Ground states and excitation spectra of baryons in a non-relativistic model with the anharmonic potential

N. Salehi[1;1)] H. Hassanabadi[1] A. A. Rajabi[2]

[1] Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran
[2] Physics Department, Shahrood University of Technology, P. O. Box 3619995161-316, Shahrood, Iran

Abstract: In this work the mass spectra for some of the baryon resonances of the particle data group with three and four star status are obtained, and a unified description of the ground states and excitation spectra of baryons are provided in the framework of a non-relativistic potential model. For this goal we have analytically solved the radial Schrödinger equation for three identical interacting particles with the anharmonic potential by using the Ansatz method and then we have calculated the baryon resonances spectrum by using the Gürsey Radicati mass formula (GR) and with generalized Gürsey Radicati mass formula (GGR). The results of our model show that the calculated masses of baryon resonances by using the generalized Gürsey Radicati mass formula are found to be in good agreement with the tabulations of the Particle Data Group. The overall good description of the spectrum which we obtain shows that our model can also be used to give a fair description of the energies of the excited multiples up to 3 GeV mass and negative-parity resonance. Moreover, we have shown that our model reproduces the position of the Roper resonance of the nucleon.

Key words: baryons resonances, spectrum, anharmonic potential, non-relativistic quark models

PACS: 12.39.Pn, 14.20.Gk, 21.60.Fw **DOI:** 10.1088/1674-1137/37/11/113101

1 Introduction

Several papers based on the non-relativistic quark models have appeared in the literature [1–7] in connection with the study of the mass spectra of light and strange baryons. The baryon spectrum is usually described well, although the various models are quite different. The three Quark interaction can be divided in two parts: the first one, containing the confinement interaction, is spin and flavor independent and it is therefore $SU(6)$ invariant, while the second violates the $SU(6)$ symmetry [8–10]. One of the most popular ways to violate the $SU(6)$ invariance was the introduction of a hyperfine interaction [11, 12], however in many studies a spin and isospin [1, 13, 14] or a spin and flavor dependent interaction [1, 13] has been considered. It is well known that the Gürsey Radicati mass formula [15] describes quite well the way $SU(6)$ symmetry is broken, at least in the lower part of the baryon spectrum. In this paper we applied the generalized Gürsey Radicati (GR) mass formula which is presented by Giannini et al. [16] to obtain the best description of the baryons spectrum. The model we used is a simple Constituent Quark Model where the $SU(6)$ invariant part of the Hamiltonian is the same as in the hypercentral Constituent Quark Model (hCQM) [17, 18] and where the $SU(6)$ symmetry is broken by a generalized GR mass formula. This paper is organized as follows. In Section 2 we review the hypercentral coordinates and introduce the interaction potentials between three quarks in baryons and then we present the analytical solution of the radial Schrödinger equation with the anharmonic potential by using the Ansatz method. In the Ansatz approach, which is followed here, we introduce a solution consistent with the requirements of quantum mechanics and thereby the differential equation under study is solved [19, 20]. In the third section in order to describe the splitting within the $SU(3)$ and $SU(6)$ multiples we introduce the Gell-Mann-Okubo (GMO) mass formula [21], the Gürsey Radicati mass formula and the generalized GR mass formula in the hCQM. Then we give the results obtained by fitting the Gürsey Radicati mass formula and generalized GR mass formula parameters to the baryons energies and we compare the spectra with the experimental data. Finally, we give a summary and a conclusion in Section 4.

Received 14 January 2013, Revised 9 May 2013

1) E-mail: salehi@shahroodut.ac.ir

2 Exact analytical solution of the Schrödinger equation with the anharmonic potential

We consider baryons as bound states of three quarks, the Hamiltonian of this system is:

$$H=\sum_{\substack{i,j=1\\ i<j}}^{3}\left\{\frac{P_i^2}{2m_i}+V(\vec{r}_{ij})\right\}. \tag{1}$$

By introducing the Jacobi vectors as

$$\vec{R}=\frac{\vec{r_1}+\vec{r_2}+\vec{r_3}}{3},\ \ \vec{\rho}=\frac{\vec{r_1}-\vec{r_2}}{\sqrt{2}},\ \ \vec{\lambda}=\frac{\vec{r_1}+\vec{r_2}-2\vec{r_3}}{\sqrt{6}}. \tag{2}$$

After removing the center of mass coordinate R, the Hamiltonian will be

$$H=\frac{P_\rho^2}{2m}+\frac{P_\lambda^2}{2m}+V(\rho,\lambda). \tag{3}$$

In order to describe the three-quark dynamics, it is convenient to introduce the hypercentral coordinates, which are obtained by substituting the absolute values of the Jacobi coordinates ρ and λ, by:

$$r=\sqrt{\rho^2+\lambda^2},\ \ \xi=\arctan\left(\frac{\rho}{\lambda}\right), \tag{4}$$

where r is the hyper-radius and ξ is the hyper-angle. The quark dynamics has a dominant $SU(6)$ invariant part, which accounts for the average multiple energies; in the hypercentral Constituent Quark Model it is assumed to be given by the hypercentral potential.

The exact solutions to the fundamental dynamical equations play crucial roles in physics. It is well-known that the exact solutions to the Schrödinger equation are possible only for several potentials and some approximation methods are frequently applied to arrive at the solutions. On the other hand, the higher order anharmonic potentials have attracted much more attention from physicists and mathematicians [22–24]. Interest in anharmonic oscillator-like interactions stems from the fact that, in many cases, the study of the relevant Schrödinger equation, for example in atomic and molecular physics, provides us with insight into the physical problem in question. In our model the interaction potential is assumed as follows:

$$V(r)=a_1r^2+b_1r^{-4}+c_1r^{-6}, \tag{5}$$

where the parameters a_1, b_1, and c_1 in the potential satisfy some constraints. This potential has been investigated recently for the one-particle problem. We mention that a simplified version of a more general form of the Ansatz for the eigenfunction, with the anharmonic potential like Eq. (5), offers a short-cut and simpler method and also provides an exact closed form solution to the Schrödinger equation for both the ground and the excited states. The quark potential $V(r)$ is in general a three-body potential, since the hyper-radius depends on the coordinates of all three quarks. First we have solved the Schrödinger equation exactly and find eigenvalue and eigenfunction of the potential, then by using the GR mass formula and the generalized GR mass formula we can try to find the baryons spectrum.

For hypercentral potential, the Schrödinger equation, in the hyperspherical coordinates, is simply reduced to a single hyper-radial equation (Eq. (6)), while the angular and hyperangular parts of the 3q-states are the known hyperspherical harmonics [24].

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\left(\frac{5}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\gamma(\gamma+4)}{r^2}\right)\psi_{\nu\gamma}(r)$$
$$=(-2m)[E-V(r)]\psi_{\nu\gamma}(r), \tag{6}$$

where m is the reduced mass [25–27], γ is the grand angular quantum number (given by $\gamma=2n+l_\rho+l_\lambda$, $n=0,1,\cdots;l_\rho$ and l_λ are the angular momenta associated with the $\vec{\rho}$ and $\vec{\lambda}$ variables) and ν denotes the number of nodes of the space three quark wave functions. The transformation $\psi_{\nu\gamma}(r)=r^{\frac{-(D-1)}{2}}R_{\nu\gamma}(r)=r^{\frac{-5}{2}}R_{\nu\gamma}(r)$ (D represents the dimension) reduces Eq. (6) to the following form:

$$-\frac{\mathrm{d}^2}{\mathrm{d}r^2}R_{\nu\gamma}(r)+\left[\frac{(2\gamma-3)(2\gamma+5)}{4r^2}\right]R_{\nu\gamma}(r)$$
$$+\left(ar^2+\frac{b}{r^4}+\frac{c}{r^6}\right)R_{\nu\gamma}(r)=2mER_{\nu\gamma}(r). \tag{7}$$

Let

$$a=2ma_1,\ \ b=2mb_1\ \text{ and }\ c=2mc_1. \tag{8}$$

Let us assume

$$\frac{(2\gamma-3)(2\gamma+5)}{4}=\eta(\eta+1)\rightarrow\eta=\gamma+\frac{3}{2},\ \ \eta=-\gamma-\frac{5}{2}. \tag{9}$$

The form of Eq. (7) is symmetric with respect to the coordinate inversion $x\rightarrow\frac{1}{x}$ and the elementary bound state will be given of course with grand angular momentum. Then from Eqs. (5) and (7) the reduced hyper-radial Schrödinger equation for three identical particles in six dimensions, Eq. (7), can be written as:

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}R_{\nu\gamma}(r)+\left[2mE-2mV(r)-\frac{\eta(\eta+1)}{r^2}\right]R_{\nu\gamma}(r)=0. \tag{10}$$

For the fixed grand-angular quantum number γ, there are different solutions, which can be labeled by υ, $\upsilon+1$. For the state solution of Eq. (10), we make an Ansatz [28–35] for the radial wave function:

$$R_{\nu\gamma}(r)=\exp[P(r)]\times\text{power series}=\mathrm{e}^{P(r)}K(r). \tag{11}$$

From Eq. (11),

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}R_{\nu\gamma}(r)-\left(P''(r)+P'^2(r)+\frac{K''(r)+2P'(r)K'(r)}{K(r)}\right)$$
$$\times R_{\nu\gamma}(r)=0. \quad (12)$$

Comparing Eqs. (10) and (11) we can write:

$$2mV(r)+\frac{\eta(\eta+1)}{r^2}-2mE$$
$$=P''(r)+P'^2(r)+\frac{K''(r)+2P'(r)K'(r)}{K(r)}. \quad (13)$$

First of all, let us take $K(r)=1$ for ground state and $P(r)$ as:

$$P(r)=\frac{1}{2}\alpha r^2+\frac{1}{2}\beta r^{-2}+\kappa\ln r. \quad (14)$$

If we insert $P(r)$ in Eq. (11), it is clear that $R_{\nu\gamma}(r)$ is not singular. Substituting Eq. (14) into Eq. (13) for the ground state we obtain:

$$2mV(r)+\frac{\eta(\eta+1)}{r^2}-2mE$$
$$=\alpha^2r^2+\alpha(1+2\kappa)+\frac{\kappa(\kappa+1)-2\alpha\beta}{r^2}+\frac{\beta(3+2\kappa)}{r^4}+\frac{\beta^2}{r^6}. \quad (15)$$

On comparing both sides of Eq. (15) we find the following corresponding energy and potential parameter relations:

$$2mE=-(1+2\kappa)\alpha,\ \alpha^2=a. \quad (16a)$$

$$\kappa^2-\kappa-2\alpha\beta=\eta(\eta+1). \quad (16b)$$

$$3\beta-2\beta\kappa=b,\ \beta^2=c. \quad (16c)$$

Equation (16) immediately yields:

$$\alpha=\pm\sqrt{a},\ \beta=\pm\sqrt{c},\ \kappa=\frac{1}{2}\pm\left[\left(\eta+\frac{1}{2}\right)^2+2\sqrt{ac}\right]^{\frac{1}{2}}. \quad (17)$$

From Eq. (9) $\eta=\gamma+\frac{3}{2}$, then by substituting η into Eq. (17) we can obtain the following equation:

$$\kappa=\frac{1}{2}\pm\left[(\gamma+2)^2+2\sqrt{ac}\right]^{\frac{1}{2}}. \quad (18)$$

Further, for physical restrictions we choose the positive sign in κ and negative sign in α and β to retain the well-behaved solution at the origin and at infinity. From Eq. (16) the ground state energy is given by:

$$E_0=\sqrt{a}\left(4+\frac{b}{\sqrt{c}}\right), \quad (19)$$

while equation (16c) leads to a constraint,

$$\left(2\sqrt{c}+b\right)^2=4c\left[(\gamma+2)^2+2\sqrt{ac}\right]. \quad (20)$$

Let us assume:

$$\omega_1=\sqrt{\frac{a_1}{m}},\ \ \omega_2=\sqrt{\frac{b_1}{m}},\ \ \omega_3=\sqrt{\frac{c_1}{m}}. \quad (21)$$

Equations (16a), (18), and (20) provide the eigenvalues as:

$$E_{0\gamma}=\left[\sqrt{2}+\left[2(\gamma+2)^2+8m^2\omega_1\omega_3\right]^{\frac{1}{2}}\right]\omega_1. \quad (22)$$

Finally, by doing some calculations, the total wave function for $\upsilon=0$ is obtained as follows:

$$\psi_{0\gamma}=N_\gamma r^{\left[-2+\left[(\gamma+2)^2+4m^2\omega_1\omega_2\right]^{\frac{1}{2}}\right]}$$
$$\times\exp\left[\frac{-(m\omega_1r^2+m\omega_3r^{-2})}{\sqrt{2}}\right], \quad (23)$$

where the normalization constant for three particles in a 6-dimensional hypersphere is:

$$\int_0^\infty r^5|\psi_{0\gamma}(r)|^2\mathrm{d}r=1. \quad (24)$$

By using the standard integral tables [36], when Re $\beta_1>0$, Re $\beta_2>0$, Re $\nu>0$,

$$\int_0^\infty r^{\nu-1}\exp[-(\beta_1r^2+\beta_2r^{-2})]\mathrm{d}r$$
$$=\left(\frac{\beta_1}{\beta_2}\right)^{\frac{\nu}{4}}K_{\frac{\nu}{2}}\left(2\sqrt{\beta_1+\beta_2}\right). \quad (25)$$

In this problem we have $\beta_1=\sqrt{a}$, $\beta_2=\sqrt{c}$, and $\frac{\nu}{2}=\kappa+3$.

In a similar manner we can continue for other modes, $\nu=1,\ 2,\ 3,\cdots$.

3 The spin- and flavor- dependent *SU*(6) violations in the baryon resonances

The spin and isospin dependent interactions are not the only source of $SU(6)$ violation. In order to study the strange baryon spectrum one has to consider the $SU(3)$ violation produced by the differences in the quark masses. The Gell-Mann-Okubo (GMO) mass formula [21] made use of a λ_8 violation of $SU(3)$ in order to explain the mass splitting within the various $SU(3)$ multiples; according to this formula the mass of a baryon belonging to a given $SU(3)$ multiple can be expressed as:

$$M(\mathrm{GMO})_{\mathrm{Baryon}}=\chi Y+\xi\left[I(I+1)-\frac{1}{4}Y^2\right]+M, \quad (26)$$

where Y is the hypercharge, I is the isospin of the baryon, χ and ξ are parameters to be fitted and M is the average energy value of the $SU(3)$ multiple. A simple way to interpret the origin of such a violation is just to attribute to the strange quark a mass different from the up and down quark ones. The unknown parameters χ and ξ in the $SU(3)$ violating terms can be in principle fitted to the experimental masses, thereby providing a phenomenological way to describe the spectrum. A similar approach for description of the splitting within the $SU(6)$ baryon multiples is supplied by the Gürsey Radicati mass formula [15]:

$$M(\mathrm{GR})_{\mathrm{Baryon}}=\chi Y+\tau S(S+1)+\xi\left[I(I+1)-\frac{1}{4}Y^2\right]+M, \tag{27}$$

where S is the total spin, τ is the parameter to be fitted and M is the average energy value of the $SU(6)$ multiple. Eq. (27) can be rewritten in terms of Casimir operators in the following way:

$$\begin{aligned}M(\mathrm{GR})_{\mathrm{Baryon}} &= \chi C_1[U_{\mathrm{Y}}(1)]+\tau C_2[SU_{\mathrm{S}}(2)]\\ &+\xi\left[C_2[SU_{\mathrm{I}}(2)]-\frac{1}{4}(C_1[U_{\mathrm{Y}}(1)])^2\right]+M,\end{aligned} \tag{28}$$

where $C_2[SU_{\mathrm{S}}(2)]$ and $C_2[SU_{\mathrm{I}}(2)]$ are the $SU(2)$ (quadratic) Casimir operators for spin and isospin, respectively, and $C_1[U_{\mathrm{Y}}(1)]$ is the Casimir for the $U(1)$ subgroup generated by the hypercharge Y. This mass formula has tested to be successful in the description of the ground state baryon masses, however, as stated by the authors themselves, Eq. (28) is not the most general mass formula that can be written on the basis of a broken $SU(6)$ symmetry.

Table 1. Eigenvalues of the $C_2[SU_{\mathrm{SF}}(6)]$ and $C_2[SU_{\mathrm{F}}(3)]$ Casimir operators.

dimension ($SU(6)$)	$C_2[SU_{\mathrm{SF}}(6)]$	dimension ($SU(3)$)	$C_2[SU_{\mathrm{F}}(3)]$
56	$\frac{45}{4}$	8	3
70	$\frac{33}{4}$	10	6
20	$\frac{21}{4}$	1	0

In order to generalize Eq. (28), Giannini et al. considered a dynamical spin-flavor symmetry $SU_{\mathrm{SF}}(6)$ [16] and described the $SU_{\mathrm{SF}}(6)$ symmetry breaking mechanism by generalizing Eq. (28) as:

$$\begin{aligned}M(\mathrm{GGR})_{\mathrm{Baryon}} &= \delta C_2[SU_{\mathrm{SF}}(6)]+\vartheta C_2[SU_{\mathrm{F}}(3)]\\ &+\tau C_2[SU_{\mathrm{S}}(2)]+\chi C_1[U_{\mathrm{Y}}(1)]\\ &+\xi\left[C_2[SU_{\mathrm{I}}(2)]-\frac{1}{4}(C_1[U_{\mathrm{Y}}(1)])^2\right]+M.\end{aligned} \tag{29}$$

The parameters δ and ϑ in the $SU(6)$ violating terms can in principle be fitted to the experimental masses. The first two terms represent the quadratic Casimir operators of the $SU_{\mathrm{SF}}(6)$ spin-flavor and the $SU_{\mathrm{F}}(3)$ flavor groups. For the definition of the Casimir operators in Eq. (29), we have followed the same convention as in Ref. [37]. The eigenvalues of the Casimirs are given by:

$$C_2[SU_{\mathrm{SF}}(n)]=\frac{1}{2}\left[\sum_{i=1}^{n}f_i(f_i+n+1-2i)-\frac{1}{n}\left(\sum_{i=1}^{n}f_i\right)^2\right]. \tag{30}$$

Where f_i denotes the number of boxes in the i-th row of the Young tableau. In Table 1, we give the expectation values of the Casimir operators $SU_{\mathrm{SF}}(6)$ and $SU_{\mathrm{F}}(3)$ for the allowed three-quark configurations.

In many studies of multiquark configurations, effective spin-flavor hyperfine interactions have been used in CQM which schematically represents the Goldstone Boson Exchange (GBE) interaction between constituent quarks [38–41]. An analysis of the strange and non-strange qqq baryon resonances in the collective string-like model [1] and the hypercentral CQM [26] also showed evidence for the need of such type of interaction terms. The generalized Gürsey Radicati mass formula (GGR) Eq. (29) can be used to describe the baryons spectrum, provided that two conditions are fulfilled. The first condition is the feasibility of using the same splitting coefficients for different $SU(6)$ multiples. This seems actually to be the case, as shown by the algebraic approach to the baryon spectrum [1], where a formula similar to Eq. (29) has been applied. The second condition is given by the feasibility of getting reliable values for the unperturbed mass values M [16]. For this goal we regarded the $SU(6)$ invariant part of the hCQM, which provides a good description of the baryons spectrum and used the Gürsey Radicati inspired $SU(6)$ breaking interaction to describe the splitting within each $SU(6)$ multiple. Therefore, the baryons masses are obtained by three quark masses and the eigenenergies ($E_{\nu\gamma}$) of the radial Schrödinger equation with the expectation values of H_{GGR} as follows:

$$\begin{aligned}M_{\mathrm{Baryon}} &= 3m_{\mathrm{q}}+E_{\nu\gamma}+\delta\langle C_2[SU_{\mathrm{SF}}(6)]\rangle+\vartheta\langle C_2[SU_{\mathrm{F}}(3)]\rangle\\ &+\tau\langle C_2[SU_{\mathrm{S}}(2)]\rangle+\chi\langle C_1[U_{\mathrm{Y}}(1)]\rangle\\ &+\xi\left[\langle C_2[SU_{\mathrm{I}}(2)]\rangle-\frac{1}{4}\langle C_1[U_{\mathrm{Y}}(1)]\rangle^2\right].\end{aligned} \tag{31}$$

In order to simplify the solving procedure, the constituent quarks masses are assumed to be the same for all the quark flavors. Therefore, within this approximation, the $SU(3)$ symmetry is only broken dynamically by the spin and flavor dependent terms in the Hamiltonian. In the previous section we determined eigenenergies ($E_{\nu\gamma}$) by exact solution of the Schrödinger radial equation for the hypercentral Potential (Eq. (5)). The expectation values of the Casimir operators are identified in Ref. [42]. In this study we do not consider interaction terms that mix the spatial and internal degrees of freedom. Therefore, the model is expected to be unsuccessful at the description of all those observables where an excellent description of the three quark wave function is crucial. For calculating the baryons mass according to Eq. (31), we need to find the unknown parameters. In the case of the qqq system, the coefficients τ, χ and ξ can be obtained from the mass differences of selected pairs of baryon resonances:

$$N(1650)P11-N(1535)P11 = 3\tau$$

$$\Sigma(1193)P11-\Lambda(1116)P01 = 2\xi \quad (32)$$

$$4N(938)P11-\Sigma(1193)P11-3\Lambda(1116)P01 = 4\chi.$$

leading to the numerical values: $\tau = 38.3$, $\chi = -197.3$ MeV and $\xi = 38.5$ MeV. We determined m_q, ω_1, ω_3 (in Eq. (22)) and the two coefficients (δ, ϑ) of Eq. (31) in a simultaneous fit to the 3 and 4 star resonances of Table 3 which have been assigned as octet and decupled states. The fitted parameters are reported in Table 2, while the resulting spectra are shown in Figs. 1 and 2. The corresponding numerical values are given in Table 3, column $M_{\text{Our Calc (GGR)}}$. In Table 3, column $M_{\text{Our Calc (GR)}}$, we also reported the numerical values of our calculations with Gürsey Radicati mass formula for baryon masses (Eq. (28)). The percentage of relative error for the spectrum of each of the baryon resonances in our model by using generalized Gürsey Radicati mass formula is shown in Table 4. The values reported in Table 4, column 6 indicate that the percentage of relative error for our calculations are between 0 and 8 percent and the maximum amount is about 8.94 percent for Λ*(1520) D01. Comparison between our results for spectrum of baryon resonances by using Gürsey Radicati mass formula (Table 3, column $M_{\text{Our Calc (GR)}}$) and the results for baryon masses based on generalized Gürsey Radicati mass formula (Table 3, column $M_{\text{Our Calc (GGR)}}$) show that using generalized GR mass formula has certainly improved reproduction of the spectrum of strange and nonstrange baryon resonances. Comparison between results in Table 3, columns $M_{\text{Our Calc (GGR)}}$ and M_{Exp} [43] show that our results are very close to the ones obtained in experiments. These improvements in reproduction of baryon resonances masses are obtained by choosing a suitable form for confinement potential, exact analytical solution of the Schrödinger equation for our proposed potential and choosing the generalized Gürsey Radicati mass formula for studying the spin- and flavor-dependent $SU(6)$ violations in the baryon spectrum.

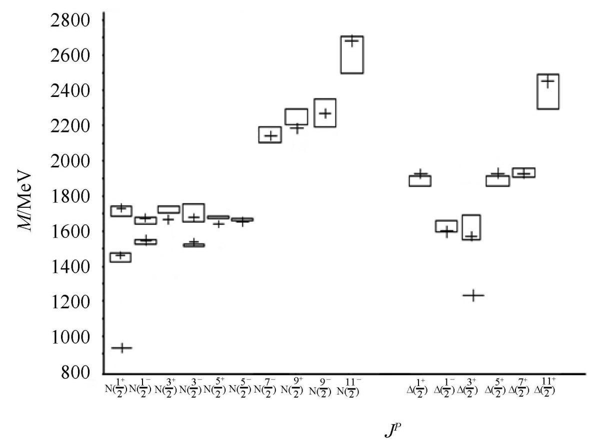

Fig. 1. Comparison between the experimental mass spectrum of three and four star N and Δ resonances [43] (boxes) and our calculated masses (+) which were obtained with the equation (31) fixing the mass relation parameters by a fitting procedure.

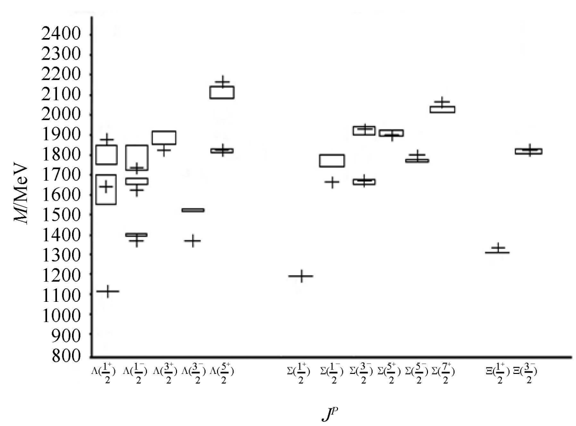

Fig. 2. Comparison between the experimental mass spectrum of three and four star Λ, Σ and Ξ resonances [43] (boxes) and our calculated masses (+) which were obtained with the equation (31) fixing the mass relation parameters by a fitting procedure.

Table 2. The fitted values of the parameters of the Eq. (31), obtained with resonances mass differences and global fit to the experimental resonance masses [43].

parameter	m_q	ω_1	ω_3	δ	ϑ	τ	χ	ξ
value	323 MeV	0.32 fm^{-1}	0.15 fm^{-1}	−18.94 MeV	17.17 MeV	38.3	−197.3 MeV	38.5 MeV

Table 3. Masses of baryon resonances (all values expressed in MeV) calculated with the mass formulas. The column $M_{\mathrm{Ourcalc(GGR)}}$ contains our calculations with the parameters of Table 2 and according to Eq. (31), while the column $M_{\mathrm{OurCalc(GR)}}$ shows our calculations according to Gürsey Radicati mass formula.

baryon	status	$M_{\mathrm{Exp[43]}}$	state	$M_{\mathrm{Our\ Calc(GR)}}$	$M_{\mathrm{Our\ Calc(GGR)}}$
N(938) P11	****	938	$^{2}8_{1/2}[56, 0^{+}]$	1099.5	938
N(1440) P11	****	1420–1470	$^{2}8_{1/2}[56, 0^{+}]$	1628.9	1467.4
N(1520) D13	****	1515–1525	$^{2}8_{3/2}[70, 1^{-}]$	1718.2	1541.5
N(1535) S11	****	1525–1545	$^{2}8_{1/2}[70, 1^{-}]$	1718.2	1541.5
N(1650) S11	****	1645–1670	$^{4}8_{1/2}[70, 1^{-}]$	1833.1	1668.4
N(1675) D15	****	1670–1680	$^{4}8_{5/2}[70, 1^{-}]$	1833.1	1668.4
N(1680) F15	***	1680–1690	$^{2}8_{5/2}[56, 2^{+}]$	1807.6	1660.4
N(1700) D13	***	1650–1750	$^{4}8_{3/2}[70, 1^{-}]$	1833.1	1668.4
N(1710) P11	***	1680–1740	$^{2}8_{1/2}[70, 0^{+}]$	1897	1735.4
N(1720) P13	****	1700–1750	$^{2}8_{3/2}[56, 2^{+}]$	1807.6	1660.4
N(2190) G17	****	2100–2200	$^{2}8_{7/2}[70, 3^{-}]$	2254.8	2150.1
N(2220) H19	****	2200–2300	$^{2}8_{9/2}[56, 4^{+}]$	2344.3	2182.8
N(2250) G19	****	2170–2310	$^{4}8_{9/2}[70, 3^{-}]$	2369.7	2265
N(2600) I1,11	***	2550–2750	$^{2}8_{11/2}[70, 5^{-}]$	2792	2687.3
Δ (1232) P33	****	1231–1233	$^{4}10_{3/2}[56, 0^{+}]$	1329.9	1232
Δ (1600) P33	***	1550–1700	$^{4}10_{3/2}[56, 0^{+}]$	1681.1	1571
Δ (1620) S31	****	1600–1660	$^{2}10_{1/2}[70, 1^{-}]$	1655.2	1602
Δ (1905) F35	****	1865–1915	$^{4}10_{5/2}[56, 2^{+}]$	2038	1927.9
Δ (1910) P31	****	1870–1920	$^{4}10_{1/2}[56, 2^{+}]$	2038	1927.9
Δ (1950) F37	****	1915–1950	$^{4}10_{7/2}[56, 2^{+}]$	2038	1927.9
Δ (2420) H3, 11	****	2300–2500	$^{4}10_{11/2}[56, 4^{+}]$	2574.7	2464.7
Λ (1116)P01	****	1116	$^{2}8_{1/2}[56, 0^{+}]$	1277.6	1116
Λ (1600)P01	***	1560–1700	$^{2}8_{1/2}[56, 0^{+}]$	1807	1645.4
Λ (1670)S01	****	1660–1680	$^{2}8_{1/2}[70, 1^{-}]$	1717.8	1620
Λ (1800)S01	***	1720–1850	$^{4}8_{1/2}[70, 1^{-}]$	1985.6	1727.9
Λ (1810)P01	***	1750–1850	$^{2}8_{1/2}[70, 0^{+}]$	1985.6	1880.9
Λ (1820)F05	****	1815–1825	$^{2}8_{5/2}[56, 2^{+}]$	1985.6	1824
Λ (1890)P03	****	1850–1910	$^{2}8_{3/2}[56, 2^{+}]$	1985.6	1824
Λ (2110)F05	****	2090–2140	$^{4}8_{5/2}[70, 2^{+}]$	2279.3	2174.6
Λ*(1405) S01	****	1402–1410	$^{2}1_{1/2}[70, 1^{-}]$	1539.9	1384
Λ*(1520)D01	****	1518–1520	$^{2}1_{3/2}[70,1^{-}]$	1539.9	1384
Σ (1193) P11	****	1193	$^{2}8_{1/2}[56, 0^{+}]$	1354.6	1193
Σ(1670)D13	****	1665–1685	$^{2}8_{3/2}[70, 1^{-}]$	1794.8	1680.1
Σ(1750)S11	***	1730–1800	$^{2}8_{1/2}[70, 1^{-}]$	1794.8	1680.1
Σ(1775) D15	****	1770–1780	$^{4}8_{5/2}[70, 1^{-}]$	1909.7	1804.9
Σ(1915)F15	****	1900–1935	$^{2}8_{5/2}[56, 2^{+}]$	2062.6	1901
Σ(1940)D13	***	1900–1950	$^{2}8_{3/2}[56, 1^{-}]$	2152	1940.4
Σ*(2030)F17	****	2025–2040	$^{4}10_{7/2}[56, 2^{+}]$	2177.5	2067.5
Ξ (1318) P11	****	1314–1316	$^{2}8_{1/2}[56, 0^{+}]$	1494.1	1331.6
Ξ (1820) D13	***	1818–1828	$^{2}8_{3/2}[70, 1^{-}]$	1934.3	1827.6

Table 4. Percentage of relative error for mass spectrum of baryon resonances calculated in our model (according to Eq. (31)).

baryon	status	$M_{\mathrm{Exp}[43]}$	state	$M_{\mathrm{Our\ Calc(GGR)}}$	percent of relative error
N(938) P11	****	938	$^2 8_{1/2}[56, 0^+]$	938	0
N(1440) P11	****	1420–1470	$^2 8_{1/2}[56, 0^+]$	1467.4	3.33%–0.17%
N(1520) D13	****	1515–1525	$^2 8_{3/2}[70, 1^-]$	1541.5	1.74%–1.08%
N(1535) S11	****	1525–1545	$^2 8_{1/2}[70, 1^-]$	1541.5	1.08%–0.22%
N(1650) S11	****	1645–1670	$^4 8_{1/2}[70, 1^-]$	1668.4	1.42%–0.09%
N(1675) D15	****	1670–1680	$^4 8_{5/2}[70, 1^-]$	1668.4	0.09%–0.69%
N(1680) F15	***	1680–1690	$^2 8_{5/2}[56, 2^+]$	1660.4	1.16%–1.75%
N(1700) D13	***	1650–1750	$^4 8_{3/2}[70, 1^-]$	1668.4	1.11%–4.66%
N(1710) P11	***	1680–1740	$^2 8_{1/2}[70, 0^+]$	1735.4	3.29%–0.26%
N(1720) P13	****	1700–1750	$^2 8_{3/2}[56, 2^+]$	1660.4	2.32%–5.12%
N(2190) G17	****	2100–2200	$^2 8_{7/2}[70, 3^-]$	2150.1	0.024%–2.26%
N(2220) H19	****	2200–2300	$^2 8_{9/2}[56, 4^+]$	2182.8	0.78%–5.09%
N(2250) G19	****	2170–2310	$^4 8_{9/2}[70, 3^-]$	2265	4.37%–1.94%
N(2600) I1,11	***	2550–2750	$^2 8_{11/2}[70, 5^-]$	2687.3	5.38%–2.28%
Δ (1232) P33	****	1231–1233	$^4 10_{3/2}[56, 0^+]$	1232	0
Δ (1600) P33	***	1550–1700	$^4 10_{3/2}[56, 0^+]$	1571	1.35%–7.5%
Δ (1620) S31	****	1600–1660	$^2 10_{1/2}[70, 1^-]$	1602	0.12%–3.49%
Δ (1905) F35	****	1865–1915	$^4 10_{5/2}[56, 2^+]$	1927.9	3.37%–0.67%
Δ (1910) P31	****	1870–1920	$^4 10_{1/2}[56, 2^+]$	1927.9	3.09%–0.41%
Δ (1950) F37	****	1915–1950	$^4 10_{7/2}[56, 2^+]$	1927.9	0.67%–1.13%
Δ (2420) H3, 11	****	2300–2500	$^4 10_{11/2}[56, 4^+]$	2464.7	7.16%–1.41%
Λ(1116)P01	****	1116	$^2 8_{1/2}[56, 0^+]$	1116	0
Λ (1600)P01	***	1560–1700	$^2 8_{1/2}[56, 0^+]$	1645.4	5.47%–3.21%
Λ (1670)S01	****	1660–1680	$^2 8_{1/2}[70, 1^-]$	1620	2.41%–3.57%
Λ (1800)S01	***	1720–1850	$^4 8_{1/2}[70, 1^-]$	1727.9	0.45%–6.6%
Λ (1810)P01	***	1750–1850	$^2 8_{1/2}[70, 0^+]$	1880.9	7.4%–1.67%
Λ (1820)F05	****	1815–1825	$^2 8_{5/2}[56, 2^+]$	1824	0.49%–0.05%
Λ (1890)P03	****	1850–1910	$^2 8_{3/2}[56, 2^+]$	1824	1.4%–4.5%
Λ (2110)F05	****	2090–2140	$^4 8_{5/2}[70, 2^+]$	2174.6	4.04%–1.61%
Λ*(1405) S01	****	1402–1410	$^2 1_{1/2}[70, 1^-]$	1384	1.28%–1.84%
Λ*(1520)D01	****	1518–1520	$^2 1_{3/2}[70,1^-]$	1384	8.82%–8.94%
Σ (1193) P11	****	1193	$^2 8_{1/2}[56, 0^+]$	1193	0
Σ(1670)D13	****	1665–1685	$^2 8_{3/2}[70, 1^-]$	1680.1	0.9%–0.29%
Σ(1750)S11	***	1730–1800	$^2 8_{1/2}[70, 1^-]$	1680.1	2.88%–6.66%
Σ(1775) D15	****	1770–1780	$^4 8_{5/2}[70, 1^-]$	1804.9	1.97%–1.39%
Σ(1915)F15	****	1900–1935	$^2 8_{5/2}[56, 2^+]$	1901	0.05%–1.75%
Σ(1940)D13	***	1900–1950	$^2 8_{3/2}[56, 1^-]$	1940.4	2.12%–0.49%
Σ*(2030)F17	****	2025–2040	$^4 10_{7/2}[56, 2^+]$	2067.5	2.09%–1.34%
Ξ (1318) P11	****	1314–1316	$^2 8_{1/2}[56, 0^+]$	1331.6	1.33%–1.18%
Ξ (1820) D13	***	1818–1828	$^2 8_{3/2}[70, 1^-]$	1827.6	0.52%–0.02%

4 Summary and conclusion

In this paper we have computed the strange and non-strange baryon resonances spectrum up to 3 GeV within a non-relativistic quark model based on the three identical quarks Schrödinger equation and the algebraic approach. Our results show that the generalized Gürsey Radicati mass formula is a good parameterization of the baryon energy splitting coming from $SU(6)$ breaking. In our model, the energy splitting within the $SU(6)$ multiples are considered as perturbations added to the $SU(6)$ invariant levels, which are given by anharmonic potential. For reproducing the spectrum of baryons resonances, we calculated the energy eigenvalues by solving exactly the Schrödinger equation by the Ansatz method for confining potential. Then, we fitted the generalized GR mass formula parameters to the baryons energies and calculated the baryons mass according to Eq. (31). The overall good description of the spectrum which we obtain by this combination of potentials shows that our model can also be used to give a fair description of the energies of the excited multiples with up to 3 GeV mass and not only for the ground state octets and decuples. Moreover, our model reproduces the position of the Roper resonance and negative-parity resonance. There are still problems with the reproduction of hyperons masses, in particular for the Λ*(1405) S01 and the Λ*(1520) D01 resonances that come out degenerate and above the experimental values. There are problems in the reproduction of the experimental masses in N (1720) P13 and Σ(1775) D15 turn out to have predicted mass about 50 MeV above the experimental value. A better agreement may be obtained either using the square of the mass [1] or using the Faddeev equation to get the eigenvalue of energy more exactly. The Faddeev equation is a three-dimensional integral equation for six variables, apparently not a trivial task. However, with modern computers and computational tools, it is possible to solve. For example in a paper about Three-Body Systems by Bellotti et al. [44], they consider three body systems in two dimensions with zero-range hence they solved Schrödinger equation in momentum-space interactions for general masses and interaction strengths. The zero-range limit provides a simplification of the Faddeev equation for the three-body bound state due to the separability of the zero-range interaction. We hope to employ this method to obtain the spectrum of baryons in our future work.

We give our most sincere thanks to the referee for a very technical and useful comment on the manuscript.

References

1 Bijker R, Iachello F. Leviatan A. Ann. Phys. (N.Y.), 1994, **236**: 69
2 Aiello M, Ferraris M, Giannini M M et al. Phys. Lett. B, 1996, **387**: 215
3 Dziembowski Z, Fabre de la Ripelle M, Miller Gerald A. Phys. Rev. C, 1996, **53**: R2038
4 Benmerrouche M, Mukhopadhyay N C, ZHANG J F. Phys. Rev. Lett., 1996, **77**: 4716
5 Chakrabarti B, Bhattacharya A, Mani S et al. Acta Physica Polonica B, 2010, **41**(1): 95–101
6 Isgur N. Karl G. Phys. Rev. D, 1978, **18**: 4187; 1979, **19**: 2653; 1979, **20**: 1191
7 Giannini M M. Rep. Prog. Phys., 1991, **54**: 453
8 Bali Gunnar S et al. Phys. Rev. D, 2000, **62**: 054503
9 Bali Gunnar S. Phys. Rep. 2001, **343**: 1
10 Alexandrou C, de Forcrand P, Jahn O. Nucl. Phys. Proc. Suppl., 2003, **119**: 667
11 Ferraris M, Giannini M M, Pizzo M et al. Phys. Lett. B, 1995, **364**: 231
12 Santopinto E, Iachello F, Giannini M M. Eur. Phys. J. A, 1998, **1**: 307
13 Glozman L Y, Riska D O. Phys. Rep., 1996, **268**: 263
14 Giannini M M, Santopinto E, Vassallo A. Eur. Phys. J. A, 2001, **12**: 447
15 Gürsey F, Radicati L A. Phys. Rev. Lett., 1964, **13**: 173
16 Giannini M M, Santopinto E, Vassallo A. Eur. Phys. J. A, 2005, **25**: 241–247
17 Salehi N, Rajabi A A. Mod. Phys. Lett. A, 2009, **24**(32): 2631–2637
18 Rajabi A A. Few-Body Sys., 2005, **37**: 197
19 Hassanabadi H, Rahimov H, Zarrinkamar S. Ann. Phys. (Berlin), 2011, **523**(7): 566
20 Hassanabadi H, Yazarloo B H, Zarrinkamar S et al. Phys. Rev. C, 2011, **84**: 064003
21 Gell-Mann M, Ne'eman Y. The Eightfold Way. New York: W. A. Benjamin, Inc., 1964
22 Khare A, Behra S N. Pramana J. Phys., 1980, **14**: 327; Bali S et al. Phys. Rev. D, 2000, **62**: 054503
23 Amin D. Phys. Today, 1982, **35**: 35. Phys. Rev. Lett., 1976, **36**: 323; Murthy M V N. Z. Phys. 1986, **31**: 81
24 Colemann S. Aspects of Symmetry. Cambridge: Cambridge Univ. Press, 1988
25 Salehi N, Rajabi A A, Ghalenovi Z. Chin. J. Phys., 2012, **50**(1): 28–39
26 Salehi N, Rajabi A A. Phys. Scr., 2012, **85**: 055101
27 Murthy M V N. Z. Phys., 1986, **31**: 81
28 Znojil M. J. Math Phys., 1990, **31**: 108
29 Rajabi A A. Indian J. Pure and Appl. Phys., 2003, **l41**: 89
30 Rajabi A A, Salehi N. Iranian Journal of Physics Research, 2008, **8**(3): 169–175
31 Hassanabadi H, Rajabi A A. Mod. Phys. Lett. A, 2009, **24**(11–13): 1043–1046
32 Causal R S, Parishes D. Phys. Lett. A, 1992, **170**: 335
33 Causal R S. Annals of Physics, 1991, **206**: 90
34 DONG S H, MA Z Q. Arxiv: preprint quant-ph/9810056
35 DONG S H. Found Phys. Lett., 2002, **15**(4): 385–395
36 Gradshteyn I S, Ryzhik I M. Table of Integrals. New York: Series and Products, Academic Press, 1965
37 Helminen C, Riska D O. Nucl. Phys. A, 2002, **699**: 624
38 Stancu F. Phys. Rev. D, 1998, **58**: 111501; Stancu F, Riska D O. Phys. Lett. B, 2003, **575**: 242
39 Helminen C, Riska D O. Nucl. Phys. A, 2002, **699**: 624
40 Carlson C E, Carone C D, Kwee H J et al. Phys. Lett. B, 2004, **579**: 52
41 Glozman L Y. Phys. Lett. B, 2003, **575**: 18
42 Bijker R, Giannini M M, Santopinto E. Eur. Phys. J. A, 2004, **22**: 319
43 Particle Data Group. J. Phys. G, 2010, **37**: 075021
44 Bellotti F F, Frederico T, Yamashita M T et al. J. Phys. B: At. Mol. Opt. Phys., 2013, **46**: 055301