Fractal geometrical properties of nuclei*

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Abstract: We present a new idea to understand the structure of nuclei and compare it to the liquid drop model. After discussing the probability that the nuclear system may be a fractal object with the characteristic of self-similarity, the irregular nuclear structure properties and the self-similarity characteristic are considered to be an intrinsic aspect of the nuclear structure properties. For the description of nuclear geometric properties, the nuclear fractal dimension is an irreplaceable variable similar to the nuclear radius. In order to determine these two variables, a new nuclear potential energy formula which is related to the fractal dimension is put forward and the phenomenological semi-empirical Bethe–Weizsäcker binding energy formula is modified using the fractal geometric theory. One important equation set with two equations is obtained, which is related to the concept that the fractal dimension should be a dynamic parameter in the process of nuclear synthesis. The fractal dimensions of the light nuclei are calculated and their physical meanings are discussed. We compare the nuclear fractal mean density radii with the radii calculated by the liquid drop model for the light stable and unstable nuclei using rational nuclear fractal structure types. In the present model of fractal nuclear structure there is an obvious additional feature compared to the liquid drop model, since the present model can reflect the geometric information of the nuclear structure, especially for nuclei with clusters, such as the α-cluster nuclei and halo nuclei.

Key words: nuclear radii, fractal dimension, binding energy, nuclear potential energy

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1 Introduction

Both in theory [1–5] and experiment [6–12], the concept of the cluster structure plays an important role in nuclear reactions, nuclear structure and nuclear excitations. The nuclear distribution is non-uniform, especially for halo nuclei. For instance, ^{11}Li has a core and valence nucleons (^{9}Li+n+n) halo structure and the matter distribution is very non-uniform [10, 13]. Its radius is much larger than that given by the usual expression ($r_0A^{\frac{1}{3}}$), which depends on the uniform-density liquid drop model.

In addition, the phenomenological semi-empirical Bethe–Weizsäcker binding energy formula [14–16] for the masses of nuclei has been derived from the liquid drop model and successfully used to calculate the binding energies for stable nuclei and nuclei very near the stable line. However, there are two points we cannot ignore. On one hand, the liquid drop model for calculating nuclear binding energies depends strictly on the experimental data. On the other hand, it is difficult to describe light and halo nuclei because it depends on statistics and cannot reflect the structure properties of the nucleus precisely. Taking into consideration the fact that, due to the nuclear particle properties and its quantum motion, it is not proper to regard the nucleus as a compact sphere with a smooth surface, and there is no explicit border. The nucleons in a nucleus are separated and there are void spaces among them. These void spaces near the surface are much bigger in halo or weakly-bounded nuclei in comparison to those in stable nuclei. The notion of radii is only a statistical average effect of nuclear matter distribution. In fact, the nuclear structures are irregular, which is the most obviously different feature from the description of the liquid drop model.

In order to describe the nuclear structure properties reasonably well, the notion of a fractal object for irregular systems with non-integral dimensions is possible. Fractal objects were introduced to science by Mandel-

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brot in 1967 [17], and their most important characteristics are self-similarity and scale invariance. Fractal self-similarity means that any part of the fractal object is similar to the whole fractal object after magnification. For an arbitrary part of the fractal object, its important properties, such as shape, complexity and irregularity, remain invariant after magnification or shrinking by a certain proportion, which is the meaning of scale invariance. For a regular fractal object, there is no characteristic size, but a characteristic fractal dimension. However, the self-similarity of an approximate or statistical fractal object exists in a finite scale range. In a nuclear system with a cluster structure, due to the similar nuclear and electromagnetic interactions between the clusters in a nucleus, it is possible that the geometric structure and physical laws are similar between the nucleons in a cluster, between clusters in the nucleus, and between the clusters and the whole nucleus. The nucleus can be deemed a statistical self-similar fractal system with finite scale range. Xavier Campi has found the existence of finite size scaling in nuclear fragmentation [18]. He introduced the fractal dimension, which gives information on the internal structure of fragments during the fragmentation process. Adamenko et al. [19] queried the liquid-like model and studied the properties in super-heavy nuclear isomers using fractal theory. These studies are the inspiration for connecting the characteristics of the nuclear structure with fractal geometric theory.

As explained above, the irregular nuclear structure properties and the self-similarity characteristic may be intrinsic aspects of nuclear structure properties. It is possible to use fractal theory to describe the nuclear structure properties. The objective of the present work is to introduce this new concept of nuclear fractal structures to study nuclear properties.

2 The nuclear fractal structure model

In our present description, a more general conception of nuclear fractal clusters (NFCs) is applied, which is similar but different from the conventional one of the α-cluster structure and the core plus valence nucleons structure in halo nuclei. The latter is considered to be one kind of the former, based on the concept of the characteristics of fractal objects. The concept of NFCs is that of the nucleus as a fractal assembly of structural subunits that are themselves made up of no less than one nucleon and keep a certain correlation of similarity with the ensemble in both geometry and physics. The geometrical boundaries of the NFCs within some nuclei are less distinguishable than those of the clusters in α-cluster nuclei and halo nuclei.

Similar to the definition given in Ref. [20], an isotropic self-similarity nuclear fractal dimension $D_{\rm f}$ is defined in the following relation:

$$M(b\cdot r)=b^{D_{\rm f}}\cdot M(r), \tag{1}$$

where $M(r)$ is the mass number within the size r of the fractal object; $M(b\cdot r)$ is the mass number of b times the size r of the fractal object, where b is a scaling factor among the similar parts within the fractal object. The only solution for relation (1) is $M(r)\propto r^{D_{\rm f}}$. The nuclear average matter density $\rho(r)$ with the law of decay of isotropic spatial correlation, $\rho(r)=\dfrac{M(r)}{V(r)}\propto\dfrac{r^{D_{\rm f}}}{r^3}\propto r^{D_{\rm f}-3}$, is a basic variable function in nuclei. So far, the geometric dimension of nuclei is considered as 3, because of the concept of the liquid drop model. For a real physical nuclear object embedded in 3-dimensional Euclidean space, its dimension must be less than or equal to 3. Most of the nuclei with fractal dimensions approaching 3 are stable, and are more like liquid drops.

In Eq. (2), we assume that there are several NFCs and A_i is the mass number of an NFC within a nucleus with mass number A. $\rho_{\rm dis}(r)$ is the density distribution function. ρ_i is the fractal mean density of an NFC and R_i is its fractal mean density radius. F is the number of the NFCs in a given nucleus.

$$\begin{aligned}\rho &= \frac{A}{V}=\frac{\int\rho_{\rm dis}(r)\cdot{\rm d}v}{\int{\rm d}v}=\frac{\sum\limits_{i=1}^{F}A_i}{V}\\ &= \frac{\sum\limits_{i=1}^{F}\int\rho_{\rm dis}(r_i)\cdot{\rm d}v_i}{V}=\frac{\sum\limits_{i=1}^{F}\rho_i\cdot V_i}{V}\\ &= \sum_{i=1}^{F}\frac{V_i}{V}\cdot\rho_i=\sum_{i=1}^{F}\left(\frac{R_i}{R}\right)^3\cdot\rho_i\end{aligned} \tag{2}$$

where R is the nuclear fractal mean density radius defined by the relation $A=\dfrac{4}{3}\pi R^3\rho$ and $\rho_i=\dfrac{3A_i}{4\pi R_i^3}$.

In addition, due to the assumption of nuclear fractal structure and the mass-radius relation of self-similar fractal objects, we list several basic relations:

$$A=\sum_{i=1}^{F}A_i; \tag{3}$$

$$A\propto R^{D_{\rm f}};A_i\propto R_i^{D_{\rm f}}. \tag{4}$$

Using relations (3) and (4), we get the relation among R, R_i and $D_{\rm f}$:

$$R^{D_{\rm f}}=\sum_{i=1}^{F}R_i^{D_{\rm f}}. \tag{5}$$

Due to the two proportional relations in (4), the relation

among R, R_i, A, A_i and $D_{\rm f}$ is

$$R=R_i\left(\frac{A}{A_i}\right)^{\frac{1}{D_{\rm f}}}. \tag{6}$$

Substituting (6) into (2), we then get

$$\rho=\sum_{i=1}^{F}\left(\frac{A_i}{A}\right)^{\frac{3}{D_{\rm f}}}\rho_i. \tag{7}$$

The relation between the NFCs in a nucleus is then

$$\rho_i=\rho_j\left(\frac{A_i}{A_j}\right)^{\frac{D_{\rm f}-3}{D_{\rm f}}}. \tag{8}$$

Therefore, to describe the nuclear geometric properties, the nuclear fractal dimension is an irreplaceable variable similar to the nuclear radius. For a given nuclear system with cluster structure, the final free variables are (R, $D_{\rm f}$) or (ρ, $D_{\rm f}$), which are also (R_i, $D_{\rm f}$) or (ρ_i, $D_{\rm f}$) due to the relations (6) and (7). In order to determine these two variables, we put forward a new nuclear potential energy formula which is related to fractal dimension. Then the phenomenological semi-empirical Bethe–Weizsäcker binding energy formula is modified and the total potential energy is obtained. One important equation set with two equations is obtained, which is related to the concept that the fractal dimension should be a dynamical parameter in the process of nuclear synthesis. So, the calculations of nuclear fractal dimensions and radii can be done.

In a nuclear system, considering the self-similarity properties of the nuclear fractal system, we put forward a nuclear potential energy formula

$$u(r)=\frac{v_0D_{\rm f}}{3(D_{\rm f}-2)}\left(\frac{r}{r_s}\right)^{D_{\rm f}-3},\quad 2<D_{\rm f}\leqslant 3. \tag{9}$$

This is proportional to the nuclear average density $\rho(r)$. $2r_{\rm s}$ stands for the minimum scale size of a nuclear fractal system and it is also the maximum size of the minimum cluster element. v_0 is a coefficient and keeps constant, corresponding to the estimation of the depth of the nuclear potential well in the liquid drop model when $D_{\rm f}=3$.

From (9) and using the idea of self-similarity, the relation for the nuclear potential energy of one NFC within the nucleus is obtained:

$$u_i(r_i)=\frac{v_0D_{\rm f}}{3(D_{\rm f}-2)}\left(\frac{r_i}{r_s}\right)^{D_{\rm f}-3},\ 2<D_{\rm f}\leqslant 3. \tag{10}$$

Because of $u(R)\propto\rho(R)$, $u_i(R_i)\propto\rho_i(R_i)$ and (7), the relation between $u(R)$ and $u_i(R_i)$ is

$$u(R)=\sum_{i=1}^{F}\left(\frac{A_i}{A}\right)^{\frac{3}{D_{\rm f}}}u_i(R_i). \tag{11}$$

Next, we modify the phenomenological semi-empirical Bethe–Weizsäcker binding energy formula with the Fermi gas model and the fractal theory. Here we are mainly concerned with the liquid drop energy and put aside the correction term based on the microscopic method, such as the description in Ref. [21]. The original equation derived from the liquid drop model is:

$$\begin{aligned}B\ =\ &(u_{\rm depth}-c_{\rm v}-c_{\rm as}\left(1-\frac{2Z}{A}\right)^2)A-c_{\rm surf}A^{\frac{2}{3}}\\&-c_{\rm Q}\frac{Z(Z-1)}{A^{\frac{1}{3}}}+c_{\rm p}\frac{(-1)^Z+(-1)^{A-Z}}{2A^{\frac{4}{3}}};\end{aligned} \tag{12}$$

where $u_{\rm depth}\approx 58$ [22](the estimation of the depth of the nuclear potential well); $c_{\rm v}=42.27$; $c_{\rm as}=23.48$; $c_{\rm surf}=17.72$; $c_{\rm Q}=0.72$; and $c_{\rm p}=19.39$. We use the experimental mass data [23] to fit the other parameters.

The modified formulae are:

$$B_{\rm strong}=\left(v_{\rm depth}-c_1(\rho)-c_2(\rho)\left(1-\frac{2Z}{A}\right)^2\right)A; \tag{13}$$

$$B_{\rm surf}=-c_{\rm s}4\pi R^2; \tag{14}$$

$$B_{\rm Q}=-\frac{3}{5}\frac{Z(Z-1)e^2}{R}; \tag{15}$$

$$B_{\rm p}=c_{\rm p}\frac{(-1)^Z+(-1)^{A-Z}}{2A^{\frac{4}{3}}}; \tag{16}$$

$$B=B_{\rm strong}+B_{\rm surf}+B_{\rm Q}+B_{\rm p}; \tag{17}$$

where

$$c_1(\rho)=\frac{3}{5}\varepsilon(\rho)\rho^{\frac{2}{3}};\ c_2(\rho)=\frac{1}{3}\varepsilon(\rho)\rho^{\frac{2}{3}};$$

$c_{\rm s}$=0.98 MeV·fm^2; $c_{\rm p}$=19.39 MeV;

$$\varepsilon(\rho)=\varepsilon_0\left(\frac{\rho}{\rho_0}\right)^{\alpha};$$

ε_0=264.12 MeV^{-1}; and ρ_0=0.138 fm^{-3}. Eq. (13) is the volume energy, consisting of three parts. The first part is the depth of nuclear potential energy, which is not a constant, but depends on the structure type. The second and third parts, derived from the Fermi gas model, are the kinetic energy of the nuclear system, because of which the binding energy decreases. $\varepsilon(\rho)$ shares the kinetic energy, which decreases the contribution of kinetic energy. ε_0 was explained as a constant related to the virial coefficient in Ref. [19]. Since we consider that the nuclear system has a fractal structure which is very different from the Fermi gas, ε_0 cannot be kept constant any more, but depends on the mean density, which is related to the mean density of every cluster in the nucleus as shown by Eq. (7). In the formula for $\varepsilon(\rho)$, α is a parameter. Formulas (14) and (15) for surface energy and the energy because of Coulomb interaction do not have any parameters. $c_{\rm s}$, the coefficient of surface tension, is about 1 MeV/fm^2. Here we still keep the pair energy because the binding energy is larger when the proton and neutron numbers are even, smaller when one of the numbers are odd and even more so when both are odd, which

may have nothing to do with the nuclear fractal structure. Thus all the terms contributing to binding energy depend on the fractal dimension $D_{\rm f}$ except the pair term. When $D_{\rm f}\to 3$, the modified formula degenerates to the ordinary binding energy formula (12), and then the state of the nucleus shifts from the non-uniform density, which has NFCs, to the uniform state of the liquid drop model. So, the parameters $c_{\rm s}$, $c_{\rm p}$, ε_0 and ρ_0 can all be derived from (12–17) when $D_{\rm f}\to 3$.

A possible formula for the depth of the nuclear potential well in the nuclear fractal system is found to be

$$v_{\rm depth} = \frac{1}{2}\left(1-\left(\frac{A-(1+s)Z}{A}\right)^2\right) \times\sum_{i=1}^{F}\frac{A_i}{A}\left(58+3u_i(R_i)\frac{D_{\rm f}-2}{D_{\rm f}}\right), \quad (18)$$

which depends on the nuclear fractal structure. In fact, the depth of nuclear potential well cannot be 58 MeV for all nuclei because of the different fractal structure for different nuclei. The product factor is introduced to describe the difference between the number of neutrons and protons in a nucleus, and $s=\dfrac{N_s}{Z_s}$, N_s and Z_s are the number of neutrons and protons in a nucleus on the stable line, which makes sure the depth of the nuclear potential well is at a maximum for nuclei on the stable line. For light stable nuclei, $s\approx 1$, and for the nuclei ^{4}He, ^{8}Be, ^{12}C, ^{16}O, ^{20}Ne, ^{24}Mg, ^{28}Si, ^{32}S, ^{36}Ar and ^{40}Ca, $s=1$. When $D_{\rm f}\to 3$, $u=u_i=v_0=-u_{\rm depth}$.

We define the total potential energy $U=U(A,\ Z,\ D_{\rm f},\ \rho)$, which is the sum of the total nuclear potential energy $U_A=U_A(A,\ D_{\rm f},\ \rho)$ and the total electromagnetic potential energy $U_Z=U_Z(Z,\ D_{\rm f},\ \rho)$. Namely,

$$U=U_A+U_Z. \quad (19)$$

The interaction among clusters in a given nucleus is shown in Fig. 1. The total nuclear potential energy is

$$U_A=\sum_{i=1}^{F}\left(\frac{A_i}{A}\right)^{\frac{3}{D_{\rm f}}}U_i+\sum_{i=1}^{F}\sum_{i\neq j}^{F}\left(\frac{A_j}{A}\right)^{\frac{3}{D_{\rm f}}}U_{ij}, \quad (20)$$

where,

$$U_i = 4\pi\frac{v_0 D_{\rm f}}{3(D_{\rm f}-2)}\int_0^{R_i}\left(\frac{r_i}{r_s}\right)^{D_{\rm f}-3}\rho_{\rm dis}(r_i)r_i^2{\rm d}r_i = \frac{4\pi v_0\rho_i R_i^{D_{\rm f}}}{3(D_{\rm f}-2)r_s^{D_{\rm f}-3}}. \quad (21)$$

$$U_{ij} = 2\pi\int_0^{\pi}\int_0^{R_j}u_i(r)\rho_{\rm dis}(r_j)r_j^2\sin(\theta){\rm d}r_j{\rm d}\theta = 2\pi\frac{v_0 D_{\rm f}}{3(D_{\rm f}-2)}\frac{\rho_j(R_{ij}+R_j)^{D_{\rm f}+1}G(D_{\rm f})}{(D_{\rm f}+D_{\rm f}^3)R_{ij}(R_{ij}^2-R_j^2)r_s^{D_{\rm f}-3}}. \quad (22)$$

In (22), $G(D_{\rm f})=(R_{ij}^2+D_{\rm f}R_j^2)\left(\left(\dfrac{R_{ij}-R_j}{R_{ij}+R_j}\right)^{D_{\rm f}+1}-1\right)+(D_{\rm f}+1)R_{ij}R_j\left(\left(\dfrac{R_{ij}-R_j}{R_{ij}+R_j}\right)^{D_{\rm f}+1}+1\right)$; the distance $r=\sqrt{r_j^2+R_{ij}^2-2r_jR_{ij}\cos\theta}$ as shown in Fig. 1; $R_{ij}\approx R_i+R_j$. The total electromagnetic potential energy is

$$U_Z=\sum_{i=1}^{F}U_{Z_i}+\sum_{i=1}^{F}\sum_{i\neq j}^{F}U_{Z_{ij}}, \quad (23)$$

where,

$$U_{Z_i}=\frac{3}{5}Z_i(Z_i-1)\frac{e^2}{R_i}; \quad (24)$$

$$U_{Z_{ij}}=\frac{Z_iZ_je^2}{R_{ij}}. \quad (25)$$

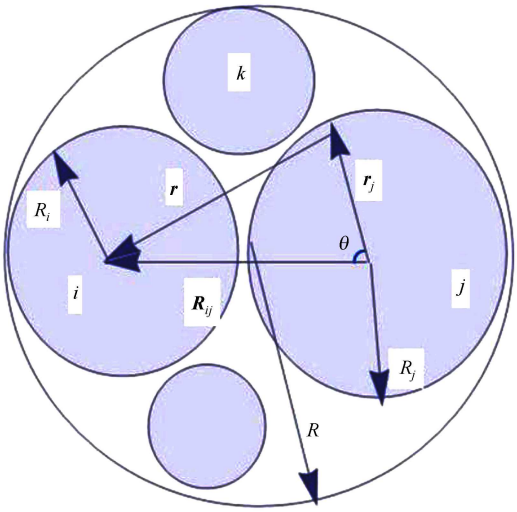

Fig. 1. The interaction among clusters in a given nucleus. The notations i, j, k stand for different NFCs.

For a given nucleus containing NFCs, we consider that its fractal dimension $D_{\rm f}$ is constant. However, in the process of nuclear synthesis it should be a dynamic parameter. In a given nuclear reaction the binding energy $B=B(A,\ Z,\ D_{\rm f},\ \rho)$ and the total potential energy $U=U(A,\ Z,\ D_{\rm f},\ \rho)$ are changing with $D_{\rm f}$. When $D_{\rm f}$ reaches its fixed value, then $B=B(A,\ Z,\ D_{\rm f},\ \rho)$ and $U=U(A,\ Z,\ D_{\rm f},\ \rho)$ reach their minimum value, which corresponds to an interacting system becoming relatively stable. So one equation set is obtained:

$$\begin{cases}\partial_{D_{\rm f}}B(A,Z,D_{\rm f},\rho)=0\\ \partial_{D_{\rm f}}U(A,Z,D_{\rm f},\rho)=0.\end{cases} \quad (26)$$

Because of the uncertain parameter α, we need more than two equations to study the structure properties in the nucleus. The additional one is

$$B(A,Z,D_{\rm f},\rho)=B_{\rm exp}+E_{\rm excited}, \quad (27)$$

where $B_{\rm exp}$ is the experimental value of the binding energy, and $E_{\rm excited}$ is the change in binding energy due to the nucleus being excited from the ground state, which corresponds to the situation where the NFC structure is forming in the excited nucleus. If only the nuclei in ground states are considered, $E_{\rm excited}$=0.

Finally, we arrive at the modified binding energy formula $B(A, Z, D_{\rm f}, \rho)$ and the total potential energy $U(A, Z, D_{\rm f}, \rho)$, which are functions of A, Z, $D_{\rm f}$ and ρ. With the important equation set (26) obtained, (26) and (27) can be used, assuming rational fractal structure types in the nucleus, to determine the nuclear fractal dimension and radius.

3 Results and discussion

In the present work, we have done some calculations and obtained some results for light nuclei in ground states. Combining the analyses given in Refs. [1–12] and [24–26] and the consideration that the rational fractal structure of one nucleus is determined by making the separated energy of NFCs as small as possible, the possible NFC structures for these nuclei are shown in Table 1. We consider that the scale r_s is about 1.25 fm, which is half the maximum size of one nucleon. We then compare the radii ($R_{\rm L}=r_0A^{\frac{1}{3}}$, r_0=1.25 fm) from the liquid drop model with the nuclear fractal mean density radii.

Table 1. The possible NFC structures for light nuclei.

nuclei	NFCs	nuclei	NFCs
^{5}He	^{4}He+n	^{14}B	^{13}B+n
^{6}He	^{4}He+n+n	^{15}B	^{14}B+n
^{7}He	^{6}He+n	^{16}B	^{15}B+n
^{8}He	^{6}He+n+n	^{9}C	^{8}B+p
^{5}Li	^{4}He+p	^{10}C	^{8}Be+p+p
^{6}Li	^{4}He+^{2}H	^{11}C	^{7}Be+^{4}He
^{7}Li	^{6}Li+n	^{12}C	^{8}Be+^{4}He
^{8}Li	^{7}Li+n	^{13}C	^{12}C+n
^{9}Li	^{8}Li+n	^{14}C	^{13}C+n
^{10}Li	^{9}Li+n	^{15}C	^{14}C+n
^{11}Li	^{9}Li+n+n	^{16}C	^{15}C+n
^{8}Be	^{4}He+^{4}He	^{13}N	^{12}C+p
^{9}Be	^{8}Be+n	^{14}N	^{10}B+^{4}He
^{10}Be	^{9}Be+n	^{15}N	^{11}B+^{4}He
^{11}Be	^{10}Be+n	^{16}N	^{15}N+n
^{12}Be	^{11}Be+n	^{17}N	^{16}N+n
^{13}Be	^{12}Be+n	^{13}O	^{12}N+p
^{14}Be	^{12}Be+n+n	^{14}O	^{13}N+p
^{8}B	^{7}Be+n	^{15}O	^{14}N+p
^{9}B	^{8}Be+n	^{16}O	^{12}C+^{4}He
^{10}B	^{6}Li+^{4}He	^{17}O	^{13}C+^{4}He
^{11}B	^{7}Li+^{4}He	^{18}O	^{14}C+^{4}He
^{12}B	^{11}B+n		

In order to describe the essential feature of the nuclear fractal self-similarity symmetry, the fractal dimension is one of the basic geometric parameters. It is related to the nuclear homogeneity and the ingredients in the nucleus, which reflect the degree of nuclear irregularity. It can be seen that the values of the dimensions are quite close to 3, if the mean densities of every NFC in a nucleus and the mean density of that nucleus are all not too different and the degree of irregularity of the nuclear structures is low. Based on the NFC structures (Table 1) and the values of the dimensions (Fig. 2), some interesting results may be discussed. For isobars, because the degree of homogeneity of the stable nuclei is generally higher than the nuclei far from the stable line, the values of the fractal dimension for the former are greater than for the latter. The fractal dimensions of isobars are comparable or the same, if the NFC structure types of these nuclei are similar. For example, the fractal dimensions of the isobars with two-body structure type, (^{5}Li, ^{5}He), (^{8}Li, ^{8}Be, ^{8}B), (^{10}Be, ^{10}B), (^{11}C, ^{11}B), (^{13}N, ^{13}C) and (^{16}C, ^{16}N), are respectively almost the same. For nuclei with the same A, the fractal dimensions of nuclei with a three-body structure, such as, ^{6}He, ^{8}He, ^{10}C, ^{11}Li and ^{14}Be, are distinctly lower than the nuclei with two-body structure. The values of fractal dimension decrease due to the increase of nuclear mass number, which may be associated with the scale r_s. The fractal scale variables amount to the resolution of the measurement [33], which are related to the scale relativity. For instance, if half the maximum size of one nucleon serves as the fractal scale, using a 3-dimensional bulk whose radius equals this scale to cover a nucleon, the dimension of one nucleon is 3. However, along with the increase of the nuclear mass number, the nuclear structure becomes more irregular. Therefore, using this 3D structure to measure the nucleus, the dimension of this nucleus is lower than 3, which reflects the information of the irregular nuclear structures. In brief, as the results of the effects of all structure variables and binding energy, the fractal dimension can generally describe the nuclear structure well. Besides, in the process of nuclear synthesis $D_{\rm f}$ is a

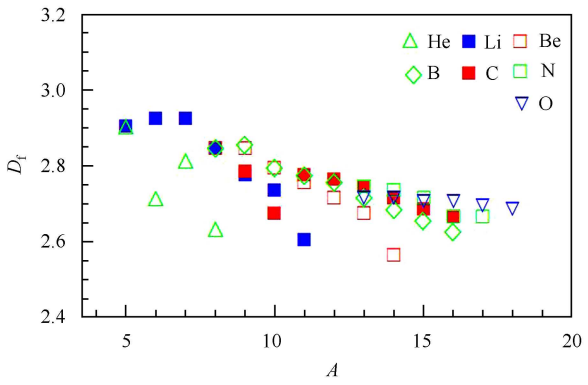

Fig. 2. (color online) Values of the fractal dimensions of certain nuclei.

dynamic parameter; when $D_{\rm f}$ reaches a fixed value, this explains the existence of relatively stable nuclei.

The nuclear fractal mean density radii, depending on the scale variable r_s in the process of measurement, can represent the nuclear size statistically; this is mainly determined by the binding energy and NFC structure types and related to the fractal dimension by Equation (5). For isobars, the higher the binding energy, the smaller the radii. Generally, as shown in Fig. 3, for stable nuclei and the nuclei very near the stable line, the values of the nuclear fractal mean density radii approach the line of $R_{\rm L}$ from the liquid drop model. When the nuclei are far from the line of stability, the values of the nuclear fractal mean density radii will be greater than $R_{\rm L}$ and this will increase as the deviation from the line of stability increases. In some special cases, nuclei with the same NFC structure and almost the same binding energy, such as ^{5}Li and ^{5}He, have almost the same radii. The same situation arises in the pairs of isobars (^{8}B, ^{8}Li), (^{9}Be, ^{9}B) and (^{13}C, ^{13}N). As a matter of fact, the liquid drop model cannot successfully calculate the radii of nuclei far from the stable line, as they have much larger radii than the predictions of the liquid drop model. The present fractal structure model, however, can successfully predict the radii of such nuclei far from the line of stability. Moreover, the comparison of the calculated values of the nuclear fractal mean density radii using the nuclear fractal structure model with the experimental results of root-mean-square (rms) matter ($R_{\rm m}$) radii are listed in Table 2, where, $R_{\rm m}=\sqrt{\frac{Z}{A}R_{\rm p}^2+\frac{N}{A}R_{\rm n}^2}$ [31]. For the stable nuclei, the results of the calculated radii using both the liquid drop model and the present method are larger than the experimental results of root-mean-square (rms) matter ($R_{\rm m}$). Generally, the results of the calculated radii using the present method are larger than the experimental results of rms matter radii ($R_{\rm m}$). Fig. 3 also shows the experimental results of rms matter radii ($R_{\rm m}$), which are corresponding to the experimental values shown in Table 2.

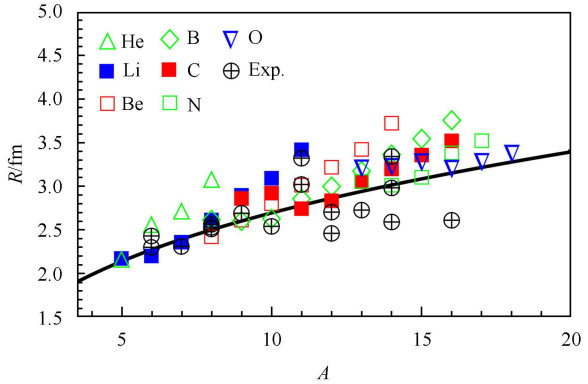

Fig. 3. (color online) Comparison of the calculated values of radii using the nuclear fractal structure model and the liquid drop model. The symbols (excluding the black quadrant circles) stand for the nuclear fractal mean density radii and the line is for the liquid drop model. The experimental points (black quadrant circles) are corresponding to the experimental values of rms matter radii in Table 2.

Table 2. Comparison of the calculated values of nuclear fractal mean density radii (here denoted $R_{\rm f}$ for convenience) using the nuclear fractal structure model with the experimental results of root-mean-square (rms) matter ($R_{\rm m}$) radii. The cluster structures of the nuclei listed here are the same as those in Table 1.

nuclei	$R_{\rm f}$/fm	$R_{\rm m}$/fm	Ref.
^{6}Li	2.217	2.32±0.03	[27]
^{6}He	2.583	2.45(10)	[28]
^{7}Li	2.374	2.33±0.02	[27]
^{8}He	3.100	2.53(8)	[28]
^{8}B	2.635	2.55±0.08	[29]
^{8}Li	2.629	2.583±0.023	[27]
^{9}C	2.874	2.71(32)	[30]
^{10}B	2.647	2.56±0.23	[27]
^{11}Li	3.432	$3.34^{+0.04}_{-0.08}$	[31]
^{11}Be	3.027	3.039±0.038	[32]
^{12}B	3.016	2.723±0.050	[27]
^{12}C	2.848	2.48±0.08	[27]
^{13}B	3.193	2.746±0.050	[27]
^{14}Be	3.736	3.36±0.19	[27]
^{14}B	3.383	3.00±0.10	[27]
^{14}N	3.027	2.61±0.10	[27]
^{16}O	3.194	2.63±0.06	[27]

With regard to the parameter α introduced in Section 2, it can be solved as the same time as the fractal dimension $D_{\rm f}$ and the nuclear fractal mean density radius R through the three equations explained at the end of Section 2. It varies from 0 to 1 in present calculations, generally corresponding to the range of nuclei from stability to instability.

In summary, the NFC structures are determined by the interactions within nuclear systems, and the fractal dimension can generally describe such structural features. The nuclear fractal mean density radii represents approximately the nuclear size, which is associated with the scale variables. Actually, the relations among the nuclear structure geometric variables are complex and correlative.

4 Conclusions

In this, we consider the importance of the irregular nuclear structure properties and self-similarity character-

istics as intrinsic aspects of nuclear structure properties. For the description of nuclear geometric properties, the nuclear fractal dimension is an irreplaceable variable similar to the nuclear radius. Compared with the liquid drop model, a feature of the fractal description is that it can reflect the important characteristics of the NFC structures, especially for describing the nuclei far from the line of stability and α-cluster nuclei. Similar to the liquid drop model, the present model can get the same results for the relation between A and Z on the β stable line for light stable nuclei, but cannot predict the existence of the magic nuclei. For heavier nuclei whose fractal structure is more complicated, more realistic formulae for the depth of nuclear potential energy need to be obtained. These heavy nuclei may have multilevel fractal structure with several clusters in every level. An anisotropic description for the nuclear fractal structure and the scale-dependent properties of the nuclear fractal system will be further studied in future. Further studies will also focus on combining the properties of the nuclear fractal structure and quantum mechanics.

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