Exact solutions of the Spinless-Salpeter equation under Kink-Like potential

H. Hassanabadi B. H. Yazarloo[1)]

Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran

Abstract: The exact solution of Spinless-Salpeter equation (SSE) in the presence of Kink-Like potential is investigated. By using the basic concepts of the supersymmetric quantum mechanics (SUSYQM) formalism and the functional analysis method, we have obtained the bound state solutions in the closed form and the eigenfunctions of the system are reported in the term of hypergeometric function. We have also reported some numerical results.

Key words: Spinless-Salpeter equation, SUSYQM Method, functional analysis method, Kink-Like potential

PACS: 03.65.Ca, 03.65.Pm, 03.65.Nk **DOI:** 10.1088/1674-1137/37/12/123101

1 Introduction

The two-body SSE, which stems from the Bethe-Salpeter equation (BTE) [1–3] by some simplifications and ignoring the spin degrees of freedom, could be considered as the generalization of the non-relativistic Schrödinger equation to the relativistic regime and is therefore of great importance due to its semi-relativistic nature and two-body formulation which finds worthwhile applications in particle and nuclear physics.

Contrary to the common wave equations of quantum mechanics such as Dirac, Klein-Gordon and Schrödinger equations, the semi-relativistic Salpeter equations have been studied only by a few authors. This is definitely because of the mathematical complexity we face due to the nonlocal nature of the equation. Schoberl, Hall and Lucha are amongst the theoretical physicists who have investigated different approximate schemes to analyze the equation [4–6]. Many analytical techniques of quantum mechanics have been applied to the equation to overcome the nonlocal nature of the equation such as SUSYQM method, Nikiforov-Uvarov (NU) technique, ansatz method, proper quantization rules and so on [7–12].

In this work, we have tried to obtain the energy spectra and the eigenfunctions of the SSE under the Kink-Like potential [13]. The exact solutions of the Klein–Gordon equation with position-dependent mass for mixed vector and scalar Kink-Like potentials are given in Ref. [14]. The PT-symmetric version of the Kink-Like potential has also been investigated within the framework of the Dirac equation with a vector potential coupling [15]. We first review the two-body SSE. Then, by considering SUSYQM we solve the SSE under Kink-Like potential and also by introducing some transformations; we bring the problem into a form which can be solved by functional analysis method to obtain the wave function of the system.

2 The two-body-Hamiltonian

The SSE for two particles interacting in a spherically symmetric potential in the center of mass system appears as [16, 17]

$$\left[\sum_{i=1,2}\left(\sqrt{-\Delta+m_i^2}-m_i\right)+V(r)-E_{n,l}\right]\chi(\vec{r})=0,\ \Delta=\nabla^2, \tag{1}$$

where $\chi(\vec{r})=R_{nl}(r)Y_{lm}(\theta,\phi)$. In the case of heavy interacting particles, we can write

$$\begin{aligned}&\sum_{i=1,2}\sqrt{-\Delta+m_i^2}\\&=\sqrt{-\Delta+m_1^2}+\sqrt{-\Delta+m_2^2}\\&=m_1\left(1-\frac{\Delta}{m_1^2}\right)^{\frac{1}{2}}+m_2\left(1-\frac{\Delta}{m_2^2}\right)^{\frac{1}{2}}\\&=m_1\left(1-\frac{1}{2}\frac{\Delta}{m_1^2}-\frac{1}{8}\frac{\Delta^2}{m_1^4}-\cdots\right)+m_2\left(1-\frac{1}{2}\frac{\Delta}{m_2^2}-\frac{1}{8}\frac{\Delta^2}{m_2^4}-\cdots\right)\\&=m_1+m_2-\frac{\Delta}{2}\left(\frac{m_1+m_2}{m_1m_2}\right)-\frac{\Delta^2}{8}\left(\frac{m_1^3+m_2^3}{m_1^3m_2^3}\right)-\cdots,\end{aligned} \tag{2}$$

Received 17 January 2013, Revised 9 May 2013

1) Corresponding author's. E-mail: hoda.yazarloo@gmail.com

where

$$\left(\frac{m_1^3+m_2^3}{m_1^3m_2^3}\right) = \left(\frac{m_1^3+m_2^3}{(m_1+m_2)^3}\right)\left(\frac{(m_1+m_2)^3}{m_1^3m_2^3}\right)$$

$$= \frac{1}{\mu^3}\frac{(m_1+m_2)^3-3m_1^2m_2-3m_2^2m_1}{(m_1+m_2)^3}$$

$$= \frac{1}{\mu^3}\frac{(m_1+m_2)^3-3m_1m_2(m_1+m_2)}{(m_1+m_2)^3},$$

$$= \frac{1}{\mu^3}\frac{\frac{(m_1m_2)^2}{\mu^2}-3m_1m_2}{\frac{(m_1m_2)^2}{\mu^2}}=\frac{1}{\mu^3}\frac{m_1m_2-3\mu^2}{m_1m_2}$$

$$= \frac{1}{\eta^3}, \tag{3}$$

therefore,

$$\sum_{i=1,2}\sqrt{-\Delta+m_i^2}=m_1+m_2-\frac{\Delta}{2\mu}-\frac{\Delta^2}{8\eta^3}-\cdots. \tag{4}$$

With

$$\mu=\frac{m_1m_2}{m_1+m_2},\quad \eta=\mu\left(\frac{m_1m_2}{m_1m_2-3\mu^2}\right)^{1/3}. \tag{5}$$

From Eqs. (1) to (5) in units where ($\hbar=c=1$) we have

$$\left[-\frac{\Delta}{2\mu}-\frac{\Delta^2}{8\eta^3}+V(r)\right]R_{n,l}(r)=E_{n,l}R_{n,l}(r). \tag{6}$$

From the well-known relations

$$\Delta^2=p^4=4\mu^2(E_{n,l}-V(r))^2$$

and

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{L^2}{r^2},\quad \text{with}\quad L^2=l(l+1) \tag{7}$$

Eq. (6) appears as

$$\left[\begin{array}{l}-\frac{1}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial_r}-\frac{l(l+1)}{r^2}\right)\\ -\frac{\mu^2}{2\eta^3}(V(r)-E_{n,l})^2+(V(r)-E_{n,l})\end{array}\right]R_{n,l}(r)=0. \tag{8}$$

Applying the well-known transformation

$$R_{n,l}(r)=\frac{\psi_{n,l}(r)}{r}, \tag{9}$$

gives

$$\left[\frac{-1}{2\mu}\frac{d^2}{dr^2}+\frac{l(l+1)}{2\mu r^2}+W_{nl}(r)-\frac{W_{nl}^2(r)}{2\tilde{m}}\right]\psi_{nl}(r)=0, \tag{10}$$

where

$$W_{nl}(r) = V(r)-E_{nl}, \tag{11}$$

$$\tilde{m} = \eta^3/\mu^2=(m_1m_2\mu)/(m_1m_2-3\mu^2). \tag{12}$$

Here, we consider the Kink-Like potential [14]

$$V(r)=\alpha\beta\tanh(\alpha r), \tag{13}$$

where the parameter, α and the coupling constants, β are real numbers. Substitution of Kink-Like potential in Eq. (10) for $l=0$ gives

$$\left[-\frac{d^2}{dr^2}+V_{\text{eff}}(r)\right]\psi_n(r)=\tilde{E}_n\psi_n(r), \tag{14}$$

where

$$V_{\text{eff}}(r) = -V_1\text{sech}^2(\alpha r)+V_2\tanh(\alpha r), \tag{15a}$$

$$\tilde{E}_n = \frac{\mu\alpha^2\beta^2}{\tilde{m}}+\frac{\mu E_n^2}{\tilde{m}}+2\mu E_n, \tag{15b}$$

and

$$V_1=-\frac{\mu\alpha^2\beta^2}{\tilde{m}},\quad V_2=2\mu\alpha\beta+\frac{2\mu E_n\alpha\beta}{\tilde{m}}. \tag{16}$$

In order to obtain the solution of Eq. (14), we use the powerful SUSY method [18, 19]. Writing the ground state wave function $\psi_0(r)$ in the form of $\psi_0(r)=\exp\left(-\int W(r)dr\right)$, substituting it into Eq. (14), we arrive at the following non-linear Riccati equation for $W(r)$ as

$$W^2(r)-\frac{dW(r)}{dr}=-V_1\text{sech}^2(\alpha r)+V_2\tanh(\alpha r)-\tilde{E}_0, \tag{17}$$

where $\tilde{E}_0$ is the effective ground state energy, and $W(r)$ is a superpotential. By choosing $W(r)$ as follows

$$W(r)=A+B\tanh(\alpha r), \tag{18}$$

and substituting it into $\psi_0(r)=\exp\left(-\int W(r)dr\right)$, one can find,

$$\psi_0(r)=\exp(-Ar)(\cosh(\alpha r))^{-\frac{B}{\alpha}}. \tag{19}$$

Substitution of $W(r)$ in Eq. (17) and comparing equal powers we arrive at

$$B = \frac{\alpha}{2}\left(-1+\sqrt{1+\frac{4V_1}{\alpha^2}}\right), \tag{20a}$$

$$A = \frac{V_2}{2B}, \tag{20b}$$

$$\tilde{E}_0 = -\left(\frac{V_2^2}{4B^2}+B^2\right). \tag{20c}$$

Therefore, our partner potentials are

$$\begin{aligned} V_{\text{eff}+}(r) &= W(r)^2+\frac{\mathrm{d}W(r)}{\mathrm{d}r} \\ &= -(B^2-\alpha B)\mathrm{sec}h^2(\alpha r)+V_2\tanh(\alpha r) \\ &\quad +\left(\frac{V_2^2}{4B^2}+B^2\right), \end{aligned} \tag{21a}$$

$$\begin{aligned} V_{\text{eff}-}(r) &= W(r)^2-\frac{\mathrm{d}W(r)}{\mathrm{d}r} \\ &= -(B^2+\alpha B)\mathrm{sec}h^2(\alpha r)+V_2\tanh(\alpha r) \\ &\quad +\left(\frac{V_2^2}{4B^2}+B^2\right). \end{aligned} \tag{21b}$$

So, we have the following relationship between two supersymmetric partner potentials

$$V_{\text{eff}+}(r,a_0)=V_{\text{eff}-}(r,a_1)+R(a_1), \tag{22}$$

which are shape invariant via a mapping of the $a_0 \to a_0-\alpha$, $a_0=B$ and $R(a_1)$ does not depend on r, $R(a_1)=\frac{V_2^2}{4a_0^2}+a_0^2$.

By using the shape invariant approach, the exact energy spectra is given by

$$\begin{aligned} \tilde{E}_0^{(-)} &= 0, \\ \tilde{E}_n^{(-)} &= \sum_{k=1}^{n} R(a_k)=\left(\frac{V_2^2}{4a_0^2}+a_0^2\right)-\left(\frac{V_2^2}{4a_n^2}+a_n^2\right). \end{aligned} \tag{23}$$

The effective energy $\tilde{E}_n$ in Eq. (14) can be written as

$$\tilde{E}_n=\tilde{E}_n^{(-)}+\tilde{E}_0=-\left(\frac{V_2^2}{4a_n^2}+a_n^2\right), \tag{24}$$

where n=0, 1, 2$\cdots$ and

$$a_n=a_0-n\alpha,\ a_0=B. \tag{25}$$

From equations (15b) and (20a) one can obtain

$$\left(\frac{\mu}{\tilde{m}}+\frac{4\mu^2\alpha^2\beta^2}{4a_n^2\tilde{m}^2}\right)E_n^2+\left(2\mu+\frac{8\mu^2\alpha^2\beta^2}{4a_n^2\tilde{m}}\right)E_n +\left(a_n^2+\frac{4\mu^2\alpha^2\beta^2}{4a_n^2}+\frac{\mu\alpha^2\beta^2}{\tilde{m}}\right)=0. \tag{26}$$

Finally, the energy spectra can be found as

$$E_n = \frac{1}{2\left(\frac{\mu}{\tilde{m}}+\frac{4\mu^2\alpha^2\beta^2}{4a_n^2\tilde{m}^2}\right)}\times\left(\begin{array}{l} -\left(2\mu+\frac{8\mu^2\alpha^2\beta^2}{4a_n^2\tilde{m}}\right) \\ \pm\left[\begin{array}{l}\left(2\mu+\frac{8\mu^2\alpha^2\beta^2}{4a_n^2\tilde{m}}\right)^2-4\left(\frac{\mu}{\tilde{m}}+\frac{4\mu^2\alpha^2\beta^2}{4a_n^2\tilde{m}^2}\right)\times \\ \left(a_n^2+\frac{4\mu^2\alpha^2\beta^2}{4a_n^2}+\frac{\mu\alpha^2\beta^2}{\tilde{m}}\right)\end{array}\right]^{\frac{1}{2}}\end{array}\right). \tag{27}$$

To obtain the wave function of the system, we start from Eq. (14). A change of variable of the form

$$z=-\tanh(\alpha r), \tag{28}$$

gives

$$\left(\begin{array}{l}(1-z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2}-2z\frac{\mathrm{d}}{\mathrm{d}z}+\frac{V_2}{\alpha^2}\frac{z}{(1-z^2)} \\ +\frac{\tilde{E}_n}{\alpha^2}\frac{1}{(1-z^2)}+\frac{V_1}{\alpha^2}\end{array}\right)\psi_n(z)=0. \tag{29}$$

We introduce $z=1-2s$, so the above equation changes into

$$\left\{\begin{array}{l}s(1-s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}+(1-2s)\frac{\mathrm{d}}{\mathrm{d}s}+\frac{V_2}{4\alpha^2}\frac{(1-2s)}{s(1-s)} \\ +\frac{\tilde{E}_n}{4\alpha^2}\frac{1}{s(1-s)}+\frac{V_1}{\alpha^2}\end{array}\right\}\psi_n(s)=0. \tag{30}$$

Here, we consider $\psi_n(s)$ as below

$$\psi_n(s)=s^{\upsilon}(1-s)^{\gamma}f_n(s). \tag{31}$$

Substituting Eq. (31) in Eq. (30), we arrive at

$$\left\{\begin{array}{l}s(1-s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}+(c'-(1+a'+b')s)\frac{\mathrm{d}}{\mathrm{d}s} \\ -\left(\upsilon(\upsilon+1)+\gamma(\gamma+1)+2\gamma\upsilon-\frac{V_1}{\alpha^2}\right)\end{array}\right\}f_n(s)=0, \tag{32}$$

where

$$\begin{aligned} a' &= \frac{1}{2}\left[(1+2\upsilon)+2\gamma+\sqrt{1+\frac{4V_1}{\alpha^2}}\right], \\ b' &= \frac{1}{2}\left[(1+2\upsilon)+2\gamma-\sqrt{1+\frac{4V_1}{\alpha^2}}\right], \\ c' &= 1+2\upsilon. \end{aligned} \tag{33}$$

With

$$\upsilon^2 = -\left(\frac{V_2+\tilde{E}_n}{4\alpha^2}\right), \tag{34a}$$

$$\gamma^2 = \frac{V_2-\tilde{E}_n}{4\alpha^2}. \tag{34b}$$

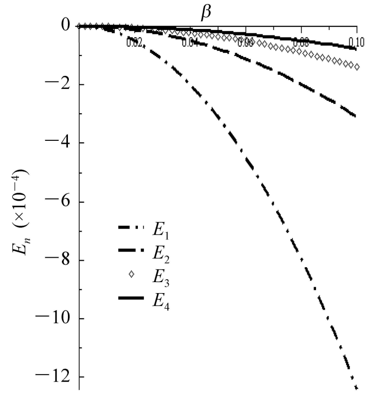

Fig. 1. Energy of the system for different states versus β for $m_1=m_2=1/2,\alpha=0.0001$.

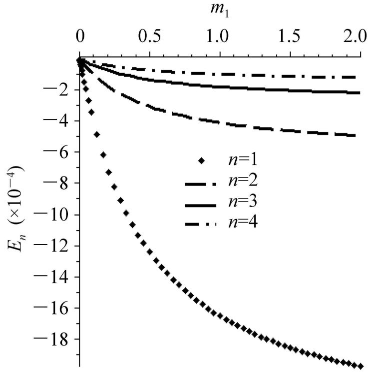

Fig. 2. Energy of the system for different states versus m_1 for $m_2=1/2$, $\alpha=0.0001$, $\beta=0.1$.

Equation (32) is just a hypergeometric equation, and its solution is the hypergeometric function

$$f_n(s)={}_2F_1(a',b',c';s). \tag{35}$$

So we have

$$\psi_n(z)=\left(\frac{1-z}{2}\right)^{\upsilon}\left(\frac{z+1}{2}\right)^{\gamma}{}_2F_1\left(a',b',c';\frac{1-z}{2}\right), \tag{36}$$

or equivalently

$$\psi_n(r) = \left(\frac{1+\tanh(\alpha r)}{2}\right)^{\upsilon}\left(\frac{1-\tanh(\alpha r)}{2}\right)^{\gamma} \times{}_2F_1\left(a',b',c';\frac{1+\tanh(\alpha r)}{2}\right). \tag{37}$$

In Table 1, we have reported numerical results for different states.

The behavior of the energy versus β, m_1 and m_2 is plotted in Figs. 1–3.

Table 1. The energy of the system for different states and $m_1=2$, $m_2=1/2$, $\alpha=0.01$, $\beta=0.01$.

$\lvert n,l=0\rangle$	E_n	$\lvert n,l=0\rangle$	E_n
$\lvert 1,0\rangle$	−1.53831651	$\lvert 6,0\rangle$	−1.53394766
$\lvert 2,0\rangle$	−1.53795634	$\lvert 7,0\rangle$	−1.53231145
$\lvert 3,0\rangle$	−1.53733345	$\lvert 8,0\rangle$	−1.53041908
$\lvert 4,0\rangle$	−1.53645762	$\lvert 9,0\rangle$	−1.52826864
$\lvert 5,0\rangle$	−1.53532929	$\lvert 10,0\rangle$	−1.52585795

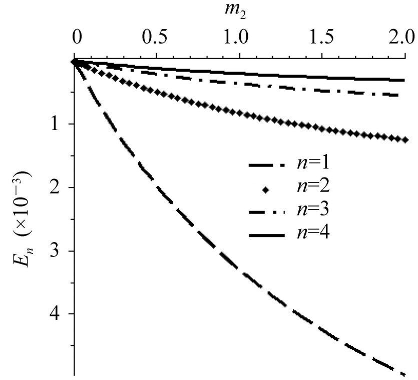

Fig. 3. Energy of the system for different states versus m_2 for $m_1=2$, $\alpha=0.0001$, $\beta=0.1$.

3 Conclusion

In this work, we have investigated the SSE for the Kink-Like potential in the case of $l=0$. We have seen that the s-wave SSE for the Kink-Like potential can be solved exactly. The relativistic bound-state energy spectrum, by using SUSYQM method and the corresponding eigenfunctions in terms of hypergeometric function are obtained. Due to the semi-relativistic nature of the equation, its two-body formulation and the choice of Kink-Like potential, results do find applications in nuclear and particle physics.

It is a great pleasure for authors to thank the kind referee for his many useful comments on the manuscript.

References

1 Salpeter E E et al. Phys. Rev., 1951, **84**; 1232
2 Wick G C. Phys. Rev., 1954, **96**: 1124
3 Chang L et al. Rev. Lett., 2009, **103**: 081601
4 Lucha W et al. Int. J. Mod. Phys. A, 2002, **17**: 2233
5 Hall R et al. J. Phys. A: Math. Gen., 2005, **38**: 7997
6 Lucha W et al. Phys. Rev. A, 1996, **54**: 3790
7 WEI G F et al. Can. J. Phys., 2011, **89**: 1225
8 Hassanabadi H et al. Commun. Theor. Phys., 2012, **57:** 339
9 DONG S H et al. Phys. Lett. A, 2003, **314**: 261
10 DONG S H et al. J. Math. Chem., 2012, **50**: 881
11 LU L L et al. Few-Body Syst., 2012, **53**: 573
12 Hassanabadi S et al. Advances in High Energy Physics, DOI: 10.1155/2012/804652
13 Castro A S De et al. Phys. Lett. A, 2006, **351**: 379
14 JIA C S et al. Few-Body Syst., 2012, **52**: 11
15 JIA C S et al. Int. J. Theor. Phys., 2008, **47**: 664
16 Zarrinkamar S et al. Few-Body Syst., DOI: 10.1007/s00601-011-0272-3
17 Zarrinkamar S et al. Phys. Scr., 2011, **84:** 065008
18 QIANG W C et al. Phys. Scr., 2005, **72**: 127
19 DONG S H. Wave Equations in Higher Dimensions, Springer, 2011