

Spacetime structure of MOND with Tully-Fisher relation and Lorentz invariance violation^{*}

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Abstract: It is believed that the modification of Newtonian dynamics (MOND) is a possible alternative for dark matter hypothesis. Although Bekenstein's TeVeS supplies a relativistic version of MOND, one may still wish for a more concise covariant formulism of MOND. In this paper, within covariant geometrical framework, we present another version of MOND. We show the spacetime structure of MOND with properties of Tully-Fisher relation and Lorentz invariance violation.

Key words: Finsler gravity, MOND

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1 Introduction

In 1932, physicists [1, 2] found from the observations of galaxies and galaxy clusters that Newton's gravity could not provide enough force to attract the matter of galaxies. The recent astronomical observations show that the rotational velocity curves of all spiral galaxies tend to some constant values [3]. These include the Oort discrepancy in the disk of the Milky Way [4], the velocity dispersions of dwarf Spheroidal galaxies [5], and the flat rotation curves of spiral galaxies [6]. These facts violate sharply the prediction of Newton's gravity. The most widely adopted way to resolve these difficulties is the dark matter hypothesis. It is assumed that all visible stars are surrounded by massive nonluminous matter.

The dark matter hypothesis has dominated astronomy and cosmology for almost 80 years. However, up to now, no direct observations have been substantially tested. Some models have been built for an alternative to the dark matter hypothesis. Their main ideas are to assume that the Newtonian gravity or Newton's dynamics are invalid in galactic scale. The most successful and famous model is MOND [7]. It assumed that the Newtonian dynamics does not hold in galactic scale. The particular form of MOND is given as

$$m\mu\left(\frac{a}{a_0}\right)\mathbf{a}=\mathbf{F}, \quad (1)$$

$$\lim_{x \gg 1} \mu(x) = 1, \quad \lim_{x \ll 1} \mu(x) = x,$$

where a_0 is at the order of 10^{-8} cm/s². At beginning, as a phenomenological model, MOND explains well the flat rotation curves with a simple formula and a new parameter. In particular, it deduce naturally a well-known global scaling relation for spiral galaxies, the Tully-Fisher relation [8]. The Tully-Fisher relation is an empirical relation between the total luminosity of a galaxy and the maximum rotational speed. The relation is of the form $L \propto v_{\max}^a$, where $a \approx 4$ if luminosity is measured in the near-infrared. Tully and Pierce [9] showed that the Tully-Fisher relation appears to be convergence in the near-infrared. McGaugh [10] investigated the Tully-Fisher relation for a large sample of galaxies, and concluded that the Tully-Fisher relation is the fundamental relation between the total baryonic mass and the rotational speed. MOND [7] predicts that the rotational speed of a galaxy has an asymptotic value $\lim_{r \rightarrow \infty} v^4 = GMa_0$, which explains the Tully-Fisher relation.

By introducing several scalar, vector and tensor fields, Bekenstein [11] rewrote the MOND into a covariant formulation (TeVeS). He showed that the MOND satisfies all four classical tests on Einstein's general relativity in the Solar system. Beside Bekenstein's theory, there are other MOND theories (for example, Einstein-aether theory [12]). These MOND theories modify gravity with additional scalar/vector/tensor fields. Bekenstein's theory and Einstein-aether theory both admit a preferred reference frame and break local Lorentz invariance. It means that local Lorentz symmetry violation is a feat-

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ure of MOND.

The local Lorentz symmetry violation implies that the space structure of a galaxy is not Minkowskian at large scale, and the relationship between the Tully-Fisher relation and MOND implies that the space structure of a galaxy depends on rotational speed. Finsler gravity based on Finsler geometry involves the above features. It is natural to assume that the Finsler gravity is a covariant formulism of MOND. Finsler geometry [13] as a natural extension of Riemann geometry involves Riemann geometry as its special case. The length element of Finsler geometry depends not only on the positions but also the velocities. Finsler gravity naturally preserves fundamental principles and results of general relativity. A new geometry (Finsler geometry) involves new spacetime symmetry. The Lorentz invariance violation is intimately linked to Finsler geometry. Kostelecky [14] has studied effective field theories with explicit Lorentz invariance violation in Finsler spacetime. In this paper, within the covariant geometric framework, we try to present a covariant formulism of MOND with explicit Tully-Fisher relation and Lorentz invariance violation.

2 Vacuum field equation in Finsler spacetime

Instead of defining an inner product structure over the tangent bundle in Riemann geometry, Finsler geometry is based on the so called Finsler structure F with the property $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$, where x represents position and $y \equiv \frac{dx}{d\tau}$ represents velocity. The Finsler metric is given as [13]

$$g_{\mu\nu} \equiv \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} \left(\frac{1}{2} F^2 \right). \quad (2)$$

Finsler geometry has its genesis in integrals of the form

$$\int_s^r F \left(x^1, \dots, x^n; \frac{dx^1}{d\tau}, \dots, \frac{dx^n}{d\tau} \right) d\tau. \quad (3)$$

The Finsler structure represents the length element of Finsler space.

The parallel transport has been studied in the framework of Cartan connection [15–17]. The notation of parallel transport in Finsler manifold means that the length $F \left(\frac{dx}{d\tau} \right)$ is constant. The geodesic equation for Finsler manifold is given as [13]

$$\frac{d^2 x^\mu}{d\tau^2} + 2G^\mu = 0, \quad (4)$$

where

$$G^\mu = \frac{1}{4} g^{\mu\nu} \left(\frac{\partial^2 F^2}{\partial x^\lambda \partial y^\nu} y^\lambda - \frac{\partial F^2}{\partial x^\nu} \right) \quad (5)$$

is called geodesic spray coefficient. Obviously, if F is

Riemannian metric, then

$$G^\mu = \frac{1}{2} \tilde{\gamma}_{\nu\lambda}^\mu y^\nu y^\lambda, \quad (6)$$

where $\tilde{\gamma}_{\nu\lambda}^\mu$ is the Riemannian Christoffel symbol. Since the geodesic Eq. (4) is directly derived from the integral length

$$L = \int F \left(x, \frac{dx}{d\tau} \right) d\tau, \quad (7)$$

the inner product $\left(\sqrt{g_{\mu\nu}} \left(x, \frac{dx}{d\tau} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = F \left(x, \frac{dx}{d\tau} \right) \right)$ of two parallel transported vectors is preserved.

In Finsler manifold, there exists a linear connection – the Chern connection [18]. It is torsion free and almost metric-compatible,

$$\Gamma_{\mu\nu}^\alpha = \gamma_{\mu\nu}^\alpha - g^{\alpha\lambda} \left(A_{\lambda\mu\beta} \frac{N_\nu^\beta}{F} - A_{\mu\nu\beta} \frac{N_\lambda^\beta}{F} + A_{\nu\lambda\beta} \frac{N_\mu^\beta}{F} \right), \quad (8)$$

where $\gamma_{\mu\nu}^\alpha$ is the formal Christoffel symbols of the second kind with the same form of Riemannian connection, N_ν^μ is defined as $N_\nu^\mu \equiv \gamma_{\nu\alpha}^\mu y^\alpha - A_{\nu\lambda}^\mu \gamma_{\alpha\beta}^\lambda y^\alpha y^\beta$ and $A_{\lambda\mu\nu} \equiv \frac{F}{4} \frac{\partial}{\partial y^\lambda} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} (F^2)$ is the Cartan tensor (regarded as a measurement of deviation from the Riemannian Manifold). In terms of Chern connection, the curvature of Finsler space is given as

$$R_{\kappa\mu\nu}^\lambda = \frac{\delta \Gamma_{\kappa\nu}^\lambda}{\delta x^\mu} - \frac{\delta \Gamma_{\kappa\mu}^\lambda}{\delta x^\nu} + \Gamma_{\alpha\mu}^\lambda \Gamma_{\kappa\nu}^\alpha - \Gamma_{\alpha\nu}^\lambda \Gamma_{\kappa\mu}^\alpha, \quad (9)$$

where $\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - N_\mu^\nu \frac{\partial}{\partial y^\nu}$.

The gravity in Finsler spacetime has been investigated for a long time [19–22]. In this paper, we introduce a vacuum field equation by the way discussed first by Pirani [23, 24]. In Newton’s theory of gravity, the equation of motion of a test particle is given as

$$\frac{d^2 x^i}{dt^2} = -\eta^{ij} \frac{\partial \phi}{\partial x^j}, \quad (10)$$

where $\phi = \phi(x)$ is the gravitational potential and η^{ij} is Euclidean metric. For an infinitesimal transformation $x^i \rightarrow x^i + \epsilon \xi^i$ ($|\epsilon| \ll 1$), the Eq. (10) becomes, up to first order in ϵ ,

$$\frac{d^2 x^i}{dt^2} + \epsilon \frac{d^2 \xi^i}{dt^2} = -\eta^{ij} \frac{\partial \phi}{\partial x^j} - \epsilon \eta^{ij} \xi^k \frac{\partial^2 \phi}{\partial x^j \partial x^k}. \quad (11)$$

Combining the above Eqs. (10) and (11), we obtain

$$-\frac{d^2 \xi^i}{dt^2} = \eta^{ij} \xi^k \frac{\partial^2 \phi}{\partial x^j \partial x^k} \equiv \xi^k H_k^i. \quad (12)$$

In Newton’s theory of gravity, the vacuum field equation is given as $H_i^i = \nabla^2 \phi = 0$. It means that the tensor H_k^i is traceless in Newton’s vacuum.

In general relativity, the geodesic deviation gives a similar equation

$$-\frac{D^2\xi^\mu}{d\tau^2}=\xi^\nu\tilde{R}^\mu{}_\nu, \quad (13)$$

where

$$\tilde{R}^\mu{}_\nu=\tilde{R}^\mu{}_{\lambda\nu\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}.$$

Here, $\tilde{R}^\mu{}_{\lambda\nu\rho}$ is Riemannian curvature tensor, D denotes the covariant derivative along the curve $x^\mu(\tau)$. The vacuum field equation in general relativity gives $\tilde{R}^\lambda{}_{\lambda\nu}=0$ [25]. It implies that the tensor $\tilde{R}^\mu{}_\nu$ is also traceless, $\tilde{R}\equiv\tilde{R}^\mu{}_\mu=0$.

In Finsler spacetime, the geodesic deviation gives [13]

$$-\frac{D^2\xi^\mu}{d\tau^2}=\xi^\nu R^\mu{}_\nu, \quad (14)$$

where $R^\mu{}_\nu=R^\mu{}_{\lambda\nu\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}$. Here, $R^\mu{}_{\lambda\nu\rho}$ is Finsler curvature tensor defined in (9), D denotes covariant derivative

$$\frac{D\xi^\mu}{D\tau}=\frac{d\xi^\mu}{d\tau}+\xi^\nu\frac{dx^\lambda}{d\tau}\Gamma^\mu{}_{\nu\lambda}\left(x,\frac{dx}{d\tau}\right).$$

Since the vacuum field equations of Newton's gravity and general relativity have similar form, we may assume that the vacuum field equation in Finsler spacetime holds a similar requirement as the case of Newton's gravity and general relativity. It implies that the tensor $R^\mu{}_\nu$ in Finsler geodesic deviation equation should be traceless, $R\equiv R^\mu{}_\mu=0$. In fact, we have proved that the analogy from the geodesic deviation equation is valid at least in Finsler spacetime of Berwald type [26]. We suppose that this analogy is still valid in general Finsler spacetime.

It should be noticed that H is called the Ricci scalar, which is a geometrical invariant. For a tangent plane $\Pi\subset T_xM$ and a non-zero vector $y\in T_xM$, the flag curvature is defined as

$$K(\Pi,y)\equiv\frac{g_{\lambda\mu}R^\mu{}_{\nu\rho}u^\nu u^\rho}{F^2g_{\rho\theta}u^\rho u^\theta-(g_{\sigma\kappa}y^\sigma y^\kappa)^2}, \quad (15)$$

where $u\in\Pi$. The flag curvature is a geometrical invariant and a generalization of the sectional curvature in Riemannian geometry. It is clear that the Ricci scalar R is the trace of $R^\mu{}_\nu$, which is the predecessor of flag curvature. Therefore, the value of Ricci scalar R is invariant under the coordinate transformation. Furthermore, the predecessor of flag curvature could be written in terms of the geodesic spray coefficient

$$R^\mu{}_\nu=2\frac{\partial G^\mu}{\partial x^\nu}-y^\lambda\frac{\partial^2 G^\mu}{\partial x^\lambda\partial y^\nu}+2G^\lambda\frac{\partial^2 G^\mu}{\partial y^\lambda\partial y^\nu}-\frac{\partial G^\mu}{\partial y^\lambda}\frac{\partial G^\lambda}{\partial y^\nu}. \quad (16)$$

Thus, the Ricci scalar R is insensitive to specific connection form. It only depends on the length element F . The gravitational vacuum field equation $R=0$ is universal in all types of theories of Finsler gravity. Pfeifer

et al. [27] have constructed gravitational dynamics for Finsler spacetimes in terms of an action integral on the unit tangent bundle. Their results also show that the gravitational vacuum field equation in Finsler spacetime is $R=0$.

3 The Newtonian limit in Finsler spacetime

It is well known that the Minkowski spacetime is a trivial solution of Einstein's vacuum field equation. In the Finsler spacetime, the trivial solution of Finslerian vacuum field equation is called locally Minkowski spacetime. A Finsler spacetime is called a locally Minkowski spacetime if there is a local coordinate system (x^μ) , with induced tangent space coordinates y^μ , such that F depends only on y and not on x . Using the formula (16), one knows obviously that locally Minkowski spacetime is a solution of the Finslerian vacuum field equation.

In Ref. [28], we supposed that the metric is close to the locally Minkowski metric $\eta_{\mu\nu}(y)$,

$$g_{\mu\nu}=\eta_{\mu\nu}(y)+h_{\mu\nu}(x,y), \quad |h_{\mu\nu}|\ll 1. \quad (17)$$

To the first order in h , the geodesic spray coefficient is

$$G^\mu=\frac{1}{4}\eta^{\mu\nu}\left(2\frac{\partial h_{\alpha\nu}}{\partial x^\lambda}y^\alpha y^\lambda-\frac{\partial h_{\alpha\beta}}{\partial x^\nu}y^\alpha y^\beta\right). \quad (18)$$

Then we obtain that the gravitational vacuum field equation is of the form

$$\eta^{\mu\nu}\frac{\partial^2 h_{\alpha\beta}}{\partial x^\mu\partial x^\nu}-\frac{1}{2}\frac{\partial\eta^{\mu\nu}}{\partial y^\mu}\frac{\partial^2 h_{\alpha\beta}}{\partial x^\nu\partial x^\lambda}y^\lambda+\frac{\partial\eta^{\mu\nu}}{\partial y^\mu}\frac{\partial^2 h_{\alpha\nu}}{\partial x^\lambda\partial x^\beta}y^\lambda=0, \quad (19)$$

where the lowering and raising of indices are carried out by $\eta_{\mu\nu}$ and its matrix inverse $\eta^{\mu\nu}$. To make contact with Newton's gravity, we consider the gravitational field $h_{\mu\nu}$ is stationary and the particle moving very slowly ($|y^k|\ll 1$). In general relativity, gravitational field h_{00} corresponds to Newton's gravitational potential. Here, we consider that only h_{00} is the non vanishing component of $h_{\mu\nu}$. Then, the non trivial equations of vacuum field Eq. (19) are given as

$$\eta^{ij}\frac{\partial^2 h_{00}}{\partial x^i\partial x^j}-\frac{1}{2}\frac{\partial\eta^{ij}}{\partial y^i}\frac{\partial^2 h_{00}}{\partial x^j\partial x^k}y^k=0, \quad (20)$$

$$\frac{\partial\eta^{\mu 0}}{\partial y^\mu}\frac{\partial^2 h_{00}}{\partial x^j\partial x^k}y^k=0. \quad (21)$$

In this paper, we consider the deviation from Minkowski spacetime to be very small, and the quantity

$$\frac{\partial\eta^{\mu\nu}}{\partial y^\mu}=-2\bar{A}^\mu{}_\nu/\sqrt{\eta_{\alpha\beta}y^\alpha y^\beta}$$

relates to the Cartan tensor of Finsler metric $\eta_{\mu\nu}$ which describes the deviation from Minkowski spacetime. We will show that the magnitude of deviation is several

times larger than the Newtonian gravitational potential. Therefore, the second term of Eqs. (20) and (21) just provides high order corrections. If we ignore it, then the vacuum field Eq. (19) could be simplified as

$$\eta^{ij} \frac{\partial^2 h_{00}}{\partial x^i \partial x^j} = 0. \quad (22)$$

The solution of (22) is given as

$$h_{00} = \frac{C}{R} \eta_{00}, \quad (R \neq 0), \quad (23)$$

where C is a constant and $R^2 = -\eta_{ij} x^i x^j$. The gravitational theory should reduce to Newton's gravity, if the Finsler metric $g_{\mu\nu}$ reduces to Riemann metric. Thus, for a given gravitational source with mass at $R=0$, the gravitational field equation is supposed to take the form

$$\eta^{ij} \frac{\partial^2 h_{00}}{\partial x^i \partial x^j} = -8\pi_F G \rho \eta_{00}, \quad (24)$$

where ρ is the energy density of gravitational source. In Finsler spacetime, the space volume of $\eta_{\mu\nu}(y)$ [13] is different from the one in Euclidean space. We used π_F in (24) to represent the difference, where

$$\pi_F \equiv \frac{3}{4} \int_{R=1} \sqrt{g} dx^1 \wedge dx^2 \wedge dx^3 \quad (25)$$

and g is the determinant of η_{ij} . The solution of (24) is given as

$$h_0^0 = \frac{2GM}{R}. \quad (26)$$

Here, we have used h_0^0 to denote the gravitational field instead of h_{00} . It is due to the fact that $\eta_{00} \neq 1$ and we want to obtain the formula (26) which is insensitive to the spacetime index. The signature of Minkowski spacetime is set to be $(+, -, -, -)$.

In the approximation of Newton's limit, by making use of the formula of geodesic spray coefficient (18), the geodesic Eq. (4) reduces to

$$\frac{d^2 x^0}{d\tau^2} - \frac{\eta^{0i}}{2} \frac{\partial h_{00}}{\partial x^i} \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} + \eta^{00} \frac{\partial h_{00}}{\partial x^j} \frac{dx^j}{d\tau} \frac{dx^0}{d\tau} = 0, \quad (27)$$

$$\frac{d^2 x^i}{d\tau^2} - \frac{\eta^{ij}}{2} \frac{\partial h_{00}}{\partial x^j} \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} = 0. \quad (28)$$

The Eq. (27) implies that $\frac{dx^0}{d\tau}$ is a function of h_{00} . Since η^{0i} is a high order quantity compared to η^{00} and η^{ij} , $\frac{dx^0}{d\tau}$ could be treated as a constant in Eq. (28). Then we find from (28) that

$$\frac{d^2 x^i}{dx^{02}} = -\frac{GM}{R^2} \frac{x^i}{R}, \quad (29)$$

where $dx^{02} = \eta_{00} dx^0 dx^0$. The formula (29) means that the law of gravity in Finsler spacetime is similar to Newton's gravity. The difference is that space length is Finslerian. It is what we expect from Finslerian gravity,

because the length difference is the major attribute of Finsler geometry.

It may be expected that the Finsler gravity (29) domains in large scale. The empirical Tully-Fisher relation [8] between galaxy luminosity and rotational speed implies that the rotational speed of the galaxy has a limitation. It hints that we may consider the space length of galaxies to be Finslerian

$$\eta_{ij} = -\delta_{ij} \left(1 - \left(GM a_0 y^{04} / (\delta_{mn} y^m y^n)^2 \right)^2 \right), \quad (30)$$

where a_0 is the constant of MOND. In Finsler spacetime, the speed of a particle is given as $v^i \equiv \frac{dx^i}{dx^0} = \frac{y^i}{y^0}$.

The space length of galaxies (30) could be written into a concise form

$$R = \sqrt{-\eta_{ij} x^i x^j} = r \sqrt{1 - \left(\frac{GM a_0}{v^4} \right)^2}, \quad (31)$$

where $r^2 \equiv \delta_{ij} x^i x^j$ and $v^2 \equiv \delta_{ij} v^i v^j$. By making use of formula (29), we obtain the approximate dynamical equation in the galaxy system. It is given as

$$\frac{GM}{R^2} = \frac{v^2}{R}. \quad (32)$$

Substituting formula (31) into Eq. (32), we obtain that

$$\frac{GM}{r \sqrt{1 - (GM a_0 / v^4)^2}} = v^2. \quad (33)$$

One could find from (33) that the rotational speed of galaxies has an asymptotic value

$$\lim_{r \rightarrow \infty} v^4 = GM a_0. \quad (34)$$

The asymptotic speed stems from the Tully-Fisher relation. The formula (33) could be written into a familiar form

$$\frac{GM}{r^2} = \frac{v^2}{r} \mu \left(\frac{v^2}{r a_0} \right), \quad (35)$$

where $\mu(x) = x / \sqrt{x^2 + 1}$ is an interpolating function of MOND. It indicates that the law of gravity (33) in Finsler spacetime is MOND and the spacetime structure of MOND is "Tully-Fisher" like (31). It should be noticed that the formula (33) is given in natural units ($c=1$). The term $L_g \equiv GM/c^2$ corresponds to the typical galaxy scale where the gravity of the galaxy is dominated. The term $L_0 \equiv c^2/a_0 \approx 2\pi L_H \approx 10^{29}$ cm [29], where L_H is the Hubble radius. Thus, the constant term $GM a_0$ in (33) equals the ratio of galaxy scale L_g to cosmological scale L_0 . The law of gravity (33) in Finsler spacetime hints that there is a connection between MOND and cosmology.

4 Conclusions

In this paper, we presented a Finsler geometry origin of MOND. We showed that the spacetime structure of galaxies may be Finslerian (31). The “Tully-Fisher” like length (31) could be the reason for the empirical Tully-Fisher relation. The law of gravity in galaxies was shown as formula (33). It hints that there is a connection between MOND and cosmology.

The strong and weak gravitational lensing observations of Bullet Cluster 1E0657-558 [30] could not be explained well by MOND and Bekenstein’s relativistic version of MOND [31]. The surface density Σ -map and the convergence κ -map of Bullet Cluster 1E0657-558 show that the center of baryonic matter separates from the center of gravitational force, and the distribution of grav-

itational force does not possess spherical symmetry. One should notice that Finsler metric (30) satisfies

$$\eta_{ij}(-y) = \eta_{ij}(y). \quad (36)$$

Most of the galaxies possess spherical symmetry, so the length element should satisfy (36). There is a class of Finsler spacetime that does not satisfy (36). For example, the Randers spacetime [32]

$$F(x, y) = \sqrt{a_{\mu\nu}y^\mu y^\nu} + b_\mu y^\mu. \quad (37)$$

The space structure of the Bullet Cluster may be one of this class, and its Finslerian gravitational behavior may account for the observations of the Bullet Cluster.

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