \to 0$ limit, the AdS superalgebra with scaled θ contracts to $\mathbf{h}_-$ superalgebra if $\epsilon \to 0$ as $\varepsilon_b := \sqrt{c_r/c} \to 0$ and if the fermionic generator is chosen as

$$
\begin{aligned}
Q^{\mathbf{h}_-} &:= \lim_{c_r \to 0} \varepsilon_b Q_r \\
&= \Lambda^{\mathbf{h}_-} \left(\mathrm{i} \frac{\partial}{\partial \bar{\theta}} - \frac{\sqrt{\sigma_\mathbf{b}}}{2c} \gamma_0 \theta \frac{\partial}{\partial t} \right),
\end{aligned} \tag{75}
$$

$$
\Lambda^{\mathbf{h}_-} = \left(\frac{1+\sqrt{\sigma_b}}{\sqrt{\sigma_b}} \right)^{\frac{1}{2}} \left(I + \frac{1}{l} \frac{\mathrm{i} x^i \gamma_i}{1+\sqrt{\sigma_\mathbf{b}}} \right). \tag{76}
$$

The algebraic relations involving the fermionic generators contract to

$$
\{Q^{\mathbf{h}_-}, \bar{Q}^{\mathbf{h}_-}\} = \mathrm{i} \gamma_0 \frac{H}{c} - 2\nu \Sigma^{0i} K_i^c, \tag{77}
$$

$$
[J_{ij}, Q^{\mathbf{h}_-}] = -\Sigma_{ij} Q^{\mathbf{h}_-}, \quad [K_i^{\mathbf{c}}, Q^{\mathbf{h}_-}] = 0, \tag{78}
$$

$$
[H, Q^{\mathbf{h}_-}] = 0, \quad [P_i^-, Q^{\mathbf{h}_-}] = \frac{\mathrm{i}}{2l} \gamma_i \bar{Q}^{\mathbf{h}_-}. \tag{79}
$$

On the other hand, when $c_r \to \infty$, the domain condition $\sigma > 0 \to (c_r/c)^2 \nu^2 t^2 > 0$, which is always valid. The AdS superalgebra with scaled θ contracts to $\mathbf{p}'$ superalgebra if $\epsilon \to 0$ as $\varepsilon_c := \sqrt{c/c_r} \to 0$ and if the fermionic generator takes the form

$$
\begin{aligned}
Q^{\mathbf{p}'} &:= \lim_{c_r \to \infty} \varepsilon_c Q_r \\
&= \Lambda^{\mathbf{p}'} \left(\mathrm{i} \frac{\partial}{\partial \bar{\theta}} - \frac{\nu t}{2l} \gamma^0 \theta x^\mu \partial_\mu \mp \frac{\nu t}{2} \gamma^i \theta \frac{\partial}{\partial x^i} \right),
\end{aligned} \tag{80}
$$

with

$$
\Lambda^{\mathbf{p}'} = \frac{I \pm \mathrm{i} \gamma_0}{\sqrt{2}}. \tag{81}
$$

The sign in $Q^{\mathbf{p}'}$ is opposite to the sign of t. The fermionic parts of algebraic relations of $\mathbf{p}'$ superalgebra are

$$
\{Q^{\mathbf{p}'}, \bar{Q}^{\mathbf{p}'}\} = \mathrm{i} \gamma_0 \frac{H'}{c} - 2\nu \Sigma^{0i} K_i^{\mathbf{g}}, \tag{82}
$$

$$
[J_{ij}, Q^{\mathbf{p}'}] = -\Sigma_{ij} Q^{\mathbf{p}'}, \quad [K_i^{\mathbf{g}}, Q^{\mathbf{p}'}] = 0, \tag{83}
$$

$$
[H', Q^{\mathbf{p}'}] = 0, \quad [P_i^-, Q^{\mathbf{p}'}] = \frac{\mathrm{i}}{2l} \gamma_i Q^{\mathbf{p}'}. \tag{84}
$$

4.4 Supersymmetric extension of Galilei algebra

The generators of Galilei algebra ($\mathbf{g}$) can be contracted from AdS algebra as follows [9]:

$$
\mathbf{g}: \quad H = \lim_{\substack{c_r,l_r \to \infty \\ \nu_r \to 0}} H_r^-, \qquad \boldsymbol{P} = \lim_{\substack{c_r,l_r \to \infty \\ \nu_r \to 0}} \boldsymbol{P}_r^-,
$$

$$
\boldsymbol{K}^{\mathbf{g}} = \lim_{\substack{c_r,l_r \to \infty \\ \nu_r \to 0}} \boldsymbol{K}_r, \qquad \boldsymbol{J}. \tag{85}
$$

The supersymmetric extensions of H, $\boldsymbol{P}$ and $\boldsymbol{K}$ are the same as the generators before supersymmetric extension. The fermionic generator of supersymmetric extension of $\mathbf{g}$ is

$$
\begin{aligned}
Q^{\mathbf{g}} &:= \lim_{\substack{c_r,l_r \to \infty \\ \nu_r \to 0}} Q_r = \lim_{c_r \to \infty} Q^{\mathbf{P}} = \lim_{\nu_r \to 0} Q^{\mathbf{n}-} \\
&= \mathrm{i} \frac{\partial}{\partial \bar{\theta}} - \frac{\gamma^i \theta}{2} \frac{\partial}{\partial x^i}.
\end{aligned} \tag{86}
$$

They span the Galilei superalgebra. In particular,

$$
\{Q^{\mathbf{g}}, \bar{Q}^{\mathbf{g}}\} = \mathrm{i} \gamma^i P_i, \tag{87}
$$

$$
[J_{ij}, Q^{\mathbf{g}}] = -\Sigma_{ij} Q^{\mathbf{g}}, \quad [K_i^{\mathbf{g}}, Q^{\mathbf{g}}] = 0, \tag{88}
$$

$$
[H, Q^{\mathbf{g}}] = 0, \quad [P_i, Q^{\mathbf{g}}] = 0. \tag{89}
$$

In the limit of $c_r, l_r \to 0$, $\nu_r \to \infty$, AdS algebra ($\mathbf{d}_-$) contracts to the second Galilei algebra ($\mathbf{g}_2$). But, since $\sqrt{\sigma}$ cannot remain real in the limiting process, no supersymmetric extension of $\mathbf{g}_2$ can be obtained in this way.

4.5 Supersymmetric extension of c and $\mathbf{c}_2$

As the contraction from $\mathbf{d}_-$, the generators of Carroll algebra $\mathbf{c}$ are [9]:

$$
\mathbf{c}: \quad H = \lim_{\substack{l_r \to \infty \\ c_r \to 0}} H_r^-, \qquad \boldsymbol{P} = \lim_{\substack{l_r \to \infty \\ c_r \to 0}} \boldsymbol{P}_r^-,
$$

$$
\boldsymbol{K}^{\mathbf{c}} = \lim_{\substack{l_r \to \infty \\ c_r \to 0}} \frac{c_r^2}{c^2} \boldsymbol{K}_r, \qquad \boldsymbol{J}, \tag{90}
$$

$$
\mathbf{c}_2: \quad H' = \lim_{\substack{l_r \to 0 \\ c_r \to \infty}} \frac{\nu^2}{\nu_r^2} H_r^-, \qquad \boldsymbol{P}' = \lim_{\substack{l_r \to 0 \\ c_r \to \infty}} \frac{l_r^2}{l^2} \boldsymbol{P}_r^-,
$$

$$
\boldsymbol{K}^{\mathbf{g}} = \lim_{\substack{l_r \to 0 \\ c_r \to \infty}} \boldsymbol{K}_r, \qquad \boldsymbol{J}. \tag{91}
$$

The expressions for H, H', $\boldsymbol{P}$, $\boldsymbol{P}'$, $\boldsymbol{K}^{\mathbf{c}}$ and $\boldsymbol{K}^{\mathbf{g}}$ after supersymmetric extension remain their forms before supersymmetric extension.

In the limit of $l_r \to \infty$ and $c_r \to 0$, the fermionic generators tend to

$$
\begin{aligned}
Q^{\mathbf{c}} &:= \lim_{\substack{l_r \to \infty \\ c_r \to 0}} \varepsilon_\mathbf{b} Q_r = \lim_{c_r \to 0} \varepsilon_b Q_r^{\mathbf{P}} = \lim_{l_r \to \infty} Q_r^{\mathbf{h}-} \\
&= \mathrm{i} \frac{\partial}{\partial \bar{\theta}} - \frac{\gamma^0 \theta}{2c} \frac{\partial}{\partial t}.
\end{aligned} \tag{92}
$$

and the AdS and Poincaré superalgebras with scaled θ ($\epsilon = \varepsilon_b$), and the anti-Hooke-Newton superalgebra contract to Carroll superalgebra,

$$
\{Q^{\mathbf{c}}, \bar{Q}^{\mathbf{c}}\} = \frac{\mathrm{i}}{c} \gamma^0 H, \tag{93}
$$

$$
[J_{ij}, Q^{\mathbf{c}}] = -\Sigma_{ij} Q^{\mathbf{c}}, \quad [K_i^{\mathbf{c}}, Q^{\mathbf{c}}] = 0, \tag{94}
$$

$$
[H, Q^{\mathbf{c}}] = 0, \quad [P_i, Q^{\mathbf{c}}] = 0. \tag{95}
$$

On the other hand, if the following scale transformations are made,

$$\theta \to \varepsilon_c \varepsilon_m^2 \theta = \sqrt{\frac{\nu l_r}{\nu_r l}} \theta, \tag{96}$$

in AdS superalgebra,

$$\theta \to \varepsilon_c \theta, \tag{97}$$

in the second Poincaré superalgebra, or

$$\theta \to \varepsilon_m^2 \theta, \tag{98}$$

in para-Poincaré superalgebra, in the limit of $c_r \to \infty$ and $l_r \to 0$ the fermionic generators will tend to

$$Q^{\mathbf{c}_2} := \lim_{\substack{l_r \to 0 \\ c_r \to \infty}} \varepsilon_c \varepsilon_m^2 Q = \lim_{c_r \to \infty} \varepsilon_c Q_r^{\mathbf{P}_2} = \lim_{l_r \to 0} \varepsilon_m^2 Q_r^{\mathbf{P}'}$$

$$= \Lambda^{\mathbf{P}'} \left(\mathrm{i} \frac{\partial}{\partial \bar{\theta}} - \frac{\nu t}{2l} \gamma^0 \theta x^\mu \partial_\mu \right), \tag{99}$$

where $\Lambda^{\mathbf{P}'}$ is given by Eq.(81), and the corresponding superalgebras contract to the second Carroll superalgebra,

$$\{Q^{\mathbf{c}_2}, \bar{Q}^{\mathbf{c}_2}\} = \frac{\mathrm{i}\gamma^0}{c} H', \tag{100}$$

$$[J_{ij}, Q^{\mathbf{c}_2}] = -\Sigma_{ij} Q^{\mathbf{c}_2}, \quad [K_i^{\mathbf{g}}, Q^{\mathbf{c}_2}] = 0, \tag{101}$$

$$[H', Q^{\mathbf{c}_2}] = 0, \quad [P_i', Q^{\mathbf{c}_2}] = 0. \tag{102}$$

4.6 Supersymmetric extension of para-Galilei algebra

The generators of para-Galilei algebra $\mathbf{g}'$ can be deduced from the contraction from $\mathbf{d}_-$ [9].

$$\mathbf{g}': \quad H' = \lim_{\substack{l_r, c_r \to \infty \\ \nu_r \to \infty}} \frac{\nu^2}{\nu_r^2} H_r^-, \quad \boldsymbol{P} = \lim_{\substack{l_r, c_r \to \infty \\ \nu_r \to \infty}} \boldsymbol{P}_r^-,$$

$$\boldsymbol{K}^{\mathbf{g}} = \lim_{\substack{l_r, c_r \to \infty \\ \nu_r \to \infty}} \boldsymbol{K}_r, \quad \boldsymbol{J}. \tag{103}$$

Again, after the supersymmetric extension, H', $\boldsymbol{P}$ and $\boldsymbol{K}^{\mathbf{g}}$ have the same expressions as those before supersymmetric extension. Also, suppose that the scale factor $\varepsilon = \varepsilon_\nu := \sqrt{\nu/\nu_r}$ in AdS and anti-Newton-Hooke superalgebras, $\varepsilon = \varepsilon_m$ in para-Poincaré superalgebras. In the limits of $l_r, c_r, \nu_r \to \infty$, the fermionic generators for $\mathbf{g}'$ superalgebra and the algebraic relations are attained,

$$Q^{\mathbf{g}'} := \lim_{c_r, l_r, \nu_r \to \infty} \sqrt{\frac{\nu}{\nu_r}} Q_r = \lim_{\nu_r \to \infty} \sqrt{\frac{\nu}{\nu_r}} Q_r^{\mathbf{n}-} = \lim_{l_r \to \infty} \sqrt{\frac{l_r}{l}} Q_r^{\mathbf{P}'}$$

$$= \Lambda^{\mathbf{P}'} \left(\mathrm{i} \frac{\partial}{\partial \bar{\theta}} \mp \frac{\nu t}{2} \gamma^i \theta \frac{\partial}{\partial x^i} \right), \tag{104}$$

and

$$\{Q^{\mathbf{g}'}, \bar{Q}^{\mathbf{g}'}\} = -2\nu \Sigma^{0i} K_i^{\mathbf{g}}, \tag{105}$$

$$[J_{ij}, Q^{\mathbf{g}'}] = -\Sigma_{ij} Q^{\mathbf{g}'}, \quad [K_i^{\mathbf{g}}, Q^{\mathbf{g}'}] = 0, \tag{106}$$

$$[H', Q^{\mathbf{g}'}] = 0, \quad [P_i, Q^{\mathbf{g}'}] = 0. \tag{107}$$

During the limit of $l_r, c_r \to 0$ but $c_r/l_r^2 = c/l^2$ is taken, σ cannot preserve positive for all x^μ. Therefore, the second para-Galilei algebra has no supersymmetric extension.

4.7 Summary

The fermionic generators in Beltrami realizations are gathered together,

$$Q = \left(1 - \frac{1}{4l}\bar{\theta}\theta\right) \Lambda \left\{ \mathrm{i}\left(1 + \frac{1}{4l}\bar{\theta}\theta\right) \frac{\partial}{\partial \bar{\theta}} + \frac{\mathrm{i}}{l}\theta\bar{\theta}\frac{\partial}{\partial\theta} \right.$$

$$- \frac{\mathrm{i}}{4l}(\gamma^\nu\theta)\bar{\theta}\left(\gamma_\nu - \frac{2\mathrm{i}}{l}\frac{x^\mu}{1+\sqrt{\sigma}}\Sigma_{\mu\nu}\right)\frac{\partial}{\partial\theta}$$

$$\left. - \frac{1}{2}\sqrt{\sigma}\gamma^\nu\theta\left(\frac{\partial}{\partial x^\nu} + \frac{l^{-2}x_\nu x^\mu}{1+\sqrt{\sigma}}\partial_\mu\right) \right\},$$

$$Q^{\mathbf{P}} = \mathrm{i}\frac{\partial}{\partial\bar{\theta}} - \frac{1}{2}\gamma^\mu\theta\frac{\partial}{\partial x_\mu},$$

$$Q^{\mathbf{P}_2} = \Lambda^{\mathbf{P}_2}\left(\mathrm{i}\frac{\partial}{\partial\bar{\theta}} - \frac{1}{2l^2}(\gamma^\nu\theta)x_\nu x^\mu\partial_\mu\right),$$

$$Q^{\mathbf{n}-} = \Lambda^{\mathbf{n}-}\left(\mathrm{i}\frac{\partial}{\partial\bar{\theta}} - \frac{1}{2}\sqrt{\sigma_{\mathbf{n}}}(\gamma^i\theta)\frac{\partial}{\partial x^i}\right),$$

$$Q^{\mathbf{h}-} = \Lambda^{\mathbf{h}-}\left(\mathrm{i}\frac{\partial}{\partial\bar{\theta}} - \frac{\sqrt{\sigma_{\mathbf{b}}}}{2c}\gamma_0\theta\frac{\partial}{\partial t}\right),$$

$$Q^{\mathbf{P}'} = \Lambda^{\mathbf{P}'}\left(\mathrm{i}\frac{\partial}{\partial\bar{\theta}} - \frac{\nu t}{2l}\gamma^0\theta x^\mu\partial_\mu - \frac{\nu t}{2}\gamma^i\theta\frac{\partial}{\partial x^i}\right),$$

$$Q^{\mathbf{g}} = \mathrm{i}\frac{\partial}{\partial\bar{\theta}} - \frac{\gamma^i\theta}{2}\frac{\partial}{\partial x^i},$$

$$Q^{\mathbf{c}} = \mathrm{i}\frac{\partial}{\partial\bar{\theta}} - \frac{\gamma^0\theta}{2c}\frac{\partial}{\partial t},$$

$$Q^{\mathbf{c}_2} = \Lambda^{\mathbf{P}'}\left(\mathrm{i}\frac{\partial}{\partial\bar{\theta}} - \frac{\nu t}{2l}\gamma^0\theta x^\mu\partial_\mu\right),$$

$$Q^{\mathbf{g}'} = \Lambda^{\mathbf{P}'}\left(\mathrm{i}\frac{\partial}{\partial\bar{\theta}} \mp \frac{\nu t}{2}\gamma^i\theta\frac{\partial}{\partial x^i}\right).$$

They are spinor representation of $\mathbf{so}(3)$ algebra, satisfying

$$[J_{ij}, \mathcal{Q}] = -\Sigma_{ij}\mathcal{Q}.$$

All realizations of the kinematical superalgebras satisfy the Jacobi identities. In addition, under the parity and time-reversal transformations the fermionic coordinates θ and $\bar{\theta}$ transform as

$$\Pi^{-1}\theta\Pi = \mathrm{i}\gamma^0\theta, \quad \Pi^{-1}\bar{\theta}\Pi = -\mathrm{i}\bar{\theta}\gamma^0,$$

$$\Theta^{-1}\theta\Theta = \mathrm{i}\gamma^1\gamma^3\theta, \quad \Theta^{-1}\bar{\theta}\Theta = \mathrm{i}\bar{\theta}\gamma^1\gamma^3,$$

and the fermionic generators Q and $\bar{Q}$ transform as

Table 2.   Supersymmetric extension of AdS algebra and its contractions.

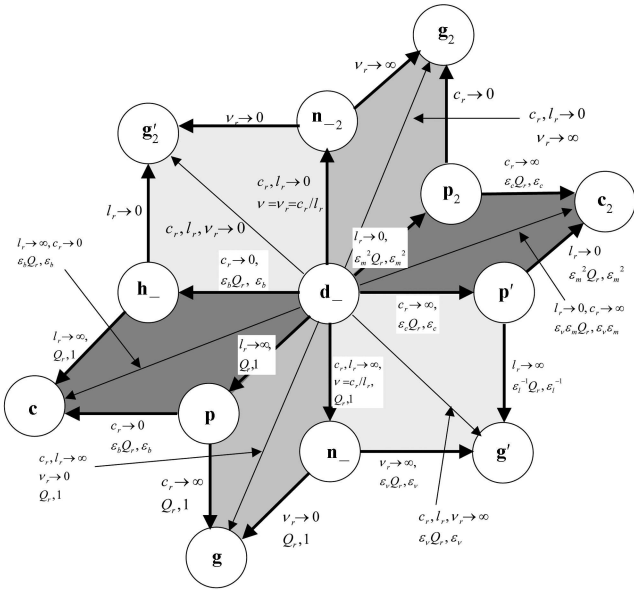| symbol	fermionic generators	$[\mathcal{H},\mathcal{Q}]$	$[\mathcal{P},\mathcal{Q}]$	$[\mathcal{K},\mathcal{Q}]$	$\mathcal{Q},\bar{\mathcal{Q}}\}$	scale of θ	limit
$\mathbf{d_-}$	$(Q,\bar{Q})$	$\frac{\nu}{2}\gamma_0 Q$	$\frac{1}{2l}\gamma_i Q$	$-\frac{1}{c}\Sigma_{0i}Q$	$i\gamma^\mu P_\mu - \frac{1}{l}\Sigma^{\mu\nu}J_{\mu\nu}$	1	
$\mathbf{p}$	$(Q^{\mathbf{p}},\bar{Q}^{\mathbf{p}})$	0	0	$-\frac{1}{c}\Sigma_{0i}Q^{\mathbf{p}}$	$i\gamma^\mu P_\mu$	1	$l_r\to\infty$
$\mathbf{p_2}$	$(Q^{\mathbf{p2}},\bar{Q}^{\mathbf{p2}})$	0	0	$-\frac{1}{c}\Sigma_{0i}Q^{\mathbf{p2}}$	$i\gamma^\mu P'_\mu$	ε_m^2	$l_r\to 0$
$\mathbf{n_-}$	$(Q^{\mathbf{n-}},\bar{Q}^{\mathbf{n-}})$	$\frac{\nu}{2}\gamma_0 Q^{\mathbf{n-}}$	0	0	$i\gamma^i P_i - 2\nu\Sigma^{0i}K_i^{\mathbf{g}}$	1	$l_r,c_r\to\infty,$ $\nu_r=\nu$
$\mathbf{h_-}$	$(Q^{\mathbf{h-}},\bar{Q}^{\mathbf{h-}})$	0	$\frac{1}{2l}\gamma_i Q^{\mathbf{h-}}$	0	$\frac{i\gamma^0}{c}H - 2\nu\Sigma^{0i}K_i^{\mathbf{c}}$	ε_b	$c_r\to 0$
$\mathbf{p'}$	$(Q^{\mathbf{p'}},\bar{Q}^{\mathbf{p'}})$	0	$\frac{1}{2l}\gamma_i Q^{\mathbf{p'}}$	0	$\frac{i\gamma^0}{c}H' - 2\nu\Sigma^{0i}K_i^{\mathbf{g}}$	ε_c	$c_r\to\infty$
$\mathbf{g}$	$(Q^{\mathbf{g}},\bar{Q}^{\mathbf{g}})$	0	0	0	$-i\gamma^i P_i$	1	$l_r,c_r\to\infty,$ $\nu_r\to 0$
$\mathbf{c}$	$(Q^{\mathbf{c}},\bar{Q}^{\mathbf{c}})$	0	0	0	$-\frac{i}{c}\gamma^0 H$	ε_b	$l_r\to\infty,$ $c_r\to 0$
$\mathbf{c_2}$	$(Q^{\mathbf{c2}}_\alpha,\bar{Q}^{\mathbf{c2}}_{\dot\alpha})$	0	0	0	$-\frac{i}{c}\gamma^0 H'$	$\varepsilon_\nu\varepsilon_m$	$l_r\to 0,$ $c_r\to\infty$
$\mathbf{g'}$	$(Q^{\mathbf{g'}}_\alpha,\bar{Q}^{\mathbf{g'}}_{\dot\alpha})$	0	0	0	$-2\nu\Sigma^{0i}K_i^{\mathbf{g}}$	ε_ν	$l_r,c_r\to\infty,$ $\nu_r\to\infty$

Fig. 1.   Contraction scheme for the AdS superalgebra and its contractions. The second version of anti-Newton-Hooke, Galilei, para-Galilei algebras have no supersymmetric extension. Thus, $\mathbf{n_{-2}}$, $\mathbf{g_2}$ and $\mathbf{g'_2}$ are denoted in light grey.

$$\Pi^{-1}\mathcal{Q}\Pi = i\gamma^0\mathcal{Q}, \quad \Pi^{-1}\bar{\mathcal{Q}}\Pi = -i\bar{\mathcal{Q}}\gamma^0,$$
$$\Theta^{-1}\mathcal{Q}\Theta = -i\gamma^1\gamma^3\mathcal{Q}, \quad \Theta^{-1}\bar{\mathcal{Q}}\Theta = -i\bar{\mathcal{Q}}\gamma^1\gamma^3,$$

respectively.

In the contraction approach, the Beltrami time t and Beltrami spatial coordinates x^i always remain unchanged, while the fermionic coordinates θ and $\bar{\theta}$ may change.

The characteristic (anti-)communitators of all supersymmetric extensions of the AdS algebra and its contractions are listed in Table 2.

The relationship of these superalgebras can be seen more clearly in Fig. 1.

The realizations of possible kinematical algebras without supersymmetric extension are depicted in light grey.

5   Concluding remarks

It has been shown in Ref. [9] that if the realizations of possible kinematical algebras are taken into account, more possible kinematics possessing the same

so(3) isotropy generated by $\boldsymbol{J}$ will be obtained than in the Bacry-Lévy-Leblond's classical work [2], having dramatically different physical and geometrical meanings [4–6, 9]. For example, the second realization of Poincaré group will not preserve the Minkowksi metric. In this paper, we apply the same method as in Ref. [9] to study the supersymmetric extension of the AdS algebra and present more explicit Beltrami realizations of the AdS superalgebra and its contractions. In particular, the supersymmetric extensions of the $\mathbf{p}_2$, $\mathbf{h}_-$, $\mathbf{p}'$, $\mathbf{c}_2$ and $\mathbf{g}'$ algebras are obtained. It is remarkable that the $\mathbf{p}_2$ and $\mathbf{c}_2$ superalgebras have the same algebraic structure as $\mathbf{p}$ and $\mathbf{c}$ superalgebras but have very different physical and geometrical meanings.

In this paper, we also show that not all Beltrami realizations of the possible kinematical algebras contracted from the AdS algebra have supersymmetric extension. The $\mathbf{n}_{-2}$, $\mathbf{g}_2$ and $\mathbf{g}'_2$ algebras are such a kind of algebras. The immediate causes are that the Beltrami fermionic generators of AdS superalgebra contain $\sigma = 1 + l^{-2}\eta_{\mu\nu}x^\mu x^\nu > 0$ and that it cannot keep positive when the corresponding limits are taken. In fact, the geometries invariant under the transformations generated by $\mathbf{n}_{-2}$, $\mathbf{g}_2$ or $\mathbf{g}'_2$ come from the contraction from the double-time de Sitter space [9]

$$\mathrm{d}s^2 = -\frac{1}{\sigma}\left(\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu - \frac{\eta_{\mu\lambda}\eta_{\nu\sigma}x^\lambda x^\sigma \mathrm{d}x^\mu\mathrm{d}x^\nu}{l^2\sigma}\right),$$

with $\sigma < 0$, which is also invariant under the transformations generated by **so**(2,3) but has the signature $(+,+,-,-)$. In other words, there is no $\mathbf{n}_{-2}$-, $\mathbf{g}_2$- or $\mathbf{g}'_2$-invariant geometries contracted from the AdS spacetime. Therefore, it is reasonable that the AdS superalgebra cannot give rise to the supersymmetric extensions of $\mathbf{n}_{-2}$, $\mathbf{g}_2$ and $\mathbf{g}'_2$ algebra by the contraction approach. The supersymmetric extension of $\mathbf{n}_{-2}$, $\mathbf{g}_2$ and $\mathbf{g}'_2$ might be attained by the contraction from the supersymmetric extension of **so**(2,3) realized on the double-time de Sitter spacetime (i.e. $\sigma < 0$). But, in that case, $\mathbf{p}$ will fail to be extended supersymmetrically.

Finally, since the central charge is not taken into account in the present paper, the fermionic generators (e.g.(86)) for $\mathbf{g}$) are different from the expressions in the literature [12].

References

1 Inönü E, Wigner E P. PNAS, 1953, **39**: 510–524
2 Bacry H, Lévy-Leblond J. M. J. Math. Phys, 1968, **9**: 1605–1614
3 Izmestev A A, Pogosyan G S, Sissakian A N et al. J. Math. Phys., 1999, **40**: 1549–1573
4 HUANG C G. Principle of Relativity, 24 Possible Kinematical Algebras and New Geometries with Poincaré Symmetry. In: Luo J, Zhou Z B, Yeh H C et al. Proc. of the 9th Asia-Pacific Int'l. Conf. Singapore: World Scientific, 2010. 118–129
5 HUANG C G, TIAN Y, WU X N et al. Chin. Phys. Lett., 2012, **29**: 040303
6 HUANG C G, T Y, WU X N et al. Commun. Theor. Phys., 2012, **57**: 553–562
7 GUO H Y, HUANG C G, XU Z et al. Mod. Phys. Lett. A, 2004, **19**: 1701–1709
8 GUO H Y, HUANG C G, XU Z et al. Phys. Lett. A, 2004 , **331**: 1–7
9 HUANG C G, TIAN Y, WU X N et al. Sci. China - Phys. Mech. Astron, 2012 , **55**: 1978–2003
10 GUO H Y, HUANG C G, WU H T et al. Sci. China-Phys. Mech. Astro, 2010, **53**: 591–597
11 Puzalowski R. Acta Phys. Austriaca, 1978, **50**: 45–52
12 Clark T E, Love S T. Nucl. Phys. B, 1984, **231**: 91–108
13 de Azcárraga J A, Ginestar D. J. Math. Phys, 1991, **32**: 3500–3508
14 Hussin V, Negro J. M. A. del Olmo. J. Phys. A: Math. Gen, 1999, **32**: 5097–5121
15 Rembielinski J, Tybor W. Acta Physica Polonica B, 1984, **15**: 611–615
16 Campoamor-Stursberg R, de Traubenberg M R. Phys. Atomic Nucl, 2010, **73**: 264–268
17 Konolpel'chenko B G. JETP Lett., 1975, **21**: 287–288
18 Gilmore R. Lie Group, Lie Algebras, and Some of Their Applications. New York: John Wiley & Sons Inc., 1974. 446–447
19 Ivanov E A, Sorini A S. J. Phys. A: Math. Gen, 1980, **13**: 1159–1188