CPC(HEP & NP), 2012, **36**(9): 797–804 Chinese Physics C Vol. 36, No. 9, Sep., 2012

A note on the soliton picture in a Skyrme-like model*

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Abstract: The role of the anti-commutator term of the chiral current in a Skyrme-like model was studied associated with the symmetric Skyrmion and the nucleon properties in terms of the zero-mode quantization. It is shown that the Skyrmion is stable only when the anti-commutator term in the model has a negative coupling constant($-k^2$) while a QCD functional analysis gives a positive coupling constant. This implies either the coupling is negligibly small and negative, or the soliton picture for the baryons is beyond the approximation of QCD at the level of the quark loop.

Key words: Skyrme model, soliton, stability

PACS: 12.38.-t, 11.15.Tk, 12.38.Aw **DOI:** 10.1088/1674-1137/36/9/001

1 Introduction

The soliton picture has been a successful description of the baryons at the phenomenological level, and it occurs in many effective models of baryons, such as the sigma model [1], the Skyrme model [2], the chromo-dielectric soliton model [3], the chiral quark soliton model [4] and the Nambu-Jona-Lasinio model [5]. The solitons in these hadron descriptions can be regarded as the dynamical and soft counterpart of the bag in the MIT bag model that confines quarks in it, and as such these solitons are hoped to be stable in order to have a consistent description of hadrons as a useful approximation to the confinement in QCD (see Ref. [6] for review). However, the stability of the soliton is not assured in priori and some enforcing constraints are needed to associate with the chiral symmetry breaking (CSB) and confinement. The typical examples of this are the "chiral circle" constraint ($\sigma^2+\pi^2=f_\pi^2$) in the sigma model and the boundary condition for the dielectric factor $\epsilon(\sigma)=\{1,0\}$ in the chromo-dielectric soliton model mimicking the two phases: the confining phase and asymptotic phase of QCD.

In the case of the Skyrme model [2], which is the effective field theory of QCD at a low-energy limit [7], the Skyrmion (chiral) field $U(x)$ approximates the baryons by manifesting itself as the localized field (soliton) made of mesonic cloud around the core of baryon, carrying the nontrivial baryon number B, and the similar constraint imposed is the condition $U=\pm1$ which corresponds to the chirally symmetric and broken vacua, respectively. Despite the limitation that the Skyrmion stability is not manifestly ensured in the light of the effective chiral theory of QCD [8], the progress on the Skyrmion description to the few-body nucleus [9, 10] provides a sustainable support for the Skyrme model in describing hadrons, or an effective theory [11, 12] for baryon interaction [13] as soon as the stable condition is fulfilled. This indicates, in a sense, that the Skyrmion picture of baryons can be a reliable description of the hadron at the qualitative level [14].

In this work, we address the role of the anti-

Received 21 December 2011, Revised 12 January 2012
* Supported by National Natural Science Foundation of China (10965005), SRF for ROCS, SEM, and Project of Key Laboratory of Atomic and Molecular Physics and Functional Materials of Gansu Province
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commutator term of the chiral current in a Skyrme-like model associated with the Skyrmion stability and the nucleon property predictions in terms of the zero-mode quantization. It is shown that the Skyrmion is stable only when the anti-commutator term of the model has a negative coupling constant which is not manifestly supported by the QCD functional analysis. Considering that the model has fine-tunable parameters which can vary in a finite region, it infers that either the choice of vanishing(or negligibly small and negative) coupling for the anti-commutator term is reasonable, or the soliton picture for the baryons is beyond the chiral expansion approximation of QCD up to the fourth derivative order, in consistence with the demonstration by Simic [8].

2 The stability of the Skyrmion in a Skyrme-like model

The $SU(2)$ Skyrme-like model to be considered reads [2]

$$\mathcal{L}^{\mathrm{SK}} = -\frac{f_\pi^2}{4}\mathrm{tr}(L_\mu L^\mu)+\frac{\varepsilon^2}{4}\mathrm{tr}[L_\mu,L_\nu]^2 -\frac{k^2}{4}\mathrm{tr}[L_\mu,L_\nu]_+^2+\frac{m_\pi^2}{2}f_\pi^2\mathrm{tr}(U-1), \quad (1)$$

where $L_\mu=U^\dagger\partial_\mu U$ is the left chiral current, $U(x,t)\in SU(2)$ the nonlinear realization of the chiral field describing the σ field and π mesons under the constraint $U^\dagger U=1$, $2f_\pi$ the pion decay constant, and ε a dimensionless constant characterizing nonlinear coupling. Here, the last term associated with the anti-commutator $[L_\mu,L_\nu]_+\equiv L_\mu L_\nu+L_\nu L_\mu$ of the chiral current, can be added to the standard Skyrme model, in general, up to the fourth derivative terms, and will be examined in detail in this paper.

The energy associated with (1) is

$$E = 4\pi\left(\frac{f_\pi}{e}\right)\int\mathrm{d}^3x\left\{\frac{1}{2}\mathrm{tr}(\partial^i U\partial^i U^\dagger) -\frac{1}{16}\mathrm{tr}[\partial^i UU^\dagger,\partial^j UU^\dagger]^2 -\frac{K}{16}\mathrm{tr}[\partial^i UU^\dagger,\partial^j UU^\dagger]_+^2+m^2\mathrm{tr}(1-U)\right\}, \quad (2)$$

with $K=-k^2/\varepsilon^2$ the reduced coupling, $m=m_\pi/(ef_\pi)$ the mass parameter in the unit of ef_π, and ∂^i the derivatives in the unit of $1/ef_\pi$, in which $e=1/(\sqrt{8}\varepsilon)$. We use notation L_j^a defined by $U^\dagger\partial_j U=\mathrm{i}\tau^a L_j^a$ and its dot product

$$\boldsymbol{L}_j\cdot\boldsymbol{L}_k=L_j^aL_k^a,$$

to rewrite Eq. (2) as

$$E=\left(\frac{\pi f_\pi}{e}\right)\left[E_2+\frac{1}{2}E_4+2m^2E_m\right]. \quad (3)$$

Here

$$E_2=\int\mathrm{d}x x^2(\boldsymbol{L}_j\cdot\boldsymbol{L}_j),$$
$$E_4= E_4^+-KE_4^-, \quad (4)$$
$$E_m=\int\mathrm{d}x x^2\mathrm{tr}(1-U),$$

with

$$E_4^+=\int\mathrm{d}x x^2\left[(\boldsymbol{L}_j\cdot\boldsymbol{L}_j)(\boldsymbol{L}_k\cdot\boldsymbol{L}_k)-(\boldsymbol{L}_j\cdot\boldsymbol{L}_k)(\boldsymbol{L}_j\cdot\boldsymbol{L}_k)\right], \quad (5)$$
$$E_4^-=\int\mathrm{d}x x^2\left[(\boldsymbol{L}_j\cdot\boldsymbol{L}_k)(\boldsymbol{L}_j\cdot\boldsymbol{L}_k)\right].$$

In D-dimensianl space, a rescaling $x\to\lambda x$ for (3) yields the rescaled energy given by

$$E=\left(\frac{\pi f_\pi}{e}\right)\left[\lambda^{2-D}E_2+\frac{\lambda^{4-D}}{2}E_4+2m^2E_m\right]. \quad (6)$$

The rescaling stability requires

$$\left.\frac{\mathrm{d}E}{\mathrm{d}\lambda}\right|_{\lambda=1}=0,\left.\frac{\mathrm{d}^2E}{\mathrm{d}\lambda^2}\right|_{\lambda=1}\geqslant 0,$$

which, in the case of (6), implies

$$(2-D)E_2=(D-4)\frac{E_4}{2},$$
$$(D^2-5D+7)E_2\geqslant 0. \quad (7)$$

When $D=3$, Eq. (7) becomes

$$E_2=\frac{1}{2}E_4, E_4\geqslant 0, \quad (8)$$

in which the first condition is nothing but the virial theorem, and $E_2\geqslant 0$ is guaranteed by the inequality $\boldsymbol{L}_j^2\equiv(\boldsymbol{L}_j\cdot\boldsymbol{L}_j)\geqslant 0$. For the condition $E_4\geqslant 0$, one can use the inequality

$$3(\boldsymbol{L}_j\cdot\boldsymbol{L}_k)^2\geqslant(\boldsymbol{L}_j\cdot\boldsymbol{L}_j)^2\geqslant(\boldsymbol{L}_j\cdot\boldsymbol{L}_k)^2, \quad (9)$$

to rewrite it as

$$\boldsymbol{L}_j^2\boldsymbol{L}_k^2-(1+K)(\boldsymbol{L}_j\cdot\boldsymbol{L}_k)^2\geqslant 0. \quad (10)$$

Combining (10) with (9), one finds that the stability condition (10) is fulfilled when

$$K\leqslant 0. \quad (11)$$

When $\varepsilon^2>0$, as is the case in the standard Skyrme model, the condition (11) implies

$$k^2\geqslant 0. \quad (12)$$

We note that the stability condition (11) or (12) for the Skyrmions is quite general and may change when the extra fourth-derivative terms are added in the model (1). A typical term considered is

$$L_{\rm e}=k_3{\rm tr}(\partial_\mu L_\mu)^2,$$

which introduces a new parameter into the condition (10) and may change the stability condition. Adding $L_{\rm e}$, the model (1) can be rewritten, in terms of U, as

$$\mathcal{L}^{\rm SK} = \frac{f_\pi^2}{4}{\rm tr}[\partial_\mu U^\dagger\partial^\mu U]+\frac{\varepsilon^2-k^2}{2}{\rm tr}[\partial_\mu U^\dagger\partial^\nu U]^2 +\left(k_3-\frac{\varepsilon^2+k^2}{2}\right){\rm tr}[\partial_\mu U^\dagger\partial^\mu U]^2 -k_3{\rm tr}[(\partial^2 U^\dagger)(\partial^2 U)]+\frac{m_\pi^2}{2}f_\pi^2{\rm tr}(U-1). \quad (13)$$

Though this model (13) includes more terms which happen in the chiral perturbation theory [15], and thus it is linked directly to QCD at low energy, the stability of the soliton in it can not be inferred manifestly when the coefficients are constrained to reconcile with QCD in its form of the derivative expansion of the chiral field up to the fourth order, as will be discussed in Section 5 in detail.

3 The Skyrmions in the stable region

As usual, we search for the symmetric Skyrmion using the hedgehog ansatz for the chiral field

$$U(x)=\cos F(r)+{\rm i}\hat{r}\cdot\vec{\tau}\sin F(r), \quad (14)$$

with $F(r)$ the radial chiral angle, subjected to the boundary condition: $F(0)=\pi$, $F(\infty)=0$, and $\vec{\tau}$ the Pauli matrices. With (14), the static energy in (2) becomes

$$M^{\rm SK} = \pi\int r^2{\rm d}r\left\{f_\pi^2\left[F_r^2+2\frac{\sin^2(F)}{r^2}\right] +\varepsilon^2\left[\sin^4(F)\left(16\frac{\sin^2(F)}{r^2}+32F_r^2\right)\right] +k^2\left[8F_r^2+16\frac{\sin^4(F)}{r^4}\right]+\frac{2f_\pi}{e}4m(1-\cos F)\right\}, \quad (15)$$

that is,

$$M^{\rm SK} = 2\pi\left(\frac{f_\pi}{e}\right)\int_0^\infty x^2{\rm d}x\left[F_x^2+\frac{2}{x^2}\sin^2(F)(1+F_x^2) +\frac{\sin^4(F)}{x^4}-\frac{K}{2}\left(F_x^2+\frac{2\sin^4(F)}{x^4}\right) +2m^2(1-\cos F)\right], \quad (16)$$

where the dimensionless variable $x=ef_\pi r$ and the notation $F_x\equiv {\rm d}F/{\rm d}x$ have been used.

The equation of motion for the model (16) is

$$\left(1+2\frac{\sin^2 F}{x^2}-3KF_x^2\right)F_{xx}+\frac{2}{x}F_x-2K\frac{F_x^3}{x} +\frac{\sin(2F)}{x^2}\left(F_x^2-1+(K-1)\frac{\sin^2 F}{x^2}\right)=m^2\sin F, \quad (17)$$

where only two parameters $(K,\ m)$ are involved.

To study the stability through the profile solution $F(x)$ to Eq. (17), we consider two cases for the choice of the reduced pian mass m, one is that the model is chirally symmetric $m=0$, and the other is that the chiral symmetry of the model is broken down to the physically interested value $m=0.526$. The relaxation procedure is applied to solve (17) numerically, with the numerical Skyrmion profile shown in Fig. 1 for the case of $m=0.526$ and $K=-0.2$, in which the analytical profile is also shown. The analytical solution to approximate the Skyrmion is given by the profile ansatz [16]

$$F_m(x) = 4w\arctan[\exp(-cx)] +\pi(1-w)\left[1-\left(\frac{\sinh^2(dmx)}{a^2+\sinh^2(dmx)}\right)^{1/2}\right], \quad (18)$$

where $m\neq 0$, and $(a,\ c,\ d,\ w)$ are the parameters to be determined variationally, a is related to the instanton scale λ through [17]

$$a=\frac{2\lambda}{1+\lambda^2}. \quad (19)$$

Given $F(x)$ specified by (18), we fix the parameters for $K=-8.6$, using the downhill simplex method to (16), as described in Ref. [18]. The fixed parameters are listed in Table 1. It is found that the best agreement between the numerical and analytical profiles is achieved for the case of $K=-8.6$, for which

$$w=3.78847,\ c=0.81208,\ d=0.5125,\ a=0.94223.$$

We further numerically solve (17) for the various values of $|K|$ and present in Fig. 2 the plot of the profile for a set of typical values of $|K|$. It is found by numerical calculations that the Skyrmion is stable only when the effective coupling K (i. e., $-k^2$) of the anti-commutator term in the model (2) is negative($k^2\geqslant 0$). To check this behavior, we use a different initial setup for $F(x)$ in the relaxation algorithm, such as the Gaussian distribution and the kink configuration, and the same result is achieved.

Table 1. The calculated static property of nucleons calculated for nonzero K. The inputs are $M_{\rm N}=938.9$, $M_{\triangle}$=1232 (MeV), and m_{π}=138 MeV.

K	−1.00	−3.00	−5.00	−8.60	−10.00
$E(/(2f_{\pi}/e))$	49.697	58.233	79.061	99.548	106.75
$M(/(6\pi^2 f_{\pi}/e))$	1.678	1.967	2.670	3.362	3.605
$2f_{\pi}$/MeV	128.32	139.90	164.92	186.033	192.83
e	5.396	5.883	6.935	7.823	8.109
Λ	49.257	69.61	134.40	217.61	251.25
$\langle r^2\rangle^{1/2}_{I=0}$/fm	0.613	0.577	0.508	0.459	0.445
$\langle r^2\rangle^{1/2}_{I=0,M}$/fm	0.894	0.849	0.752	0.680	0.659
$\mu_{\rm p}$	1.897	1.862	1.804	1.767	1.757
$\mu_{\rm n}$	−1.307	−1.341	−1.399	−1.436	−1.446
$\lvert\mu_{\rm p}/\mu_{\rm n}\rvert$	1.4512	1.389	1.289	1.230	1.215

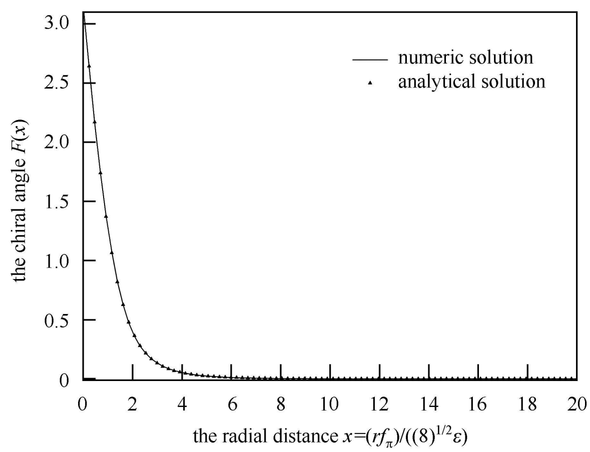

Fig. 1. The numerical profile for the chiral angle solved from (17) for $m=0.526$ and $K=-0.2$. The corresponding analytical solution (18) for the same K is also shown.

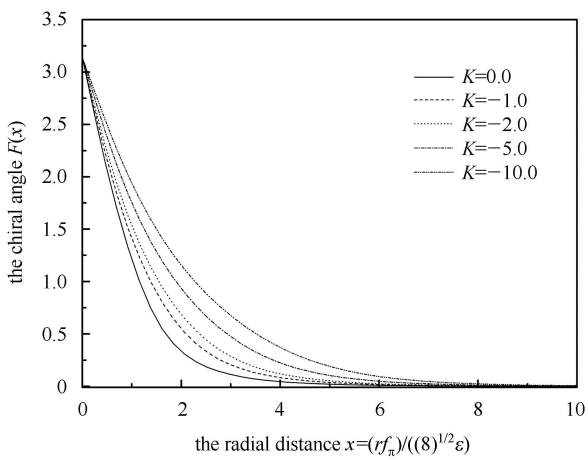

Fig. 2. The numerical solutions to (17) are shown for the different K.

By checking the available chiral angle profiles obtained analytically and numerically, both done for the different $K>0$, using the previous method, we find that the Skyrmion is stable only when the coupling $K(-k^2)$ of the anti-commutator term is negative ($K\geqslant 0$). This agrees nicely with the stability condition (11) or (12) in Section 2.

4 The static properties of nucleons as solitons

The realistic prediction for the static properties of nucleons using the Skyrmion obtained above requires the quantization of the Skyrme model. This can be done by semi-classically quantizing the spinning modes of the Skyrme Lagrangian in terms of the collective variables [11, 12]. To see how the predictions vary in a Skyrme-like model (1) with K, we use the obtained numerical hedgehog solution, obtained for different K, to compute the static properties of nucleons and nucleon-isobar (Δ) in the framework of the bosonic quantization of a soliton.

Choosing a $SU(2)$-variable $A(t)$ as the collective variables, and substituting $U=A(t)U_0(x)A(t)^{\dagger}$ into (1), the action becomes, in the adiabatic limit,

$$L=-M^{\rm SK}+I_0\Lambda{\rm Tr}\left[\frac{\partial A}{\partial t}\frac{\partial A^{\dagger}}{\partial t}\right], \tag{20}$$

with $M^{\rm SK}$ the soliton energy for the static hedgehog Skyrmion, $I_0=\pi/(3e^3 f_{\pi})$, and

$$\Lambda=8\int_0^{\infty}x^2{\rm d}x\sin^2F[1+F_x^2+(1-K)\sin^2F/x^2]. \tag{21}$$

The Hamiltonian associated with (20), when quantized collectively, yields an eigenvalue $\langle H\rangle=M^{\rm SK}+J(J+1)/(2I_0\Lambda)$. It follows that the masses of the nucleon and Δ-isobar are

$$\begin{aligned} M_{\rm N} &= M^{\rm SK}+\frac{3}{8I_0\Lambda},\\ M_{\Delta} &= M^{\rm SK}+\frac{15}{8I_0\Lambda}. \end{aligned} \tag{22}$$

Table 2. The static property of nucleons calculated for $K=0$. The inputs are $M_{\rm N}=938.9$, $M_{\triangle}=1232$ (MeV), and m_π=138 MeV.

quantities	Ref. [11]	Ref. [12]	Num.	Expt.
$E(/(2f_\pi/e))$	36.5	–	39.143	–
$M(/(6\pi^2 f_\pi/e))$	1.233	–	1.3220	–
$2f_\pi$/MeV	129	108	112.643	186
e	5.45	4.84	4.7367	–
m_π	0	138(input)	138(input)	138
$\langle r^2\rangle^{1/2}_{I=0}$/fm	0.59	0.68	0.6621	0.72
$\langle r^2\rangle^{1/2}_{I=0,M}$/fm	0.92	0.95	0.9280	0.80
$\mu_{\rm p}$	1.87	1.97	1.945	2.79
$\mu_{\rm n}$	−1.31	−1.24	−1.258	−1.91
$\lvert\mu_{\rm p}/\mu_{\rm n}\rvert$	1.43	1.59	1.546	1.46

As done in [12], we choose to adjust model parameters (f_π, e) to fit the hadron masses, namely, the N, Δ, and π masses through (22), where the other quantities involved are given by (16) and (21). Our reports for the computation of the static properties of baryons are presented in Table 2, including the experimental values as well as the results from Refs. [11, 12].

In the following, we also list the most relevant formulas for our computations, for instance, the isoscalar root mean square(r.m.s) radius and isoscalar magnetic r.m.s radius,

$$\langle r^2\rangle^{1/2}_{I=0}=\frac{1}{ef_\pi}\left\{-\frac{2}{\pi}x^2\sin^2 FF_x\right\}^{1/2},$$

$$\langle r^2\rangle^{1/2}_{M,I=0}=\frac{1}{ef_\pi}\left\{\frac{\int_0^\infty x^4\sin^2 FF_x{\rm d}x}{\int_0^\infty x^2\sin^2 FF_x{\rm d}x}\right\}^{1/2}.$$

The magnetic moments for proton and neutron are given by

$$\mu_{\rm p,n} = \mu^{I=0}_{\rm p,n}+\mu^{I=1}_{\rm p,n}=\frac{\langle r^2\rangle_{I=0}}{9}M_{\rm N}(M_\Delta-M_{\rm N}) \pm\frac{M_{\rm N}}{2(M_\Delta-M_{\rm N})}, \tag{23}$$

where plus and minus correspond to proton and neutron, respectively.

5 The Skyrmion-like action from the QCD functional

It is well known that the low-energy version of QCD is dictated by the spontaneous breaking of the chiral symmetry in which the light pseudoscalar mesons play a central role in the form of summarizing the long-wavelength properties of the vacuum state. This does not, however, manifestly justify the soliton picture of the nucleons as a Skyrmion [8]. To re-examine the soliton picture of the baryon, we consider the parameters in the Skyrme-like model within a QCD script in which the gluonic effects are reduced to a background potential for the scalar excitations, along the line of treatment by Simic [8], and Zahed and Brown [13].

We start with the QCD generating functional in the absence of the sources

$$Z^{\rm QCD}=\int[{\rm d}\bar{q}{\rm d}q{\rm d}G^a_\mu{\rm d}c{\rm d}\bar{c}]\exp[{\rm i}(S_{\rm QCD}+S_{\rm FP}+S_{\rm GF})], \tag{24}$$

where $S_{\rm QCD}$ is the QCD action in terms of the quark field $q(x)$ and gluon field $G^a_\mu(x)$, $S_{\rm FP}$ is the Faddeev-Popov action involving the ghost fields c and $\bar{c}$, and $S_{\rm GF}$ is the gauge fixing term. As is known, the full quantum calculation in terms of (24) remains notoriously involved. However, we know, at least from the lattice QCD simulation and the QCD sum rules, that the long-wavelength sector of QCD is well characterized by scalar quark and gluon condensates, e.g., $\langle\bar{q}q\rangle$ and $\langle GG\rangle$, etc., so one can divide the field variables in (24) into classes characterized by a certain vacuum structure at low energy. Here, we use that of CSB realized by the nonvanishing vacuum expectation value of the quark bilinear $\bar{q}_{\rm L}q_{\rm R}$ and $\bar{q}_{\rm R}q_{\rm L}$, by enforcing the identification $\bar{q}_{\rm L}q_{\rm R}=\sigma^\dagger U$ and $\bar{q}_{\rm R}q_{\rm L}=U^\dagger\sigma$ with σ a scalar meson matrix and U a unitary matrix such that $UU^\dagger=1$. Then, the QCD functional can be extended to the integration over the the new fields (σ,U) defined above. Following [13], we introduce auxiliary fields of a scalar $S(x)$ and a pseudoscalar $P(x)$, through

$$\delta(\bar{q}_{\rm L}q_{\rm R}-\sigma U) = \int[{\rm d}S{\rm d}P] \times\exp\left\{-{\rm i}\int d^4x{\rm tr}[\bar{q}_{\rm L}(S+{\rm i}P)q_{\rm R} -(S+{\rm i}P)\sigma^\dagger U]\right\}, \tag{25}$$

with the usual definition for the left and right handed quarks $\bar{q}_{\rm L,R}=\frac{1}{2}(1\mp\gamma_5)q$. Putting (25) into (24) with the delta function constraints and making the unitary and anomaly-free change of the variable $(S+{\rm i}P)\rightarrow U^{\dagger}(S+{\rm i}P)$, we get

$$Z^{\rm QCD} = \int[{\rm d}\bar{q}{\rm d}q{\rm d}G^a_\mu{\rm d}c{\rm d}\bar{c}][{\rm d}U{\rm d}\sigma][{\rm d}S{\rm d}P]J[U] \times\exp\{{\rm i}(S_{\rm QCD}+S_{\rm FP}+S_{\rm GF}) -{\rm i}\int{\rm d}^4x{\rm tr}[\bar{q}U_5^{\dagger}(S+{\rm i}P\gamma_5)q-2S\sigma]\}, \quad (26)$$

where we have used the relation

$$\begin{aligned}&{\rm tr}\{\bar{q}_{\rm L}U^{\dagger}(S+{\rm i}P)q_{\rm R}+\bar{q}^{\dagger}_{\rm R}(S-{\rm i}P)Uq_{\rm L}\}\\ &= {\rm tr}[\bar{q}U_5^{\dagger}(S+{\rm i}P\gamma_5)q],\end{aligned}$$

with

$$\begin{aligned}U_5^{\dagger} &= \frac{1-\gamma^5}{2}U^{\dagger}+\frac{1+\gamma^5}{2}U\\ &= \exp({\rm i}\gamma_5\pi^a(x)\lambda^a/f_\pi). \quad (27)\end{aligned}$$

Let us consider the gluon and ghost parts of $Z^{\rm QCD}$

$$\begin{aligned}\exp\{{\rm i}G[J]\} \equiv& \int[{\rm d}G_\mu{\rm d}c{\rm d}\bar{c}]\\ &\times\exp\left\{{\rm i}\int{\rm d}^4x\left[-G^a_\mu J^{\mu,a}-\frac{1}{2}{\rm tr}G^2_{\mu\nu}\right]\right\}\\ &\times\exp\left\{{\rm i}\int{\rm d}^4x[S_{\rm GF}[\partial_\mu G_\mu]\right.\\ &\left.+S_{\rm FP}[\partial_\mu D^\mu G_\mu]]\right\},\end{aligned}$$

which is a functional of the color current $J^a_\mu=g\bar{q}\gamma_\mu\lambda^a q$ and chirally invariant, $G[J^U]=G[J]$. Following Coleman et al. [19], we assume $G[J]$ is dominated by an effective potential of the quark bilinears

$$\begin{aligned}G[J] &= -\int{\rm d}^4x[V_{\rm G}(\bar{q}q)+\cdots]\\ &\approx -\int{\rm d}^4x[V_{\rm G}(\langle\bar{q}q\rangle)+\cdots]\\ &= -\int{\rm d}^4x[V_{\rm G}(\sigma)+\cdots]. \quad (28)\end{aligned}$$

Assuming (28) yields the functional (26) in the form

$$\begin{aligned}Z^{\rm L}_{\rm QCD} =& \int[{\rm d}U{\rm d}\sigma][{\rm d}\bar{q}{\rm d}q{\rm d}S{\rm d}P]\exp\{{\rm i}\int{\rm d}^4x[\bar{q}[{\rm i}\,\partial\!\!\!/\\ &-(m+U_5^{\dagger}(S+{\rm i}P\gamma^5)]q\\ &-V_{\rm G}(\sigma)+2{\rm tr}(S\sigma)]+\ln J[U]\}, \quad (29)\end{aligned}$$

in which the auxiliary fields (S, P) are used to exponentiate the delta-function constraints in (26).

Rotating to Euclidian space and integrating out the quark variables yields (29) in the form

$$\begin{aligned}Z^{\rm E}_{\rm QCD} =& N^{-1}\int[{\rm d}S{\rm d}P][{\rm d}U{\rm d}\sigma]\\ &\times\exp\{N_{\rm c}{\rm tr}\ln(\partial\!\!\!/_E+m+U_5^{\dagger}(S+{\rm i}P\gamma^5))\\ &\times\exp\left\{-\int{\rm d}^4x[V_{\rm G}(\sigma)-2{\rm tr}(S\sigma)]-\ln J[U]\right\}.\end{aligned} \quad (30)$$

In the large N_c limit, the functional (29) will be dominated by the stationary phase $(S,\ P,\ \sigma)=(\bar{S},\ \bar{P},\ \bar{\sigma})$ of the auxiliary fields. Here, $\bar{P}=0$ because of the parity conservation. One has then from (30)

$$\begin{aligned}Z^{\rm E}_{\rm QCD} =& N^{-1}\int[{\rm d}U{\rm d}S]\exp\{N_{\rm c}{\rm tr}\ln(\partial\!\!\!/_E+m+SU_5^{\dagger})\\ &\times\exp\left\{-\int{\rm d}^4x[V_{\rm G}(\bar{\sigma})-2{\rm tr}(S\bar{\sigma})]-\ln J[U]\right\},\end{aligned} \quad (31)$$

in which $\bar{\sigma}$ is, using the saddle-point approximation, determined by $\bar{S}=(1/2)\delta V_{\rm G}(\bar{\sigma})/\delta\bar{\sigma}$.

Ignoring the imaginary part of the action functional in (31), which is the non-local Wess-Zumino term for fermions, we can rewrite the real part of the action in (31) as

$$\begin{aligned}S_{\rm q} =& \frac{N_{\rm c}}{2}{\rm tr}\ln|\,\partial\!\!\!/_E+m+SU_5^{\dagger}|^2\\ =& \frac{N_{\rm c}}{2}{\rm tr}\ln[D^2_{\rm M}-m(\partial\!\!\!/_E U_5^{\dagger})+m\bar{S}(U_5+U_5^{\dagger})]\\ =& \frac{N_{\rm c}}{2}{\rm tr}\ln(D^2_{\rm M})+\frac{N_{\rm c}}{2}{\rm tr}\ln[1+D^{-2}_{\rm M}(m\bar{S}(U+U^{\dagger})\\ &-m(\partial\!\!\!/_E U_5^{\dagger}))],\end{aligned} \quad (32)$$

where $D^2_{\rm M}\equiv-\partial^2_E+M^2$, $M^2\equiv m^2+\bar{S}^2$. Omitting the first term in (32), which is the free contribution to the effective potential and taking the derivative expansion of (32), one finds, to the fourth order of derivatives of U,

$$\begin{aligned}S^{\rm L} =& \frac{N_{\rm c}}{2}{\rm tr}\ln[1+D^{-2}_{\rm M}m\bar{S}(U+U^{\dagger})]\\ &+\frac{N_{\rm c}}{2}{\rm tr}\ln[1-mD^{-2}_{\rm M}(\partial\!\!\!/_E U_5^{\dagger})]\\ =& \frac{N_{\rm c}}{2}{\rm tr}[D^{-2}_{\rm M}m\bar{S}(U+U^{\dagger})]+S_{2{\rm q}}, \quad (33)\end{aligned}$$

where

$$S_{2q} = \frac{N_c}{2}\mathrm{tr}\ln[1-mD_M^{-2}(\partial\!\!\!/_E U_5^\dagger)]$$
$$= -\frac{N_c}{4}\mathrm{tr}[(mD_M^{-2}(\partial\!\!\!/_E U_5^\dagger))^2]$$
$$-\frac{N_c}{8}\mathrm{tr}[(mD_M^{-2}(\partial\!\!\!/_E U_5^\dagger))^4]+O(\partial^4), \quad (34)$$

owing to the fact that the odd powers in the expansion series vanish ($\mathrm{tr}(\gamma^{\mathrm{odd}})=0$). Both (33) and (34) are ultraviolet divergent, as can be explicitly seen in the momentum representation. Using the proper time regularization (or heat kernal method) with a momentum cutoff Λ

$$D_M^{-2} = \int_{1/\Lambda^2}^{\infty} \mathrm{d}s \mathrm{e}^{-sD_M^2}, \quad (35)$$

we obtain

$$\frac{N_c}{2}\mathrm{tr}\ln(D_M^{-2}m\bar{S}(U+U^\dagger))$$
$$= \frac{N_c}{2}m\bar{S}\int_{1/\Lambda^2}^{\infty}\mathrm{d}s\int_E \mathrm{d}^4x\langle x|\mathrm{e}^{-sD_M^2}\mathrm{tr}(U+U^\dagger)|x\rangle$$
$$= 4N_c D_S\int_E \mathrm{d}^4x\mathrm{tr}\left[\frac{m}{2}(U+U^\dagger)\right], \quad (36)$$

where

$$D_S = \bar{S}\int_{1/\Lambda^2}^{\infty}\mathrm{d}s\int_E \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-s(k^2+M^2)}$$
$$= \frac{\bar{S}}{16\pi^2}\int_{1/\Lambda^2}^{\infty}\frac{\mathrm{d}s}{s^2}\mathrm{e}^{-sM^2}$$
$$= \frac{\Lambda^3}{16\pi^2}\left(\frac{\bar{S}}{\Lambda}\right)\int_1^{\infty}\frac{\mathrm{d}s}{s^2}\mathrm{e}^{-s(M/\Lambda)^2}.$$

With the help of (35) and a bit of algebra, we have

$$-\frac{N_c}{4}\mathrm{tr}[(mD_M^{-2}(\partial\!\!\!/_E U_5^\dagger))^2]$$
$$= -\frac{N_c m^2}{16\pi^2}C_M\int_E \mathrm{d}^4x\mathrm{tr}\left[\partial_\mu U^\dagger\partial^\mu U\right]$$
$$+\frac{N_c m^2}{16\pi^2}B_M\int_E \mathrm{d}^4x\mathrm{tr}\left[(\partial^2 U^\dagger)(\partial^2 U)\right], \quad (37)$$

and

$$-\frac{N_c}{8}\mathrm{tr}[(mD_M^{-2}(\partial\!\!\!/_E U_5^\dagger))^4]$$
$$= -\frac{N_c m^4}{2\cdot 32\pi^2}Q_M\int_E \mathrm{d}^4x\mathrm{tr}\{2(\partial_\mu U^\dagger\partial^\mu U)^2$$
$$-(\partial_\mu U^\dagger\partial^\nu U)^2\}$$
$$= \frac{N_c m^4}{2\cdot 32\pi^2}Q_M\int_E \mathrm{d}^4x\left\{\mathrm{tr}[L_\mu,L_\nu]^2-2\mathrm{tr}(L_\mu^2L_\nu^2)\right\}, \quad (38)$$

in which

$$C_M = \frac{\bar{S}}{16\pi^2}\iint_{1/\Lambda^2}^{\infty}\mathrm{d}s\mathrm{d}\tau\frac{\mathrm{e}^{-M^2(s+\tau)}}{(s+\tau)^2}$$
$$= \frac{\pi}{2}\int_{1/\Lambda^2}^{\infty}\frac{\mathrm{d}s}{s}\mathrm{e}^{-\sqrt{2}s(M/\Lambda)^2}, \quad (39)$$

$$B_M = \int_{1/\Lambda^2}^{\infty}\mathrm{d}s\mathrm{d}\tau\frac{\tau\mathrm{e}^{-M^2(s+\tau)}}{4s^2}$$
$$= \frac{\Lambda^2}{M^4}\mathrm{e}^{-(M/\Lambda)^2}\left(1+\frac{M^2}{\Lambda^2}\right)\int_1^{\infty}\frac{\mathrm{d}s}{s^2}\mathrm{e}^{-s(M/\Lambda)^2}, \quad (40)$$

$$Q_M = \int_{1/\Lambda^2}^{\infty}\prod_{i=1}^{4}\mathrm{d}s_i\frac{\mathrm{e}^{-(\Sigma_i s_i)M^2}}{(\Sigma_i s_i)^2}$$
$$\approx \frac{4\pi^2}{\Lambda^4}\int_1^{\infty}\mathrm{d}ss\mathrm{e}^{-s(M/\Lambda)^2/\sqrt{2}}. \quad (41)$$

Putting (36), (37) and (38) together, one has a Skyrme-like action for (32)

$$L^{SL} = \frac{N_c m^2}{16\pi^2}C_M\mathrm{tr}\left[\partial_\mu U^\dagger\partial^\mu U\right]+\frac{N_c m^4}{64\pi^2}Q_M\{\mathrm{tr}[L_\mu,L_\nu]^2$$
$$-2\mathrm{tr}(L_\mu^2L_\nu^2)\}+\frac{N_c m^2}{16\pi^2}B_M\mathrm{tr}\left[(\partial^2 U^\dagger)(\partial^2 U)\right]$$
$$+4N_c D_S\mathrm{tr}\left[\frac{m}{2}(U+U^\dagger)\right]+WZ\text{-term}. \quad (42)$$

in which the rotation back to Minkowski space was taken. This is an effective action at low energy up to the loop level of the quark fields. The matching of the coefficients in (42) with the extended Skyrme model (13) yields

$$\varepsilon^2 = \frac{N_c m^2}{16\pi^2}\left[\frac{m^2}{2}Q_M-B_M\right],$$
$$k^2 = -\frac{N_c m^2}{16\pi^2}\left[\frac{m^2}{2}Q_M+B_M\right],$$
$$k_3 = -\frac{N_c m^2}{16\pi^2}B_M. \quad (43)$$

Since both Q_M and B_M in (43) are positive, we find

$$k^2 \leqslant 0, \quad (44)$$

which remains valid even if setting $k_3\to 0$ is taken, as shown in (43): $k^2\sim -Q_M$.

The disagreement between (12) and (44) implies either k^2 in the extended Skyrme model (1) is negligibly small and negative if we insist on the soliton picture for baryons, or the soliton picture for the baryons is beyond the approximation of QCD at the level of the quark loop.

6 Summary and concluding remarks

We studied the role of the anti-commutator term of the chiral current in a Skyrme-like model associated with the Skyrmion stability and the nucleon properties with the framework of the zero-mode quantization. It is shown that the anti-commutator term stablizes the Skyrmion soliton only when it couples in terms of a negative coupling constant while a QCD functional analysis, at the level of quark loop, supports a positive coupling constant. This means that to have a stable soliton picture for baryons such a coupling for the anti-commutator term has to be negligiblely small and negative, or that the soliton description of baryons is beyond the approximation of QCD at the level of the quark loop. We note that this does not mean that the soliton description of baryons breaks down qualitatively, rather, that the Skyrmion soliton picture effectively describes the baryons as a rough approximation of the QCD.

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