Quantized space-time and its influence on two physical problems *

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Abstract: Based on Snyder's idea of quantized space-time, we derive a new generalized uncertainty principle and new modified density of states. Accordingly, we discuss the influence of the modified generalized uncertainty principle on the black hole entropy and the influence of the modified density of states on the Stefan-Boltzman law.

Key words: quantized space-time, generalized uncertainty principle, Stefan-Boltzmann law

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1 Introduction

Recently, considerable interest has been devoted to the study of the generalized uncertainty principle (GUP) [1–14]. The main consequence of the GUP is the existence of a minimal length scale of the order of the Planck length, which can be deduced in string theory and other theories of quantum gravity [15–22]. Although at present this kind of minimal length has not been probed experimentally, it is predicted from the form of the GUP or the general commutation relation to exist theoretically. The minimal length can provide a natural ultraviolet (UV) cut-off. The GUP can also influence the structure of phase space and modify the usual density of states to a different form with a weighted factor. With the modified density of states, one can calculate its effects on the cosmological constant and black body radiation [7] as well as the entanglement entropy of black holes by means of statistical mechanics [23].

As mentioned above, the existence of a minimum length can be the result of generalized uncertainty principle. In fact, one can also consider the problem in an opposite direction. To deal with the trouble of quantum field theory, Snyder [24] suggested that the usual four dimensional space-time may not be continuous but is discrete or quantized. This means that there is a smallest unit of length in space-time and the space-time should be noncommutative. Based on the assumption of existence of a minimal length and Lorentz invariance, Snyder introduced some operators for position, momentum and angular momentum, and obtained a sequence of commutators between them. This shows that the usual commutation relation $[\hat{\mathcal{X}},\hat{\mathcal{P}}]=\mathrm{i}$ and $[\hat{\mathcal{X}},\hat{\mathcal{X}}]=0$ no longer exist. According to the commutators obtained by Snyder, we can also derive a new GUP. In addition, one can also obtain the modified density of states different from the one obtained from the usual GUP. Thus, one can recalculate the influence of the new modified density of states on physical quantities and laws, such as the cosmological constant, black body radiation, the entanglement entropy of black holes and so on. In particular, the minimal length in Snyder's model gives a natural UV cutoff in the calculation of entanglement entropy; thus, the UV divergence caused by the infinite density near the horizon can be removed.

This paper is arranged as follows. In the next section we first introduce Snyder's quantized space-time model, and derive the GUP and modified density of states. In the third section, we calculate its influence on the black hole entropy and, especially, on the Stefan-Boltzmann laws. We shall give some concluding remarks in the final section.$(c=\hbar=G=k_{\mathrm{B}}=1)$.

2 Snyder's quantized space-time and GUP

Snyder developed the concept of quantized space-time that is invariant under Lorentz transformation, namely the quadratic form $-\eta^2=\eta_0^2-\eta_1^2-\eta_2^2-\eta_3^2-\eta_4^2$ should be invariant under Lorentz transformation. The η_b,(b=0, 1, 2, 3, 4) are the homogeneous projective coordinates of a real 4-dimensional space of constant curvature. From another point of view, Snyder's model can also be interpreted as the Beltrami-dS model in momentum space [25–27]. In particular, in Ref. [27] the authors describe a duality between Snyder's model and the de Sitter special relativity on the basis of Yang's model [28]. Snyder introduces the space-time operator $\hat{x}_\mu$(μ=0, 1, 2, 3), which

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can be defined as

$$\hat{x}_0=\mathrm{i}a\left(\eta_4\frac{\partial}{\partial\eta_0}+\eta_0\frac{\partial}{\partial\eta_4}\right);\ \hat{x}_1=\mathrm{i}a\left(\eta_4\frac{\partial}{\partial\eta_1}-\eta_1\frac{\partial}{\partial\eta_4}\right),$$
$$\hat{x}_2=\mathrm{i}a\left(\eta_4\frac{\partial}{\partial\eta_2}-\eta_2\frac{\partial}{\partial\eta_4}\right);\ \hat{x}_3=\mathrm{i}a\left(\eta_4\frac{\partial}{\partial\eta_3}-\eta_3\frac{\partial}{\partial\eta_4}\right), \tag{1}$$

where a is the minimum length.

In addition, there are another two groups of operators:

$$\hat{p}_0=\frac{1}{a}(\eta_0/\eta_4);\ \hat{p}_1=\frac{1}{a}(\eta_1/\eta_4),$$
$$\hat{p}_2=\frac{1}{a}(\eta_2/\eta_4);\ \hat{p}_3=\frac{1}{a}(\eta_3/\eta_4), \tag{2}$$

and

$$\hat{L}_1=\mathrm{i}\left(\eta_3\frac{\partial}{\partial\eta_2}-\eta_2\frac{\partial}{\partial\eta_3}\right),\ \hat{L}_2=\mathrm{i}\left(\eta_1\frac{\partial}{\partial\eta_3}-\eta_3\frac{\partial}{\partial\eta_1}\right),$$
$$\hat{L}_3=\mathrm{i}\left(\eta_2\frac{\partial}{\partial\eta_1}-\eta_1\frac{\partial}{\partial\eta_3}\right),$$
$$\hat{M}_1=\mathrm{i}\left(\eta_0\frac{\partial}{\partial\eta_1}+\eta_1\frac{\partial}{\partial\eta_0}\right),\ \hat{M}_2=\mathrm{i}\left(\eta_0\frac{\partial}{\partial\eta_2}+\eta_2\frac{\partial}{\partial\eta_0}\right),$$
$$\hat{M}_3=\mathrm{i}\left(\eta_0\frac{\partial}{\partial\eta_3}+\eta_3\frac{\partial}{\partial\eta_0}\right). \tag{3}$$

We know that Lorentz group has six generators, which can be recorded as $M_{ij}(i,j{=}1,\ 2,\ 3)$ and M_{0i}. The three generators for rotation $L_i=\frac{1}{2}\epsilon_{ijk}M_{jk}$ and the other three generators for boost can be described as $M_i=M_{0i}$. It is, therefore, easy to get the commutator below:

$$[\hat{x}_i,\hat{x}_j]=\mathrm{i}a^2\hat{M}_{ij},\quad [\hat{x}_0,\hat{x}_i]=\mathrm{i}a^2\hat{M}_{0i}. \tag{4}$$

Obviously, if one takes the limit $a\rightarrow 0$, the quantized and noncommutative space-time will turn into the usual continuous and commutative space-time. In fact, what we really care about is the commutators below:

$$[\hat{x}_i,\hat{p}_j]=\mathrm{i}(\delta_{ij}+a^2\hat{p}_i\hat{p}_j). \tag{5}$$

This relation reminds us of the more general commutator from GUP, which is [4]

$$[\hat{x}_i,\hat{p}_j]=\mathrm{i}(\delta_{ij}+\lambda\delta_{ij}\hat{p}^2+\lambda'\hat{p}_i\hat{p}_j). \tag{6}$$

One can easily find out that the commutator Eq.(5) from quantized space-time is a special case of the commutators Eq. (6) from GUP, with $\lambda{=}0$ and $\lambda'{=}a^2$.

In general, it is known that for any pair of observables A and B, the uncertainty relation

$$\Delta A\Delta B\geqslant\frac{1}{2}|\overline{[A,B]}|. \tag{7}$$

In view of $\Delta A{=}A{-}\bar{A}$ and $(\Delta A)^2=\overline{A^2}-\overline{A}^2$, we can obtain from Eq. (5) that

$$\Delta x_i\Delta p_j\geqslant\frac{1}{2}(\delta_{ij}+a^2\Delta p_i\Delta p_j+\gamma), \tag{8}$$

where γ is positive and dependent on the expectation value of p_i. We can name the GUP above as Snyder's GUP. If considering the $i{=}j$ case only, the formula above will turns into

$$\Delta x_i\Delta p_i\geqslant\frac{1}{2}[1+a^2(\Delta p_i)^2+\gamma], \tag{9}$$

or

$$\Delta x_i\geqslant\frac{1}{2}\left(\frac{1+\gamma}{\Delta p_i}+a^2\Delta p_i\right), \tag{10}$$

which is similar to the frequently used GUP. Obviously, when setting $\gamma=0$, it will give a minimal uncertainty length $\Delta x{=}a$.

According to the usual Heisenberg uncertainty principle, one can obtain the D dimensional phase space volume

$$\mathrm{d}^D\boldsymbol{x}\mathrm{d}^D\boldsymbol{p}. \tag{11}$$

Upon quantization, the corresponding number of quantum states per momentum space volume is

$$\frac{\mathrm{d}^D\boldsymbol{x}\mathrm{d}^D\boldsymbol{p}}{(2\pi)^D}. \tag{12}$$

Considering the commutator Eq. (6) from GUP, the number of quantum states changes to [6]

$$\frac{\mathrm{d}^D\boldsymbol{x}\mathrm{d}^D\boldsymbol{p}}{(2\pi)^D(1+\lambda p^2)^{D-1}[1+(\lambda+\lambda')p^2]^{1-\frac{\lambda'}{2(\lambda+\lambda')}}}. \tag{13}$$

Thus, the number of quantum states for quantized space-time should be

$$\frac{\mathrm{d}^D\boldsymbol{x}\mathrm{d}^D\boldsymbol{p}}{(2\pi)^D(1+a^2p^2)^{1/2}}. \tag{14}$$

In fact, its counterpart

$$\frac{\mathrm{d}^D\boldsymbol{x}\mathrm{d}^D\boldsymbol{p}}{(2\pi)^D(1+\lambda p^2)^D}, \tag{15}$$

which corresponds to $\lambda'=0$ in Eq. (13), is often considered by physicists. The difference between the two forms lies in the weighted factor or, more precisely, on the exponent there. The exponent in Eq. (15) is dimension-dependent, whereas the one in Eq. (14) is a constant 1/2. Thus, when the two forms of number of quantum states are used in quantum field theory, one can deduce that Eq. (15) can remove the divergence more effectively; especially, higher space dimensions will weaken divergence. Be that as it may, the Eq. (14) should be employed if the space-time is really discrete, as in Snyder's proposition. One can use the formula to recalculate many quantities, like black body radiation, cosmological constant, etc.

3 The influence of two physical problems

3.1 Black hole entropy from Snyder's GUP

Based on Snyder's GUP (10) and set $\gamma{=}0$ for simplicity, one can obtain

$$\frac{\Delta x_i}{a^2}\left[1-\sqrt{1-\frac{a^2}{(\Delta x_i)^2}}\right]\leqslant\Delta p_i\leqslant\frac{\Delta x_i}{a^2}\left[1+\sqrt{1-\frac{a^2}{(\Delta x_i)^2}}\right]. \quad (16)$$

We expand the expression above as Taylor series at $a=0$.

$$\Delta p_i\geqslant\frac{1}{2\Delta x_i}\left[1+\frac{1}{4}\left(\frac{a}{\Delta x_i}\right)^2+\frac{1}{8}\left(\frac{a}{\Delta x_i}\right)^4+\cdots\right]. \quad (17)$$

We consider the saturated case only. For a Schwarzschild black hole $\mathrm{d}A=8\pi r_+\mathrm{d}r_+=32\pi M\mathrm{d}M$ and $\mathrm{d}M\approx\Delta p$. Thus,

$$\begin{aligned}\mathrm{d}A_{\mathrm{SNY}} &\approx 32\pi M\Delta p\\ &= \frac{16\pi M}{\Delta x}\left[1+\frac{1}{4}\left(\frac{a}{\Delta x}\right)^2+\frac{1}{8}\left(\frac{a}{\Delta x}\right)^4+\cdots\right]\\ &= \mathrm{d}A\left[1+\frac{1}{4}\left(\frac{a}{\Delta x}\right)^2+\frac{1}{8}\left(\frac{a}{\Delta x}\right)^4+\cdots\right]. \quad (18)\end{aligned}$$

If taking $\Delta x=2r_+=\sqrt{\frac{A}{\pi}}$, one can integrate the equation to obtain:

$$A_{\mathrm{SNY}}=A+\frac{\pi a^2}{4}\ln A-\frac{\pi^2a^4}{8A}-\cdots. \quad (19)$$

Based on Bekenstein-Hawking area law $S=A/4$, one can derive the corrected entropy

$$S_{\mathrm{SNY}}=\frac{A}{4}+\frac{\pi a^2}{4}\ln A-\frac{\pi^2a^4}{8A}-\cdots. \quad (20)$$

This result is consistent with the general entropy-area relation of the type

$$S=\frac{A}{4l_{\mathrm{p}}^2}+\rho\ln\frac{A}{l_{\mathrm{p}}^2}+\mathcal{O}\left(\frac{l_{\mathrm{p}}^2}{A}\right). \quad (21)$$

Until now the coefficient of the logarithmic correction term ρ is controversial, whereas the first correction term, which is represented as $\rho\ln\frac{A}{l_{\mathrm{p}}^2}$, is appropriate [29–33]. Eq. (20) indicates that the quantized space-time does not influence the thermodynamic entropy of black holes, obviously the leading term is still the celebrated $A/4$.

3.2 The modified Stefan-Boltzmann law

We discuss the black-body radiation and consider the radiation field as a photon gas. In general, the quantum states with momentum from $p\sim p+\mathrm{d}p$ in volume V is

$$\frac{V}{\pi^2}p^2\mathrm{d}p=\frac{V}{\pi^2}\omega^2\mathrm{d}\omega, \quad (22)$$

here we have considered the spin degeneracy of photons and $\varepsilon=\omega=p$[1]. The average quantum number should be

$$\frac{V}{\pi^2}\frac{\omega^2\mathrm{d}\omega}{\mathrm{e}^{\omega/T}-1}. \quad (23)$$

The internal energy of the photon gas is

$$U(\omega,T)\mathrm{d}\omega=\frac{V}{\pi^2}\frac{\omega^3\mathrm{d}\omega}{\mathrm{e}^{\omega/T}-1}, \quad (24)$$

which is a Planck formula. By integrating the equation above one can obtain

$$\begin{aligned}U &= \frac{V}{\pi^2}\int_0^\infty\frac{\omega^3\mathrm{d}\omega}{\mathrm{e}^{\omega/T}-1}\overset{x=\omega/T}{=\!=\!=}\frac{VT^4}{\pi^2}\int_0^\infty\frac{x^3\mathrm{d}x}{\mathrm{e}^x-1}\\ &= \frac{\pi^2V}{15}T^4, \quad (25)\end{aligned}$$

which can give the usual Stefan-Boltzmann law.

Considering the quantized space-time, the density of states is modified by Eq. (14). Thus, the internal energy of the photon gas should be

$$\begin{aligned}U &= \frac{V}{\pi^2}\int_0^\infty\frac{\omega^3\mathrm{d}\omega}{(1+a^2\omega^2)^{1/2}(\mathrm{e}^{\omega/T}-1)}\overset{x=\omega/T}{=\!=\!=}\frac{VT^4}{\pi^2}\int_0^\infty\frac{x^3\mathrm{d}x}{(1+a^2T^2x^2)^{1/2}(\mathrm{e}^x-1)}\\ &= \frac{VT^4}{\pi^2}\int_0^\infty\left[1-\frac{1}{2}(aTx)^2+\frac{3}{8}(aTx)^4-\frac{5}{16}(aTx)^6+\cdots\right]\frac{x^3\mathrm{d}x}{\mathrm{e}^x-1}\\ &= \frac{VT^4}{\pi^2}\left[\int_0^\infty\frac{x^3\mathrm{d}x}{\mathrm{e}^x-1}-\frac{a^2T^2}{2}\int_0^\infty\frac{x^5\mathrm{d}x}{\mathrm{e}^x-1}+\frac{3a^4T^4}{8}\int_0^\infty\frac{x^7\mathrm{d}x}{\mathrm{e}^x-1}-\cdots\right]. \quad (26)\end{aligned}$$

It is known that

$$\int_0^\infty\frac{x^{\alpha-1}\mathrm{d}x}{\mathrm{e}^x-1}=\Gamma(\alpha)\zeta(\alpha), \quad (27)$$

where $\Gamma(\alpha)$ and $\zeta(\alpha)$ are the Gamma function and Riemann zeta function, respectively. Because the minimal length a should be very small, the series expansion at $a=0$ is appropriate. Thus, Eq. (26) can be calculated exactly

$$\begin{aligned}U &= \frac{VT^4}{\pi^2}\left(\frac{\pi^4}{15}-\frac{4\pi^6a^2T^2}{63}+\frac{3\pi^8a^4T^4}{15}-\cdots\right)\\ &= \frac{\pi^2VT^4}{15}\left(1-\frac{60\pi^2a^2T^2}{63}+3\pi^4a^4T^4-\cdots\right). \quad (28)\end{aligned}$$

1) In Snyder's quantized space-time model , the dispersion relation should be different from the usual one in Minkowski spacetime. Here, we just take an approximation because we are considering a photon gas that has zero rest mass.

The first term is the usual Stefan-Boltzmann relation and the latter ones are correction terms. For the usual Stefan-Boltzmann law, we know that there is a maximal frequency, $x=\omega_{\rm m}/T\approx 2.82$, corresponding to the maximal internal energy. According to the modified expression of internal energy for the photon gas, we can also find the maximal frequency $\omega_{\rm mq}$. Given different values of aT, one can find the maximal values of the modified expression. Fig. 1 shows that bigger values of aT will lead to smaller values of maximal frequency and maximal internal energy.

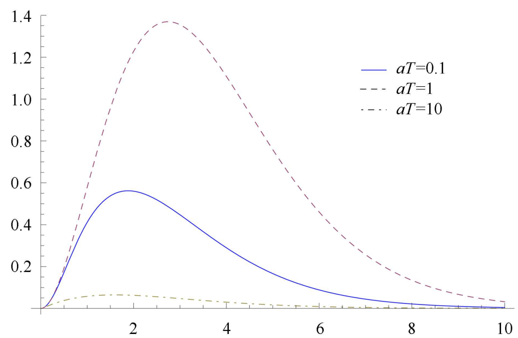

Fig. 1. For the case of aT=0.1 the maximal value lies at x=2.74, the case of aT=1 at x=1.88, and the case of aT=10 at x=1.60.

4 Conclusion

According to Snyder's idea of quantized space-time, we derive the generalized uncertainty principle and modified density of states. The density of states obtained from Snyder's model is different from those of the usual GUP. The weighted factor in the modified density of states is $1/(1+a^2p^2)^{1/2}$, with a constant exponent 1/2, whereas the weighted factor from the usual GUP is $1/(1+\lambda p^2)^D$, with a dimension-dependent exponent D. This difference leads to different modifications to some physical problems. It should be noted that Snyder's model is based on de Sitter space, which is a kind of curved space. Despite the lack of consensus on the construction of phase space in curved space-time, there is some tentative research on this subject. Caianiello [34] has already proposed a "game" that geometrizes quantum mechanics by endowing phase space and metric, connection and curvature. An interesting byproduct of the theory is the existence of a maximal acceleration [35]. Another method for constructing curved phase space is advanced by Hazboun and Wheeler [36]. They employ the quotient manifold method to obtain biconformal space with symplectic form. Although both methods try to follow Born's reciprocity principle [37], they do not give an universal expression about the volume element of curved phase space. Therefore, our work is based on the assumption that the construction of phase space in Snyder's model has a similar form to that in flat space.

Based on Snyder's GUP, we calculate the modified black hole entropy. The leading term in the result is the celebrated Hawking-Bekenstein area formula. The first correction term has a logarithmic form with the coefficient $\frac{\pi a^2}{4}$. This kind of logarithmic correction, which can be obtained in string theory, loop quantum gravity and other theories, is appropriate. However the exact value of the coefficient is still controversial. The Stefan-Boltzmann laws in thermodynamics may also need to be modified because of the modified density of states. Except for the usual $\sim T^4$ term, some correction terms also exist. Considering the modified Stefan-Boltzmann laws, the rate of a black hole's radiation will be influenced and the evolution of the universe should also be modified. These problems will be discussed in another paper.

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