D-brane superpotentials and Ooguri-Vafa invariants of compact Calabi-Yau threefolds*

XU Feng-Jun(徐锋军)[1)] YANG Fu-Zhong(杨富中)[2)]

College of Physical Sciences, University of Chinese Academy of Sciences, Yuquan Road 19 A, Beijing 100049, China

Abstract: We calculate the D-brane superpotentials for two compact Calabi-Yau manifolds $X_{14}(1,1,2,3,7)$ and $X_8(1,1,1,2,3)$ which are of non-Fermat type in the type II string theory. By constructing the open mirror symmetry, we also compute the Ooguri-Vafa invariants, which are related to the open Gromov-Witten invariants.

Key words: D-brane, superpotential, F-theory, Ooguri-Vafa invariant

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1 Introduction

When considering certain $\mathcal{N}=1$ supersymmetric string compactifications of type II string theories with space-filling D-branes, superpotential is an important quantity due to its BPS-property, which is exactly solvable. Superpotential is both important in physics and mathematics. The physical interest is served by the fact that a superpotential is also known as the holomorphic F-term in the effective lagrangian. The mathematical application is related to the non-perturbative stringy geometry. Similar to the Closed string theory, there is a geometry parameterizing the moduli space, namely N=1 special geometry. From the viewpoint of Mathematics, the presence of a superpotential describes an obstruction to continuous deformation of moduli space. Even more surprising results are derived from enumerative geometry. The superpotentials of the A-model at a large radius region count the disk invariants [1] which are related to open Gromov-Witten invariants.

The $N=1$, $d=4$ superpotential term can be computed by the open topological string amplitudes $\mathcal{F}_{g,h}$ of the A-model as follows

$$h\int \mathrm{d}^4x\mathrm{d}^2\theta\mathcal{F}_{g,h}(\mathcal{G}^2)^g(\mathcal{F}^2)^{h-1}, \tag{1}$$

where $\mathcal{G}$ is the gravitational chiral superfield and $\mathcal{F}$ is the gauge chiral superfield. The formula (1) at $g=0$, $h=1$ leads to in F-terms of N=1 supersymetric theories:

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\mathcal{W}(\Phi).$$

The papers [1–7] studied the superpotentials by the open-closed mirror symmetry for some non-compact Calabi-Yau manifolds. The work [1] gave the special Lagrangian submanifold as the geometrical picture of the classical A-brane. The papers [3, 4] calculated the superpotentials from an alternative approach of the variation of the mixed Hodge structure and the $N=1$ spacial geometry. Guided and motivated by these works, a progress on compact manifolds came from Refs. [8–11], which studied a class of involution branes independent of open deformation moduli. Furthermore, there were some related works on the superpotential and the BPS invariants, e.g. the Gromov-Witten invariants and the Ooguri-Vafa Invariants for compact Calabi-Yau manifolds depending on open-closed deformation moduli [12–33], where the works [12, 19] studied it by the direct integration，and by the matrix factorization and the conformal field theory method in Refs. [26, 27].

In this paper, we extend the available method to compute the superpotentials of the D-brane in a Fermat Calabi-Yau threefolds to the non-Fermat Calabi-Yau threefolds. In contrast to the Fermat Calabi-Yau threefolds, the non-Fermat Calabi-Yau threefolds have obtained fewer results from study so far. In fact, the best one can do for the mirror symmetry so far is to do it case by case, thus we calculate the D-brane superpotentials and extract the Ooguri-Vafa invariants for two compact non-Fermat Calabi-Yau threefolds by constructing and using the open-closed mirror symmetry and generalized GKZ system [34–39]. These superpotentials are non-perturbative in essence, and are impossibly obtained from the perturbative method. On the other hand, they determine the vacuum of the low energy effective theory,

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1) E-mail: xufengjun10@mails.ucas.ac.cn

2) E-mail: fzyang@ucas.ac.cn, corresponding author

and have potential phenomenological applications. Those Ooguri-Vafa Invariants predicted in this paper cannot be obtained from another approach, since there is not a systematic mathematical method to compute them by now, after all.

Furthermore, according to the duality between the type Ⅱ compactification with brane on the threefold and the M/F-theory compactifying on the Calabi-Yau fourfold without any branes but with fluxes [15–17, 22, 25, 40], in the weak decoupling limit $g_s \to 0$, the Gukov-Vafa-Witten superpotentials [41] $\mathcal{W}_{\rm GVW}$ of F-theory compactified on this dual fourfold agrees with superpotentials $\mathcal{W}$ of type Ⅱ string compactification on the threefold with branes at lowest order in g_s [16–18, 22, 25, 42]

$$\mathcal{W}_{\rm GVW}=\mathcal{W}+\mathcal{O}(g_{\rm s})+\mathcal{O}(\mathrm{e}^{-1/g_{\rm s}}). \tag{2}$$

Hence, in this limit, we can obtain the flux superpotential $\mathcal{W}_{\rm GVW}$ from the superpotential $\mathcal{W}$ which will be given in this paper.

Section 2 and Section 3 are an overview of toric Calabi-Yau Threefolds, Superpotentials and the generalized GKZ system. In Section 4, we extend these methods to two compact non-Fermat Calabi-Yau threefold with two deformation parameters: $X_{14}(1,1,2,3,7)$ and $X_8(1,1,1,2,3)$. Although some relevant works on the compact Calabi-Yau manifolds are available as mentioned above, the works about non-fermat cases are few. For the two non-Fermat Calabi-Yau threefolds, we study the superpotential, construct the open-closed mirror symmetry and extract some Ooguri-Vafa invariants for D-brane with a single open deformation moduli. Section 5 is a short summary.

2 Some known results

In this section, we collect some known results on the D-brane superpotential in type Ⅱ string compactification.

2.1 Superpotentials on Calabi-Yau threefolds

Type Ⅱ string theory compactified to four-dimensions on Calabi-Yau threefold is described by an effective $N=1$ supergravity action with non-trivial superpotentials as a section of a line bundle on the deformation space $\mathcal{M}$ when adding D-branes and background fluxes. The superpotential $\mathcal{W}_{\rm brane}$ for N D-brane wrapped the whole Calabi-Yau threefold is given by the holomorphic Chern-Simons theory, which controls the $U(N)$ gauge field A. When reduced dimensionally to the low dimension, the superpotentials can be obtained as [1, 43, 44]

$$\mathcal{W}_{\rm brane}=N_\nu\int_{\Gamma^\nu}\Omega^{3,0}(z,\hat{z})=\sum_\nu N_\nu \Pi^\nu, \tag{3}$$

where Γ^ν is a special Lagrangian 3-chain and $(z,\hat{z})$ are closed-string complex structure moduli and D-brane moduli from the open-string sector, respectively.

On the other hand, the $N=2$ supersymmetry will be broken to $N=1$ when including the background fluxes $H^{(3)}=H^{(3)}_{\rm RR}+\tau H^{(3)}_{\rm NS}$, where the $\tau=C^{(0)}+\mathrm{i}\mathrm{e}^{-\varphi}$ is the complexified type ⅡB coupling field. The background flux $H^{(3)}$ takes its values in the cohomology group $H^3(X,\mathbb{Z})$, and gives its contribution to superpotentials [45, 46]:

$$\mathcal{W}_{\rm flux}(z)=\int_X H^{(3)}_{\rm RR}\wedge\Omega^{3,0}=\sum_\alpha N_\alpha\cdot\Pi^\alpha(z)\ , \quad N_\alpha\in Z. \tag{4}$$

The general formula of a superpotential includes the contributions from both D-brane and background flux (here the NS-flux is ignored) which is as follows [3, 4]

$$\mathcal{W}(z,\hat{z})=\mathcal{W}_{\rm brane}(z,\hat{z})+\mathcal{W}_{\rm flux}(z)=\sum_{\gamma_\Sigma\in H^3(Z^*,\mathcal{H})} N_\Sigma\Pi_\Sigma(z,\hat{z}), \tag{5}$$

where Π_Σ is a relative period defined in a relative cycle $\Gamma\in H_3(X,D)$ whose boundary is wrapped by D-branes and $N_\Sigma=n_\Sigma+\tau m_\sigma$, τ is the dilaton of type Ⅱ string theory.

Given two D-brane configurations wrapping on two curves C^+ and C^-, from the physical point of view, the difference between the value of superpotentials for the two D-branes is $\mathcal{T}(z,\hat{z})=\mathcal{W}(C^+)-\mathcal{W}(C^-)$ with $\partial\Gamma_\Sigma=C^+-C^-$.

On the other hand, from the algebraic geometry point of view, $\mathcal{T}(z,\hat{z})$ is measured by the the relative period [3, 4, 47]

$$\Pi_\Sigma=\int_{\Gamma_\Sigma}\Omega(z,\hat{z}). \tag{6}$$

At the critical points determined by $\dfrac{\mathrm{d}W}{\mathrm{d}\hat{z}}=0$, we can obtain the on-shell domain wall tension [22]

$$T(z)=\mathcal{T}(z,\hat{z})|_{\hat{z}=\text{critic points}}, \tag{7}$$

which is just the Abel-Jacobi invariants [9, 20, 22, 48, 49]. At these critical points, the $C^\pm$ are the holomorphic curves in the Calabi-Yau threefold.

The superpotential can be expressed in terms of flat coordinates $(t,\hat{t})$ on the open-closed string moduli space in the A-model side. It is a key point that the superpotential is also the generating function of the Ooguri-Vafa invariants [1, 4, 15, 50]

$$\mathcal{W}(t,\hat{t})=\sum_{\vec{k},\vec{m}}G_{\vec{k},\vec{m}}q^{\mathrm{d}\vec{k}}\hat{q}^{\mathrm{d}\vec{m}}=\sum_{\vec{k},\vec{m}}\sum_{\mathrm{d}}n_{\vec{k},\vec{m}}\frac{q^{\mathrm{d}\vec{k}}\hat{q}^{\mathrm{d}\vec{m}}}{k^2}, \tag{8}$$

where $q=\mathrm{e}^{2\pi\mathrm{i}t}$, $\hat{q}=\mathrm{e}^{2\pi\mathrm{i}\hat{t}}$ and $n_{\vec{k},\vec{m}}$ are the Ooguri-Vafa invariants [50] counting disc instantons in relative homology class $(\vec{m},\vec{k})$, where $\vec{m}$ represents the elements of $H_1(D)$ and $\vec{k}$ represents an element of $H_2(X)$. $G_{\vec{k},\vec{m}}$ are the open Gromov-Witten invariants. However, the systematic mathematical theory of these Ooguri-Vafa invariants or open Gromov-Witten invariants is in development, and the practical mathematical method to compute these invariants is unavailable so far. From the physical point of view, these terms of the superpotential are the contribution of the string world-sheet sphere and disk instantons, etc respectively. Thus the topological string theory provides an effective approach to compute these geometric invariants.

2.2 Relative periods and generalized GKZ system

The generalized GKZ hypergeometric differential equation system originated from Ref. [34]. The authors of the papers [35–39] applied the GKZ method to study the mirror symmetry.

Let (W^*,W) be two toric varieties defined by the real four-dimensional reflexive polyhedron $(\triangle,\triangle^*)$, which are dual to eath other, respectively. The fan $\Sigma(\triangle)$ can be constructed from the set of cones over the faces of $\triangle^*$. From the fan $\Sigma(\triangle)$, the toric variety W can also be constructed by the standard toric geometric method as $W=P_{\Sigma(\triangle)}$, and analogously for $W^*=P_{\Sigma(\triangle^*)}$. The compact Calabi-Yau threefold (X^*,X) are the mirror pair defined as the hypersurfaces in (W^*,W), respectively.

The charge vectors of massless fields in the gauged linear sigma model (GLSM)[51] correspond to the generators l^a of the Mori cone of the toric variety [52–56], which are the same as the generators of the Mori cone of the compact Calabi-Yau threefold hypersurfaces embedded in it for our examples.

According to the toric geometry, the polyhedron defining the ambient toric variety of the Calabi-Yau hypersurface is just the Newton polyhedron of the polynomial P defining the Calabi-Yau manifold as the hypersurface in it. The defining polynomial P of the Calabi-Yau threefold X on the B-model side is determined by p integral points of $\triangle^*$ as follows

$$P=\sum_{i=0}^{p-1}a_i\prod_{k=0}^{4}X_k^{\nu^*_{i,k}}, \tag{9}$$

where the X_k are the toric coordinates of the algebraic torus $(\mathbb{C}^*)^4\subset W$. The complex coefficients a_i determine the complex structure of X up to a complex coordinate transformation, and give the torus invariant algebraic coordinates z_a of the moduli space of the complex structure of X [34, 35]:

$$z_a=(-1)^{l_0^a}\prod_i a_i^{l_i^a}, \tag{10}$$

where l_a, $a=1,\cdots,h^{2,1}(X)$ are generators of the Mori cone. The defining polynomial P can also be represented as

$$P=\sum_{i=0}^{p-1}a_i\prod_{\nu\in\triangle}x_j^{\langle\nu,\nu_i^*\rangle+1} \tag{11}$$

in terms of homogeneous coordinates x_j on the toric ambient variety.

The open-string sector of the D-branes wrapping on the curves can be described by the family of divisors $\mathcal{D}$ in the Calabi-Yau threefold, which is the complete intersection $P=0=Q(\mathcal{D})$ in the ambient toric variety. The $Q(\mathcal{D})$ defining the family of divisors can be given as [15, 22]

$$Q(\mathcal{D})=\sum_{i=p}^{p+p'-1}b_iX_k^{\nu^*_{i,k}}, \tag{12}$$

where the complex coefficients b_i and the above a_i describe together the moduli space of the open-closed string complex deformations. The additional p' vertices v_i^* correspond to the monomials in $Q(\mathcal{D})$.

The dual F-theory Four-fold X_4 can be obtained from the so called "Enhanced polyhedron" in the one higher real dimension than that of Calabi-Yau threefold in terms of the following extended vertices

$$\underline{\overline{\nu}_i^*}=\begin{cases}(\nu_i^*,0) & i=0,\cdots,p-1\\ (\nu_i^*,1) & i=p,\cdots,p+p'-1\end{cases}. \tag{13}$$

The period integrals can be written as

$$\Pi_i=\int_{\gamma_i}\frac{1}{P(a,X)}\prod_{j=1}^{n}\frac{\mathrm{d}X_j}{X_j}. \tag{14}$$

According to Refs. [35, 38, 39], the following differential operators annihilate these period integrals

$$\mathcal{L}(l)=\prod_{l_i>0}(\partial_{a_i})^{l_i}-\prod_{l_i<0}(\partial_{a_i})^{l_i},\ \mathcal{Z}_k=\sum_{i=0}^{p-1}\nu^*_{i,k}\vartheta_i,\ \mathcal{Z}_0=\sum_{i=0}^{p-1}\vartheta_i-1 \tag{15}$$

where $\vartheta_i=a_i\partial_{a_i}$. The equations $\mathcal{Z}_k\Pi(a_i)=0$ are the result of the invariance under the torus action on the toric variety [16, 35].

The solution to the GKZ system can be written as [22, 35, 39]

$$B_{l^a}(z^a;\rho)=\sum_{n_1,\cdots,n_N\in Z_0^+}\frac{\Gamma(1-\sum_a l_0^a(n_a+\rho_a))}{\prod_{i>0}\Gamma(1+\sum_a l_i^a(n_a+\rho_a))}\prod_a z_a^{n_a+\rho_a}. \tag{16}$$

In this paper we construct the family of divisors $\mathcal{D}$ defined by[1)]

$$Q(\mathcal{D})=x_1^{b_1}+\hat{z}x_2^{b_2}=0 \tag{17}$$

where $\hat{z}$ is the open string deformation, and b_1, b_2 are some appropriate integers. The relative 3-form $\underline{\Omega}:=$

1) In Refs. [13, 23] they considered another approach which blows up along the curve C and replaces the pair (X,C) with a non-Calabi-Yau manifold $\widehat{X}$.

$(\Omega_X^{3,0},0)$ and the relative periods satisfy a set of differential equations [3, 4, 12, 16, 22]

$$\mathcal{L}_a(\theta,\hat{\theta})\underline{\Omega}=\mathrm{d}\underline{\omega}^{(2,0)} \Rightarrow \mathcal{L}_a(\theta,\hat{\theta})\mathcal{T}(z,\hat{z})=0. \tag{18}$$

with some corresponding two-form $\underline{\omega}^{(2,0)}$. The differential operators $\mathcal{L}_a(\theta,\hat{\theta})$ can be reduced as [22]

$$\mathcal{L}_a(\theta,\hat{\theta}):=\mathcal{L}_a^b-\mathcal{L}_a^{bd}\hat{\theta} \tag{19}$$

with the operator $\mathcal{L}_a^b$ acting only on a bulk part from the closed sector, $\mathcal{L}_a^{bd}$ on a boundary part from the open-closed sector and $\hat{\theta}=\hat{z}\partial_{\hat{z}}$. One can get

$$2\pi\mathrm{i}\hat{\theta}\mathcal{T}(z,\hat{z})=\pi(z,\hat{z}), \tag{20}$$

by which we can simplify the problem to the reduced subsystem determined by the family of divisors $\mathcal{D}$.

3 Superpotential of Calabi-Yau $X_{14}(1, 1, 2, 3, 7)$

The $X_{14}(1, 1, 2, 3, 7)$ is defined as a degree 14 hypersurface which is the zero locus of polynomial P

$$P=x_1^{14}+x_2^{14}+x_3^7+x_3x_4^4+x_5^2. \tag{21}$$

The GLSM charge vectors for this manifold are [57]

	0	1	2	3	4	5	6
l_1	0	−2	−2	−4	1	0	7
l_2	−2	1	1	2	0	1	−3

(22)

The mirror manifold is [57] X^* is $X^*=\widehat{X}/H$, where $\widehat{X}=X_{28}(2,\ 2,\ 3,\ 7,\ 14)$ and $H=(h_i^k)=\frac{1}{14}(1,\ 13,\ 0,\ 0,\ 0)$, $\frac{1}{2}$ (1, 0, 0, 0, 1) which act by $x_i\to\exp(2\pi\mathrm{i}h_i^k)x_i$. We consider the following curves

$$C_{\alpha,\pm}=\{x_4=\xi x_5,x_3=0,x_1^7x_4+x_2^7x_5=0\},\ \xi^3=-1, \tag{23}$$

which are on the family of divisor

$$Q(\mathcal{D})=x_4^3+z_3x_5^3 \tag{24}$$

at the critical points $z_3=1$.

By the generalized GKZ system, the period on the surface $P=0=Q(\mathcal{D})$ has the form

$$\begin{aligned}\pi &= \frac{c}{2}B_{(\hat{l}_1,\hat{l}_2)}\left(u_1,u_2;0,\frac{1}{2}\right)\\ &= \sum_{n_1,n_2}\frac{cz_1^{n_1}z_2^{\frac{1}{2}+n_2}\Gamma\left(3\left(n_1+\frac{1}{2}\right)+(n_2)+1\right)}{\Gamma(2+2n_1)\Gamma(1+n_2)^3\Gamma(n_1-2n_2)}\\ &= -\frac{4c}{\pi^{\frac{3}{2}}}\sqrt{u_1}u_2+\mathcal{O}((u_1u_2)^{3/2})\end{aligned} \tag{25}$$

which vanishes at the critical locus $u_2=0$ in terms of new parameters as

$$u_1=\frac{-z_1}{z_3}(1-z_3)^2 \quad u_2=z_2 \tag{26}$$

and $z_{1,2}$ are relevant coordinates in the large complex structure limit defined as (10) . Following the [31], the off-shell superpotentials can be obtained by integrating the π:

$$\mathcal{T}_a^{\pm}(z_1,z_2)=\frac{1}{2\pi\mathrm{i}}\int\pi(z_3)\frac{\mathrm{d}z_3}{z_3}, \tag{27}$$

with the appropriate integral constants [22], the superpotentials can be chosen as $\mathcal{W}^+=-\mathcal{W}^-$. In this convention, the on-shell superpotentials can be obtained as

$$\begin{aligned}2\mathcal{W}^+ &= \frac{1}{2\pi\mathrm{i}}\int_{-z_3}^{z_3}\pi(z_3)\frac{\mathrm{d}z_3}{z_3}, \quad W^{\pm}(z_1,z_2)\\ &= \mathcal{W}^{\pm}(z_1,z_2,z_3)|_{z_3=1}.\end{aligned} \tag{28}$$

Eventually, the superpotentials are:

$$\begin{aligned}\mathcal{W}^{\pm}(z_1,z_2,z_3) &= \mp\sum_{n_1,n_2}\frac{cz_1^{n_1}z_2^{\frac{1}{2}+n_2}z_3^{\frac{-1-2n_1}{2}}\Gamma(3n_1+n_2+1)}{4\pi(-1+4n_1^2)\Gamma(2+2n_1)\Gamma(1+n_2)^3\Gamma(n_1-2n_2)}\\ &\times\left\{(1-2n_1)\ {}_2F_1\left(-\frac{1}{2}-n_1,-2n_1,\frac{1}{2}-n_1;z_3\right)+(1+2n_1)z_3\ {}_2F_1\left(\left(\frac{1}{2}-n_1,-2n_1,\frac{3}{2}-n_1;z_3\right)\right)\right\},\end{aligned} \tag{29}$$

and those can divide into two parts

$$\mathcal{W}^{\pm}(z_1,z_2,z_3)=W^{\pm}(z_1,z_2)+f(z_1,z_2,z_3), \tag{30}$$

where the $f(z_1,z_2,z_3)$ are related to the open-string parameter, W are the on-shell superpotential defined as

$$W^{\pm}=\mp\frac{c}{8}\sum_{n_1,n_2}\frac{z_1^{n_1}z_2^{\frac{1}{2}+n_2}\Gamma\left(3(n_1)+\left(n_2+\frac{1}{2}\right)+1\right)}{\Gamma\left(n_2+\frac{1}{2}+1\right)^3\Gamma(n_1+1)^2\Gamma(n_1-2n_2)}. \tag{31}$$

The additional GLSM charge vectors corresponding to the divisor (24) are

	0	1	2	3	4	5	6	7	8
l_4	−1	1	0	0	0	0	0	1	−1

(32)

The classic A-brane in the mirror Calabi-Yau manifold X^* of X determined by the additional charge vectors (0, −1, 1, 0, 0, 0, 0) is a special Lagrangian submanifold of X^* defined as [1, 2, 6, 7, 15]

$$-|x_4|+|x_5|=\eta, \tag{33}$$

where x_i are the coordinates on X^*, η is a Kähler moduli parameter with $\hat{z}=\epsilon e^{-\eta}$ for a phase ϵ.

The flat coordinates in the A-model at a large radius regime are related to the flat coordinates of the B-model at a large complex structure regime by mirror map $t_i=\frac{\omega_i}{\omega_0}$, $\omega_i:=D_i^{(1)}\omega_0(z,\rho)|_{\rho=0}$

$$\begin{aligned} q_1 &= z_1-6z_1^2+63z_3-866z_1^4+68z_1^3z_2+\mathcal{O}(z^5) \\ q_2 &= z_2+14z_1z_2-7z_1^2z_2+294z_1^3z_2-96z_1^2z_2^2+\mathcal{O}(z^5), \end{aligned} \quad (34)$$

and we can obtain the inverse mirror map in terms of $q_i=e^{2\pi i t_i}$

$$\begin{aligned} z_1 &= q_1+6q_1^2+9q_1^3+56q_1^4-68q_1^3q_2+\mathcal{O}(q^5) \\ z_2 &= q_2-14q_1q_2-119q_1^2q_2-924q_1^3q_2+96q_1^2q_2^2\mathcal{O}(q^5). \end{aligned} \quad (35)$$

Using the modified multi-cover formula [8, 21] for this case

$$\frac{W^{\pm}(z(q))}{w_0(z(q))}=\frac{1}{(2i\pi)^2}\sum_{k\ \text{odd}}\ \sum_{d_2,d_1\text{odd}\geqslant 0} n^{\pm}_{d_1,d_2}\frac{q_1^{kd_1}q_2^{kd_2/2}}{k^2}, \quad (36)$$

the superpotentials $W^{\pm}$, at the critical points $z_3=1$, give Ooguri-Vafa invariants n_{d_1,d_2} for the normalization constants $c=1$, which are listed in Table 1.

Table 1. Ooguri-Vafa invariants $n_{(d_1,d_2)}$ for the on-shell superpotential W^+ on the 3-fold $X_{14}(1, 1, 2, 3, 7)$, the horizonal coordinates represent d_2 and vertical coordinates represent d_1.

$d_1\backslash d_2$	0	1	2	3	9
1	2	0	0	0	0
3	−2	0	0	0	0
5	10	−220	0	0	0
7	−84	400	0	0	0
9	858	−1844	−13500	0	0
11	−9878	−61760	1501528	0	0

Another interesting thing which should be mentioned is the superpotential W^+ encode — the information of the superpotential of the non-compact geometry $\mathcal{O}(-3)_{\mathbb{P}^2}$ in the limit of $q_1\to 0$. This can be shown by $n_{d_1,0}=n_k$, where n_k are the disc invariants of $\mathcal{O}(-3)_{\mathbb{P}^2}$ which were studied in work [58]. See more details in the appendix of Ref. [22].

4 Superpotential of Calabi-Yau $X_8(1, 1, 1, 2, 3)$

The $X_8(1, 1, 1, 2, 3)$ is defined as a degree 8 hypersurface which is the zero locus of polynomial P

$$P=x_1^8+x_2^8+x_3^8+x_4^4+x_4x_5^2. \quad (37)$$

The GLSM charge vectors l_a are the generators of the Mori cone as follows [57]

$$\begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline l_1 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \\ l_2 & -2 & 1 & 1 & 1 & 2 & 0 & -3 \end{array}. \quad (38)$$

The mirror manifolds X^* is $X^*=\widehat{X}/H$, where $\widehat{X}=X_8(1, 1, 1, 1, 4)$ and $H=(h_i^k)=\frac{1}{8}(7, 0, 0, 1, 0)$, $\frac{1}{8}(7, 0, 1, 0, 0)$ which act by $x_i\to x_i\exp(2\pi i h_i^k)$.

We consider the following curves

$$C_{\alpha,\pm}=\{x_2=\xi x_1, x_3=\xi x_4, x_5^2+\psi x_1x_2x_3x_4x_5=0\},\ \xi^8=-1 \quad (39)$$

which are on the family of divisor

$$Q(\mathcal{D})=x_2^8+z_3x_1^8 \quad (40)$$

at the critical points $z_3=1$.

By the GKZ system, the period on the hypersurface has the form

$$\pi=\frac{c}{2}B_{\{\hat{l}_1,\hat{l}_2\}}\left(u_1,u_2;0,\frac{1}{2}\right)=-\frac{4c}{\pi^{\frac{3}{2}}}u_1\sqrt{u_2}+\mathcal{O}((u_1u_2)^{3/2}), \quad (41)$$

which vanishes at the critical locus $u_2=0$ expressed in terms of new parameters as

$$u_1=z_1 \quad u_2=\frac{-z_2}{z_3}(1-z_3)^2. \quad (42)$$

Similarly, the off-shell superpotentials can be obtained by integrating the π:

$$\mathcal{T}_a^{\pm}(z_1,z_2,z_3)=\frac{1}{2\pi i}\int\pi(\hat{z})\frac{d\hat{z}}{\hat{z}}, \quad (43)$$

with the appropriate integral constants [22], the superpotentials can be chosen as $\mathcal{W}^+=-\mathcal{W}^-$. In this convention, the on-shell superpotentials can be obtained as

$$2\mathcal{W}^+=\frac{1}{2\pi i}\int_{-z_3}^{z_3}\pi(z_3)\frac{dz_3}{z_3}, W^{\pm}(z_1,z_2)=\mathcal{W}^{\pm}(z_1,z_2,z_3)|_{z_3=1}. \quad (44)$$

Eventually, the superpotentials are:

$$\begin{aligned} \mathcal{W}^{\pm}(z_1,z_2,z_3) = \mp\sum_{n_1,n_2}&\frac{cz_1^{n_1}z_2^{\frac{1}{2}+n_2}z_3^{\frac{-1-2n_1}{2}}\Gamma(2n_1+2n_2+1)}{4\pi(-1+4n_1^2)\Gamma(2+2n_1^2)\Gamma(1+n_2)\Gamma\left(-\frac{1}{2}-3n_1+n_2\right)\Gamma\left(\frac{3}{2}+n_2\right)} \\ &\times\left\{(1-2n_1)\ {}_2F_1\left(-\frac{1}{2}-n_1,-2n_1,\frac{1}{2}-n_1;z_3\right)+(1+2n_1)z_3\ {}_2F_1\left(\left(\frac{1}{2}-n_1,-2n_1,\frac{3}{2}-n_1;z_3\right)\right)\right\}, \end{aligned} \quad (45)$$

and those can divide into two parts

$$\mathcal{W}^{\pm}(z_1,z_2,z_3)=W^{\pm}(z_1,z_2)+f(z_1,z_2,z_3), \tag{46}$$

where the $f(z_3)$ are related to the open-string parameter, W is the on-shell superpotential defined as

$$W^{\pm}(z_1,z_2)=\mp\frac{c}{8}B_{[l_1,l_2]}\left((z_1,z_2);0,\frac{1}{2}\right), \tag{47}$$

substituting the vector l_1, l_2 in this hypersurface, the superpotentials are

$$W^{\pm}=\mp\frac{c}{8}\sum_{n_1,n_2}\frac{z_1^{n_1}z_2^{\frac{1}{2}+n_2}\Gamma\left(2n_1+2\left(n_2+\frac{1}{2}\right)+1\right)}{\Gamma\left(n_2+\frac{1}{2}+1\right)^3\Gamma(n_1+1)\Gamma\left(2\left(n_1+\frac{1}{2}\right)+1\right)\Gamma\left(n_1-3\left(n_2+\frac{1}{2}\right)+1\right)}. \tag{48}$$

The additional GLSM charge vectors corresponding to the divisor (40) are

$$\begin{array}{c|ccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline l_4 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}. \tag{49}$$

The classic A-brane in the mirror Calabi-Yau manifold X^* of X determined by the additional charge vectors (0, −1, 1, 0, 0, 0, 0) is a special Lagrangian submanifold of X^* defined as [1, 2, 6, 7, 15]

$$-|x_1|^2+|x_2|^2=\eta, \tag{50}$$

where x_i are coordinates on X^*, η is a Kähler moduli parameter with $\hat{z}=\epsilon e^{-\eta}$ for a phase ϵ.

Table 2. Ooguri-Vafa invariants $n_{(d_1,\ d_2)}$ for the on-shell superpotential W^+ on the 3-fold $X_8(1, 1, 1, 2, 3)$, the horizonal coordinates represent d_2 and vertical coordinates represent d_1.

$d_1\backslash\frac{d_2}{2}$	1	3	5	7
0	2	−2	10	−84
1	−40	40	−360	7232
2	−180	−200	5500	−215356
3	−40	−29720	66963200	2314120
4	−628	−625424	25006400960	—

The flat coordinates in the A-model at a large radius regime are related to the flat coordinates of the B-model at a large complex structure regime by mirror map $t_i=\frac{\omega_i}{\omega_0}$, $\omega_i:=D_i^{(1)}\omega_0(z,\rho)|_{\rho=0}$

$$\begin{aligned} q_1 &= z_1+2z_1^2+5z_1^3+2z_1z_2-12z_1^{12}z_2-13z_1z_2^2\mathcal{O}(z^4) \\ q_2 &= z_2-6z_2^2+12z_1z_2+100z_1^2z_2-24z_1z_2^2+\mathcal{O}(z^4), \end{aligned} \tag{51}$$

and we can obtain the inverse mirror map in terms of $q_i=\mathrm{e}^{2\pi\mathrm{i}t_i}$

$$\begin{aligned} z_1 &= q_1-2q_1^2+3q_1^3-2q_1q_2+48q_1^2q_2+5q_1q_2^2+\mathcal{O}(q^4) \\ z_2 &= q_2+6q_2^2+9q_2^3-12q_1q_2+68q_1^2q_2-168q_1q_2^2\mathcal{O}(q^4). \end{aligned} \tag{52}$$

Using the modified multi-cover formula [8, 21], as the Eq. (36) for this case, the Ooguri-Vafa invariants n_{d_1,d_2} which are listed in Table 2 can be extracted for the normalization constants $c=1$ from the superpotentials $W^{\pm}$ at the critical points $\hat{z}=1$.

In the limit $z_1=0$ we also give the superpotential of non-compact manifold $\mathcal{O}(-3)_{\mathbb{P}^2}$ which can be proved by $n_{0,d_2}=n_k$.

5 Summary

For two compact Calabi-Yau threefolds of non-fermat type constructed by using toric geometry [57], we study superpotentials of the D-branes on the these manifolds by extending the GKZ method to the Calabi-Yau manifolds of the non-Fermat type on the B-model side. Furthermore, with the mirror symmetry, we calculate the non-perturbative superpotential on the A-model side and extract the Ooguri-Vafa invariants for D-brane with a single open deformation moduli.

On the other hand, we will study these problems in this paper from an alternative approach [13, 14, 17, 23, 25] that treated the complex structure deformations of the Calabi-Yau threefolds and the open string deformations of the D-branes on an equal footing by studying the complex structure deformations of a non-Calabi-Yau manifold obtained by blowing up the original Calabi-Yau threefold along the curve wrapped by the D-brane. It is interesting to compare the results obtained from the two approaches for the same system of the Calabi-Yau threefold and D-branes.

Furthermore, we are going to treat the obstruction problem to the deformations with the quiver gauge theories [59],the matrix factorizations and the A_∞ -algebraic structure [60–68] in the derived category of coherent sheaves.

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