Off-shell superpotentials and Ooguri-Vafa invariants of type Ⅱ/F theory compactification*

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Abstract: In this paper, we calculate the off-shell superpotential of two Calabi-Yau manifolds with three parameters by integrating the period of the subsystem. We also obtain the Ooguri-Vafa invariants with open mirror symmetry.

Key words: superpotential, mirror symmetry, Ooguri-Vafa invariants

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1 Introduction

When Type Ⅱ string theory compactfying on Calabi-Yau threefold with D-brane and background flux, the superpotentials will be generated which in general can divided into two parts–one originated from D-brane and the other from flux. The superpotentials also play an important role in mathematics which generate the Ooguri-Vafa invariants and count the number of disks and sphere instantons.

For $D5$-brane wrapped the whole Calabi-Yau threefold, the holomorphic Chern-Simons theory [1]

$$\mathcal{W}=\int_X \Omega^{3,0}\wedge\mathrm{Tr}\left[A\wedge\bar{\partial}A+\frac{2}{3}A\wedge A\wedge A\right] \quad (1)$$

gives the brane superpotential $\mathcal{W}_{\rm brane}$, where A is the gauge field with gauge group $U(N)$ for N $D5$-branes. When reduced dimensionally, the low-dimenaional brane superpotentials can be obtained as [2, 3]

$$\mathcal{W}_{\rm brane}=N_\nu\int_{\Gamma^\nu}\Omega^{3,0}(z,\hat{z})=\sum_\nu N_\nu \Pi^\nu, \quad (2)$$

where Γ^ν is a special Lagrangian 3-chain and $(z,\hat{z})$ are closed-string complex structure moduli and D-brane moduli from the open-string sector, respectively.

The background fluxes $H^{(3)}=H^{(3)}_{\rm RR}+\tau H^{(3)}_{\rm NS}$, which take values in the integer cohomology group $H^3(X,\mathbb{Z})$, also break the supersymmetry $N=2$ to $N=1$. The $\tau=C^{(0)}+\mathrm{i}e^{-\varphi}$ is the complexified Type IIB coupling field. Its contribution to superpotentials is [4, 5]

$$\mathcal{W}_{\rm flux}(z)=\int_X H^{(3)}_{\rm RR}\wedge\Omega^{3,0}=\sum_\alpha N_\alpha\cdot\Pi^\alpha(z), \quad N_\alpha\in Z. \quad (3)$$

The contributions of D-brane and background flux (here the NS-flux is ignored) give together the general form of superpotential as follow [6, 7]

$$\mathcal{W}(z,\hat{z})=\mathcal{W}_{\rm brane}(z,\hat{z})+\mathcal{W}_{\rm flux}(z)=\sum_{\gamma_i\in H^3(Z^*,\mathcal{H})}N_i\Pi_i(z,\hat{z}), \quad (4)$$

where $N_i=n_i+\tau m_\sigma$, τ is the dilaton of type Ⅱ string and Π_i is a relative period defined in a relative cycle $\Gamma\in H_3(X,D)$ whose boundary is wrapped by D-branes and D is a holomorphic divisor of the Calabi-Yau space. In fact, the two-cycles wrapped by the D-branes are holomorphic cycles only, if the moduli are at the critical points of the superpotentials. Thus, the two-cycles are generically not holomorphic. However, according to the arguments of [6–8], the non-holomorphic two-cycles can be replaced by a holomorphic divisor D of the ambient Calabi-Yau space with the divisor D encompassing the two-cycles.

Geometrically speaking, when varying the complex structure of Calabi-Yau space, a generic holomorphic curve will not be holomorphic with the respect to the new complex structure, and will become obstructed to the deformation of the bulk moduli. The requirement for the holomorphy gives rise to a relation between the closed and open string moduli. Physically speaking, it turns out that the obstruction generates a superpotential for the effective theory depending on the closed and open string moduli.

The off-shell tension of D-branes, $\mathcal{T}(z,\hat{z})$, is equal to the relative period [6, 7, 9]

$$\Pi_\Sigma=\int_{\Gamma_\Sigma}\Omega(z,\hat{z}), \quad (5)$$

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which measures the difference between the value of on-shell superpotentials for the two D-brane configurations

$$\mathcal{T}(z,\hat{z})=\mathcal{W}(C^+)-\mathcal{W}(C^-), \tag{6}$$

with $\partial\Gamma_\Sigma=C^+-C^-$. The domain wall tension is [10]

$$T(z)=\mathcal{T}(z,\hat{z})|_{\hat{z}=\text{critic points}}, \tag{7}$$

where the critical points correspond to $\frac{\mathrm{d}W}{\mathrm{d}\hat{z}}=0$ [9] and the $C^\pm$ is the holomorphic curves at those critical points. The critical points are alternatively defined as the Nother-Lefshetz locus [11]

$$\mathcal{N}=\{(z,\hat{z})|\pi(z,\hat{z};\partial\Gamma(z,\hat{z}))\equiv 0\}, \tag{8}$$

where

$$\pi(z,\hat{z};\partial\Gamma(z,\hat{z}))=\int_{\partial\Gamma}\omega_{\hat{a}}^{(2,0)}(z,\hat{z}),\quad \hat{a}=1,\cdots,\dim(H^{2,0}(D)), \tag{9}$$

and $\omega_{\hat{a}}^{(2,0)}$ is an element of the cohomology group $H^{(2,0)}(D)$. At those critical points, the domain wall tensions are also known as normal functions giving the Abel-Jacobi invariants [10–14].

The Superpotential can be calculated by studying the Hodge variation on the related cohomology group. The flat Gauss-Manin connection on the Hodge bundle can give rise to a system of differential equations controlling the periods which determine the mirror map between the A-model and the B-model. The Ooguri-Vafa invariants can be obtained by using the mirror symmetry. See also [15–28] for related work, where in Refs. [17, 18, 21–23] they considered another approach which blows up along the curve C and replaces the pair (X, C) with a non-Calabi-Yau manifold $\widehat{X}$.

In this note, we will generalize the works [15–28], which only calculated on-shell superpotential, to the off-shell superpotential which at the critical point gives the domain wall tensions (on-shell superpotential).

2 Generalized GKZ system and differential operators

The period integrals can be written as

$$\Pi_i=\int_{\gamma_i}\frac{1}{P}\prod_{j=1}^{4}\frac{\mathrm{d}X_j}{X_j}. \tag{10}$$

Where P is the hypersurface equation defined as

$$P=\sum_{i=1}^{p-1}a_i\prod_{k=1}^{4}X_k^{\mu_{i,k}}, \tag{11}$$

p is the number of integer points μ_i of reflexive polyhedron Δ, a_i is the moduli determining the complex structure in the B-model.

See more in Ref. [28]. According to the Refs. [29, 30], the period integrals can be annihilated by differential operators

$$\mathcal{L}(l)=\prod_{l_i>0}(\partial_{a_i})^{l_i}-\prod_{l_i<0}(\partial_{a_i})^{l_i},$$

$$\mathcal{Z}_k=\sum_{i=0}^{p-1}\nu_{i,k}^*\vartheta_i,\quad \mathcal{Z}_0=\sum_{i=0}^{p-1}\vartheta_i-1, \tag{12}$$

where $\vartheta_i=a_i\partial_{a_i}$. As noted in Refs. [19, 31], the equations $\mathcal{Z}_k\Pi(a_i)=0$ reflex the invariance under the torus action, defining torus invariant algebraic coordinates z_a on the moduli space of the complex structure of X [10]:

$$z_a=(-1)^{l_0^a}\prod_i a_i^{l_i^a}, \tag{13}$$

where l_a, $a=1,\cdots,h^{2,1}(X)$ are generators of the Mori cone, one can rewrite the differential operators $\mathcal{L}(l)$ as [10, 30, 31]

$$\mathcal{L}(l)=\prod_{k=1}^{l_0}(\vartheta_0-k)\prod_{l_i>0}\prod_{k=0}^{l_i-1}(\vartheta_i-k)-(-1)^{l_0}z_a\prod_{k=1}^{-l_0}(\vartheta_0-k)\times\prod_{l_i<0}\prod_{k=0}^{l_i-1}(\vartheta_i-k). \tag{14}$$

The solution to the GKZ system can be written as [10, 30, 31]

$$B_{l^a}(z^a;\rho)=\sum_{n_1,\cdots,n_N\in Z_0^+}\frac{\Gamma(1-\sum_a l_0^a(n_a+\rho_a))}{\Pi_{i>0}\Gamma(1+\sum_a l_i^a(n_a+\rho_a))}\prod_a z_a^{n_a+\rho_a}. \tag{15}$$

In this paper we consider the family of divisors $\mathcal{D}$ with a single open deformation moduli $\hat{z}$

$$x_1^{b_1}+\hat{z}x_2^{b_2}=0, \tag{16}$$

where b_1, b_2 are some appropriate integers. The relative 3-form $\underline{\Omega}:=(\Omega_X^{3,0},0)$ and the relative periods satisfy a set of differential equations [6–8, 10, 19]

$$\mathcal{L}_a(\theta,\hat{\theta})\underline{\Omega}=\mathrm{d}\underline{\omega}^{(2,0)}\Rightarrow\mathcal{L}_a(\theta,\hat{\theta})\mathcal{T}(z,\hat{z})=0, \tag{17}$$

with some corresponding two-form $\underline{\omega}^{(2,0)}$. The differential operators $\mathcal{L}_a(\theta,\hat{\theta})$ can be expressed as [10]

$$\mathcal{L}_a(\theta,\hat{\theta}):=\mathcal{L}_a^{\mathrm{b}}-\mathcal{L}_a^{\mathrm{bd}}\hat{\theta}, \tag{18}$$

for $\mathcal{L}_a^{\mathrm{b}}$ acting only on the bulk part from the closed sector, $\mathcal{L}_a^{\mathrm{bd}}$ on the boundary part from the open-closed sector and $\hat{\theta}=\hat{z}\partial_{\hat{z}}$. The explicit form of these operators will be given in the following model. From the Eq. (9) one can obtain

$$2\pi\mathrm{i}\hat{\theta}\mathcal{T}(z,\hat{z})=\pi(z,\hat{z}), \tag{19}$$

for only the family of divisors $\mathcal{D}$ depending on the $\hat{z}$. Hence the off-shell superpotential can be obtained by integrating the period on subsystem $\pi(z,\hat{z})$.

3 Superpotentials of hypersurface $X_{24}(1, 1, 2, 8, 12)$

The $X_{24}(1, 1, 2, 8, 12)$ is defined as the zero locus of polynomial P

$$P = x_1^{24}+x_2^{24}+x_3^{12}+x_4^3+x_5^2+\psi x_1x_2x_3x_4x_5+\phi x_1^6x_2^6x_3^6 +\chi x_1^{12}x_2^{12}. \quad (20)$$

The GLSM charge vectors l_a are the generators of the Mori cone as follows [31]

	0	1	2	3	4	5	6	7
l_1	−6	0	0	0	2	3	0	1
l_2	0	1	1	0	0	0	−2	0
l_3	0	0	0	1	0	0	1	−2

(21)

The mirror manifolds can be constructed as an orbifold by the Greene-Plesser orbifold group acting as $x_i \to \lambda_k^{g_{k,i}} x_i$ with weights

$$\mathbb{Z}_6\colon\ g_1 = (1,-1,0,0,0),\quad \mathbb{Z}_6\colon\ g_2=(1,0,-1,0,0),$$
$$\mathbb{Z}_3\colon\ g_3 = (1,0,0,-1,0), \quad (22)$$

where we denote $\lambda_{1,2}^6=1$ and $\lambda_3^3=1$.

By the generalized GKZ system, the period on the K3 surface has the form

$$\pi = \frac{c}{2}B_{\{\hat{l}_1,\hat{l}_2,\hat{l}_3\}}\left(u_1,u_2,u_3;\frac{1}{2},\frac{1}{2},0\right) = -\frac{4c}{\pi^{\frac{3}{2}}}\sqrt{u_1u_2}\,u_3+\mathcal{O}((u_1u_2)^{3/2}), \quad (23)$$

which vanishes at the critical locus $u_2=0$. According to Eq. (19), the off-shell superpotentials can be obtained by integrating the π:

$$\mathcal{T}_a^{\pm}(z_1,z_2,z_3)=\frac{1}{2\pi\mathrm{i}}\int\pi(\hat{z})\frac{\mathrm{d}\hat{z}}{\hat{z}}, \quad (24)$$

with the appropriate integral constants [10], the superpotentials can be chosen as $\mathcal{W}^+=-\mathcal{W}^-$. In this convention, the off-shell superpotentials can be obtained as

$$2\mathcal{W}^+=\frac{1}{2\pi\mathrm{i}}\int_{-\hat{z}}^{\hat{z}}\pi(\zeta)\frac{\mathrm{d}\zeta}{\zeta},\ W^{\pm}(z_1,z_2,z_3)=\mathcal{W}^{\pm}(z_1,z_2,z_3)|_{\hat{z}=1}. \quad (25)$$

Eventually, the superpotentials are

$$\mathcal{W}^{\pm}(z_1,z_2,z_3,\hat{z}) = \sum_{n_1,n_2,n_3}\frac{\mp c z_1^{\frac{1}{2}+n_1}z_2^{\frac{1}{2}+n_2}z_3^{n_3}\hat{z}^{\frac{-1-2n_2}{2}}\Gamma(6n_1+4)}{\Gamma(2+2n_2)\Gamma(2+2n_1)\Gamma\left(\frac{5}{2}+3n_1\right)\Gamma(1+n_3)\Gamma(n_3-2n_2)\Gamma\left(n_1-2n_3+\frac{3}{2}\right)}$$
$$\frac{\left\{(1-2n_2)_2F_1\left(-\frac{1}{2}-n_2,-2n_2,\frac{1}{2}-n_2;\hat{z}\right)+\hat{z}(1+2n_2)_2F_1\left(\left(\frac{1}{2}-n_2,-2n_2,\frac{3}{2}-n_2;\hat{z}\right)\right)\right\}}{4\pi(-1+4n_2^2)}. \quad (26)$$

For the calculation of instanton corrections, one needs to know the mirror map. The fundamental period ω_0 is a solution of the Picard-Fuchs equation which we listed in Ref. [28]. The flat coordinates in the A-model at the large radius regime are related to the flat coordinates of the B-model at the large complex structure regime by the mirror map $t_i=\frac{\omega_i}{\omega_0}$, $\omega_i:=D_i^{(1)}\omega_0(z,\rho)|_{\rho=0}$. The open-string mirror maps are

$$\begin{aligned}
q_1&=z_1+312z_1^2+107604z_1^3-z_1z_3-192z_1^2z_3-z_1z_3^2+\mathcal{O}(z^4),\\
q_2&=z_2+2z_2^2+5z_2^3+z_2z_4+3z_2^2z_4+z_2^2z_4^2+\mathcal{O}(z^4),\\
q_3&=z_3+2z_3^2+3z_3^3+120z_3z_1+41580z_1^2z_3+\mathcal{O}(z^4),\\
q_4&=z_4-z_4^2+z_4^3+\mathcal{O}(z^4).
\end{aligned} \quad (27)$$

Here $q_i=\mathrm{e}^{2\pi\mathrm{i}t_i}$ and we can obtain the inverse mirror map

$$\begin{aligned}
z_1 &= q_1-312q_1^2+87084q_1^3+q_1q_3-864q_1^2q_3+q_1q_2q_3+\mathcal{O}(q^4),\\
z_2 &= q_2-2q_2^2+3q_2^3+\mathcal{O}(q^4),\\
z_3 &= q_3-2q_3^2+3q_3^3-120q_1q_3+10260q_1^2q_3+q_2q_3,\\
&\quad -120q_1q_2q_3+600q_1q_3^2-4q_2q_3^2+\mathcal{O}(q^4),\\
z_4 &= q_4+q_4^2+q_4^3+\mathcal{O}(q^4).
\end{aligned} \quad (28)$$

Using the modified multi-cover formula [2] for this case

$$\frac{\mathcal{W}^{\pm}(z(q))}{w_0(z(q))} = \frac{1}{(2\mathrm{i}\pi)^2}\sum_{k\ \mathrm{odd}}\ \sum_{d_3,d_4,d_{1,2}\mathrm{odd}\geqslant 0} n^{\pm}_{d_1,d_2,d_3,d_4} \times\frac{q_1^{kd_1/2}q_2^{kd_2/2}q_3^{kd_3}q_4^{kd_4}}{k^2}. \quad (29)$$

The superpotentials $\mathcal{W}^+$ give Ooguri-Vafa invariants n_{d_1,d_2,d_3,d_4} for the normalization constants $c=1$, which are listed in Table 1.

Table 1. Disc invariants n_{d_1,d_2,d_3,d_4} for the off-shell superpotential W_1 of the 3-fold $\mathbb{P}_{1,1,2,8,12}$ [24].

$d_4=0,d_3=1$ $d_1/2\backslash d_2/2$	1	3	5	7	9
1	1	0	0	0	$\frac{-5}{2}$
3	−848	0	0	0	2120
5	−270978	0	0	0	677445
7	−4107040	0	0	0	10267600
9	−4859101222	0	0	0	12147753055

$d_4=0,d_3=2$ $d_1/2\backslash d_2$	1	3	5	7	9
1	$\frac{-9}{16}$	$\frac{-9}{16}$	0	0	$\frac{45}{32}$
3	$\frac{521}{2}$	$\frac{521}{2}$	0	0	$\frac{-2605}{4}$
5	$\frac{-1397265}{8}$	$\frac{-2506065}{8}$	167400	−195120	$\frac{7890645}{16}$
7	100877911	205105111	−118540800	142047360	$\frac{-553418675}{2}$
9	$\frac{-160323502433}{8}$	$\frac{226729748767}{8}$	−64409331600	71920841760	$\frac{251804856805}{16}$

$d_4=1,d_3=1$ $d_1/2\backslash d_2$	1	3	5	7	9
1	$\frac{-29}{18}$	0	0	$\frac{-7}{2}$	$\frac{-35}{36}$
3	$\frac{12296}{9}$	0	0	2968	$\frac{7420}{9}$
5	$\frac{1309727}{3}$	0	5130	943293	$\frac{1611485}{6}$
7	$\frac{59552080}{9}$	0	−3734640	18109280	$\frac{2324840}{9}$
9	$\frac{70456967719}{9}$	0	−1890907740	18897762017	$\frac{50997932065}{18}$

4 Superpotential of hypersurface X_{12}(1, 1, 1, 3, 6)

The X_{12} (1, 1, 1, 3, 6) is defined as the zero locus of P:

$$P=x_1^{12}+x_2^{12}+x_3^{12}+x_4^4+x_5^2+\psi x_1x_2x_3x_4x_5+\phi x_1^4x_2^4x_3^4. \quad (30)$$

The GLSM charge vectors in this case are [31]

$$\begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline l_1 & -4 & 0 & 0 & 0 & 1 & 2 & 1 \\ l_2 & 0 & 1 & 1 & 1 & 0 & 0 & -3 \end{array}. \quad (31)$$

On the mirror manifolds, the Greene-Plesser orbifold group acts as $x_i\rightarrow\lambda_k^{g_{k,i}}x_i$ with weights

$$\mathbb{Z}_6:\ g_1=(1,-1,0,0,0),\ \ \mathbb{Z}_4:\ g_2=(0,1,2,1,0), \quad (32)$$

where we denote $\lambda_1^6=1$, $\lambda_2^4=1$.

In Ref. [28], we have obtained the period in the subsystem as follows

$$\pi(u_1,u_2)=\frac{c}{2}B_{\{\hat{l_1},\hat{l_2}\}}\left(u_1,u_2;0,\frac{1}{2}\right), \quad (33)$$

where c are some normalization constants not determined by the differential operator. According to Eq. (19), the off-shell superpotentials can be obtained by integrating the π:

$$\mathcal{T}_a^{\pm}(z_1,z_2,z_3)=\frac{1}{2\pi i}\int\pi(\hat{z})\frac{d\hat{z}}{\hat{z}}, \quad (34)$$

with the appropriate integral constants [10], the superpotentials can be chosen as $\mathcal{W}^+=-\mathcal{W}^-$.

Eventually, The superpotential are

$$\mathcal{W}^{\pm}(z_1,z_2,z_3,\hat{z}) = \sum_{n_1,n_2,n_3} \frac{\mp c z_1^{\frac{1}{2}+n_1} z_2^{n_2} \hat{z}^{\frac{-1-2n_1}{2}} \Gamma\left(4n_1+\frac{5}{2}\right)}{\Gamma(1+2n_2)\Gamma(1+n_2)\Gamma\left(\frac{3}{2}+n_1\right)\Gamma(2+2n_1)\Gamma\left(n_1-3n_2+\frac{3}{2}\right)}$$

$$\times\frac{\left\{(1-2n_1)\,{}_2F_1\left(-\frac{1}{2}-n_1,-2n_1,\frac{1}{2}-n_1;\hat{z}\right)+\hat{z}(1+2n_1)\,{}_2F_1\left(\left(\frac{1}{2}-n_1,-2n_1,\frac{3}{2}-n_1;\hat{z}\right)\right)\right\}}{4\pi(-1+4n_1^2)}. \quad (35)$$

Table 2. Disc invariants n_{d_1,d_2,d_3} for the off-shell superpotential $\mathcal{W}_1^+$ of the 3-fold $\mathbb{P}_{1,1,1,3,6}[12]$.

$d_3=0$; $d_1/2\backslash d_2$	0	1	2	3	4
1	1	$\frac{-13}{16}$	$\frac{2693}{1024}$	$\frac{19517}{9}$	$\frac{7703}{16384}$
3	$\frac{1312}{243}$	$\frac{68231}{1296}$	$\frac{-23305385}{82944}$	$\frac{-3519745}{18}$	$\frac{-1672979243}{3981312}$
5	$\frac{63544513}{28350}$	$\frac{135578197}{12960}$	$\frac{346285919719}{5806080}$	$\frac{-9330830923}{1944}$	$\frac{-1608130586479}{92897280}$
7	$\frac{172956753731}{1389150}$	$\frac{5372183267179}{3175200}$	$\frac{-21892937788889}{8128512}$	$\frac{32917422417037}{136080}$	$\frac{-282745996819463}{43352064}$
9	$\frac{13409490308809711}{600112800}$	$\frac{216480619417211431}{355622400}$	$\frac{-1964470782077819881}{13655900160}$	$\frac{-82475053081873279}{7620480}$	$\frac{31104035766372911003 3}{93640458240}$

$d_3=1$; $d_1/2\backslash d_2$	0	1	2	3	4
1	$\frac{-10}{9}$	$\frac{175}{144}$	$\frac{-18923}{2304}$	$\frac{23077}{147456}$	$\frac{-132267135}{2097152}$
3	$\frac{3380}{81}$	$\frac{-82429}{432}$	$\frac{48484423}{41472}$	$\frac{1376117443}{1327104}$	$\frac{601661687053}{42467328}$
5	$\frac{-1138840}{567}$	$\frac{403544255}{18144}$	$\frac{-534921106991}{2903040}$	$\frac{166478391791}{3440640}$	$\frac{-20526886980289679}{5945425920}$
7	$\frac{3400299058}{42525}$	$\frac{-835235546479}{272160}$	$\frac{786036635335453}{60963840}$	$\frac{4714357006892647}{650280960}$	$\frac{-441800377875169456 79}{62426972160}$
9	$\frac{-121337433752293}{33339600}$	$\frac{-99152104754391869}{152409600}$	$\frac{15374804216369862857}{8534937600}$	$\frac{-626930861424694345 27}{6242697216}$	$\frac{-923658211082431780070641}{3995326218240}$

For the calculation of instanton corrections, one needs to know the mirror map. The fundamental period ω_0 is the solution of the Picard-Fuchs equation which we listed in Ref. [28]. The flat coordinates in the A-model at the large radius regime are related to the flat coordinates of the B-model at the large complex structure regime by the mirror map $t_i=\frac{\omega_i}{\omega_0}$, $\omega_i:=D_i^{(1)}\omega_0(z,\rho)|_{\rho=0}$. The open-string mirror maps are

$$\begin{aligned} q_1&=z_1+40z_1^2+1876z_1^3+2z_1z_2-13z_1z_2^2+z_1z_2z_3+\mathcal{O}(z^4),\\ q_2&=z_2-6z_2^2+63z_2^3+z_2z_3-9z_2^2z_3+\mathcal{O}(z^4), \quad (36)\\ q_3&=z_3-z_3^2+z_3^3+\mathcal{O}(z^4), \end{aligned}$$

here $q_i=\mathrm{e}^{2\pi\mathrm{i}t_i}$ and we can obtain the inverse mirror map as follows

$$\begin{aligned} z_1&=q_1-40q_1^2+1324q_1^3-2q_1q_2+268q_1^2q_2+5q_1q_2^2+\mathcal{O}(q^4),\\ z_2&=q_2+6q_2^2+9q_2^3-36q_1q_2-468q_1q_2^2+630q_1^2q_2+\mathcal{O}(q^4), \quad (37)\\ z_3&=q_3+q_3^2+q_3^3+\mathcal{O}(q^4). \end{aligned}$$

Using the modified multi-cover formula [2] for this case

$$\frac{\mathcal{W}^{\pm}(z(q))}{w_0(z(q))} = \frac{1}{(2\pi\mathrm{i})^2}\sum_{k\ \mathrm{odd}}\ \sum_{d_1\mathrm{odd},d_{2,3}\geqslant 0} n^{\pm}_{d_1,d_2,d_3} \times\frac{q_1^{kd_1/2}q_2^{kd_2}q_3^{kd_3}}{k^2}. \quad (38)$$

The superpotentials $\mathcal{W}^+$ give Ooguri-Vafa invariants

n_{d_1,d_2,d_3} for the normalization constants c=1, which are listed in Table 2.

5 Summary

In this paper, we make a further step of previous work [28] and calculate the off-shell superpotential. By open mirror symmetry, we also compute the Ooguri-Vafa invariants from the A-model expansion.

The superpotentials of Type Ⅱ string theory are important in both physics and mathematics. It also relates to F-theory by open-closed duality [15, 19, 32]. In type Ⅱ/F-theory compactification, the vacuum structure is determined by the superpotentials, whose second derivative gives the chiral ring structure. The quantum cohomology ring structure comes from the world-sheet instanton corrections and space-time instanton corrections [6, 7]. In fact, the more general vacuum structure of type Ⅱ/F-theory/heterotic theory compactification can be tackled by the Hodge variance approach.

In the next work, we will study D-brane in the general case. We also try to calculate the D-brane superpotential with the method of A_∞ structure of the derived category $D_{\text{coh}}(X)$ and path algebras of quivers.

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