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Representations of coherent and squeezed states in an extended two-parameter Fock space

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Abstract: Recently an f-deformed Fock space which is spanned by $|n\rangle_\lambda$ was introduced. These bases are the eigenstates of a deformed non-Hermitian Hamiltonian. In this contribution, we will use rather new non-orthogonal basis vectors for the construction of coherent and squeezed states, which in special case lead to the earlier known states. For this purpose, we first generalize the previously introduced Fock space spanned by $|n\rangle_\lambda$ bases, to a new one, spanned by extended two-parameters bases $|n\rangle_{\lambda_1,\lambda_2}$. These bases are now the eigenstates of a non-Hermitian Hamiltonian $H_{\lambda_1,\lambda_2}=a^\dagger_{\lambda_1,\lambda_2}a+\frac{1}{2}$, where $a^\dagger_{\lambda_1,\lambda_2}=a^\dagger+\lambda_1 a+\lambda_2$ and a are, respectively, the deformed creation and ordinary bosonic annihilation operators. The bases $|n\rangle_{\lambda_1,\lambda_2}$ are non-orthogonal (squeezed states), but normalizable. Then, we deduce the new representations of coherent and squeezed states in our two-parameter Fock space. Finally, we discuss the quantum statistical properties, as well as the non-classical properties of the obtained states numerically.

Key words: coherent state, squeezed state, representation theory, quantum statistics, non-orthogonal bases

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1 Introduction

Orthonormal basis sets, $\{|n\rangle, n\in N, \langle m|n\rangle=\delta_{m,n}\}_{n=0}^{\infty}$, usually as the eigenvectors of hermitian Hamiltonians, are the most common basis sets and have been frequently used in the framework of the mathematical description of many areas of physics, especially in quantum mechanics. Moreover, the generalization to non-orthogonal basis vectors such as $|n\rangle_\lambda$ proposed in Ref. [1] may be preferred. The non-orthogonal basis states are recognized to be helpful in "generalized measurement", "quantum non-demolition measurement" and "quantum information theory" [2]. In the present paper, we will extend the previous work in Ref. [1] and introduce a set of two-parameter non-orthogonal bases, $\{|n\rangle_{\lambda_1,\lambda_2}\}_{n=0}^{\infty}$, which are the eigenvectors of a special deformed (non-hermitian) harmonic oscillator Hamiltonian. Also, following this idea will lead us to some new classes of generalized coherent states (CSs) and squeezed states (SSs) in the quantum optics field. Generalized CSs not only possess a beautiful mathematical structure but also can generally be useful for the description of various concepts of physics [3–5] (also see Ref. [6]). As regards the latter goal, we will construct CSs and SSs, as venerable objects in physics [7], using the bases $\{|n\rangle_{\lambda_1,\lambda_2}, n\in N\}_{n=0}^{\infty}$, instead of the usual orthonormal $\{|n\rangle\}_{n=0}^{\infty}$ and non-orthogonal $\{|n\rangle_\lambda\}_{n=0}^{\infty}$ bases in Ref. [1]. Thus, in our opinion this approach provides a more fundamental and certainly more flexible basis than that of the usual orthonormal one considered extensively in the literature, as well as the non-orthogonal basis $|n\rangle_\lambda$ in Ref. [1], for the construction of new generalized CSs and SSs [8–10]. Indeed, the explicit forms of CSs and SSs will be introduced, which contain two tunable parameters λ_1 and λ_2, by which one can obtain wide classes of states, from "standard CSs and SSs" (with $\lambda_1=0=\lambda_2$), previous non-orthogonal representations of CSs and SSs in Ref. [1] (with $\lambda_1=0$) to new two-parameter states, will be introduced here.

For a quantized harmonic oscillator (QHO), one has $H_0=a^\dagger a+\frac{1}{2}$, $[a,a^\dagger]=\hat{I}$, $(a^\dagger)^\dagger=a$, where a, $a^\dagger$, H and $\hat{I}$ are, respectively, bosonic annihilation, creation,

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Hamiltonian and unity operators. We consider the parametric harmonic oscillator, by deforming the creation operator according to the proposal in Ref. [11]

$$a^{\dagger}_{\lambda_1,\lambda_2}=a^{\dagger}+\lambda_1 a+\lambda_2\hat{I},\qquad \lambda_1,\lambda_2\in R \tag{1}$$

while the annihilation operator remains unchanged, i.e., $a_{\lambda_1,\lambda_2}=a$. So, by analogy with the harmonic oscillator, the λ_1,λ_2-Hamiltonian becomes

$$\begin{aligned}H_{\lambda_1,\lambda_2} &= \frac{1}{2}\{a^{\dagger}_{\lambda_1,\lambda_2},a\}\\ &= a^{\dagger}a+\lambda_1 a^2+\lambda_2 a+\frac{1}{2}.\end{aligned} \tag{2}$$

Note that, $(a^{\dagger}_{\lambda_1,\lambda_2})^{\dagger}\neq a_{\lambda_1,\lambda_2}$ and $H^{\dagger}_{\lambda_1,\lambda_2}\neq H_{\lambda_1,\lambda_2}$. However, as for the QHO yet we have

$$\begin{aligned}&[a,a^{\dagger}_{\lambda_1,\lambda_2}]=\hat{I},\quad [H_{\lambda_1,\lambda_2},a^{\dagger}_{\lambda_1,\lambda_2}]=a^{\dagger}_{\lambda_1,\lambda_2},\\ &[H_{\lambda_1,\lambda_2},a]=-a.\end{aligned} \tag{3}$$

Thus, the new set $\left\{a,a^{\dagger}_{\lambda_1,\lambda_2},a^{\dagger}_{\lambda_1,\lambda_2}a,\hat{I}\right\}$ still satisfies the Weyl-Heisenberg algebra as in the case of the QHO.

One of us has shown in Ref. [8] that a large class of generalized CSs can be obtained by changing the bases in the underlying Hilbert space. At this stage, we would like to illustrate that the particular deformation which we employed in this paper is a special case of the general scheme for the representation theory of CSs, and was introduced in Ref. [8]. For futher clarification, we will explain briefly the setting. Let $\mathcal{H}$ be a Hilbert space and T, T^{-1} be operators densely defined and closed on $\mathcal{D}(T)$ and $\mathcal{D}(T^{-1})$, respectively, and $F=T^{\dagger}T$. Two new Hilbert spaces $\mathcal{H}_F$, $\mathcal{H}_{F^{-1}}$ are the completions of the sets $\mathcal{D}(T)$ and $\mathcal{D}(T^{\dagger^{-1}})$ with the scalar product $\langle f|g\rangle_F=\langle f|Fg\rangle_{\mathcal{H}}$ and $\langle f|g\rangle_{F^{-1}}=\langle f|F^{-1}g\rangle_{\mathcal{H}}$, respectively. Considering the generators of the Weyl-Heisenberg algebra, as a basis on $\mathcal{H}$, one may obtain the transformed generators on $\mathcal{H}_F$, as follows

$$a_F=T^{-1}aT,\quad a^{\dagger}_F=T^{-1}a^{\dagger}T,\quad N_F=T^{-1}NT. \tag{4}$$

A similar argument may be followed for Hilbert space $\mathcal{H}_{F^{-1}}$. If we take the non-unitary T-operator as

$$T_{n,\lambda_1,\lambda_2}=\xi_n \mathrm{e}^{-\frac{1}{2}\lambda_1 a^2-\lambda_2 a}, \tag{5}$$

it is easy to check that the following deformed operators may be obtained

$$a_F=T^{-1}_{n,\lambda_1,\lambda_2}aT_{n,\lambda_1,\lambda_2}=a, \tag{6}$$

$$\begin{aligned}a^{\dagger}_F &= T^{-1}_{n,\lambda_1,\lambda_2}a^{\dagger}T_{n,\lambda_1,\lambda_2}\\ &= \mathrm{e}^{\frac{1}{2}\lambda_1 a^2\lambda_2 a}a^{\dagger}\mathrm{e}^{-\frac{1}{2}\lambda_1 a^2-\lambda_2 a}\\ &= a^{\dagger}+\lambda_1 a+\lambda_2\hat{I},\end{aligned} \tag{7}$$

and

$$\begin{aligned}N_F &= T^{-1}_{n,\lambda_1,\lambda_2}a^{\dagger}aT_{n,\lambda_1,\lambda_2}\\ &= \mathrm{e}^{\frac{1}{2}\lambda_1 a^2+\lambda_2 a}a^{\dagger}a\,\mathrm{e}^{-\frac{1}{2}\lambda_1 a^2-\lambda_2 a}\\ &= a^{\dagger}_{\lambda_1,\lambda_2}a,\end{aligned} \tag{8}$$

where ξ_n is an appropriate normalization factor, which will be determined in the continuation. In this manner, we have established the fundamental place of the particular kind of deformation proposed in Ref. [11] and will be used by us in the present work in the general framework of the representation theory of CSs in non-orthogonal basis.

In terms of $T_{n,\lambda_1,\lambda_2}$, we can rewrite the Hamiltonian in Eq. (2), as $H_{\lambda_1,\lambda_2}=T^{-1}_{n,\lambda_1,\lambda_2}H_0T_{n,\lambda_1,\lambda_2}$, where H_0 is the QHO Hamiltonian. According to the general formalism in Ref. [8], the normalized non-orthogonal bases in the new Hilbert space can be obtained as

$$\begin{aligned}|n\rangle_{\lambda_1,\lambda_2} &= T_{n,\lambda_1,\lambda_2}|n\rangle=\xi_n\sum_{r=0}^{n}\sum_{k=0}^{\left[\frac{n-r}{2}\right]}\\ &\times\left(\frac{1}{2}\right)^k\frac{\lambda_1^k\lambda_2^{n-r-2k}}{k!(n-r-2k)!}\sqrt{\frac{n!}{r!}}|r\rangle,\end{aligned} \tag{9}$$

where we have used $T_{n,\lambda_1,\lambda_2}$ in Eq. (5), $[m]$ denotes the integer part of m, and ξ_n may be derived with the normalization condition, ${}_{\lambda_1,\lambda_2}\langle n|n\rangle_{\lambda_1,\lambda_2}=1$ as

$$\begin{aligned}\xi_n &= \left[\sum_{r=0}^{n}\frac{n!}{r!}\lambda_2^{2(n-r)}\sum_{k=0}^{\left[\frac{n-r}{2}\right]}\sum_{k'=0}^{\left[\frac{n-r}{2}\right]}\left(\frac{1}{2}\right)^{k+k'}\right.\\ &\left.\times\frac{\lambda_1^{k+k'}\lambda_2^{-2(k+k')}}{k!(n-r-2k)!k'!(n-r-2k')!}\right]^{-\frac{1}{2}}.\end{aligned} \tag{10}$$

Eq. (9) suggests that every state $|n\rangle_{\lambda_1,\lambda_2}$ in this non-orthogonal Hilbert space can be regarded as a special finite superposition of $|0\rangle,|1\rangle,...,|n\rangle$ in the standard Fock space. In the limit $\lambda_1\to 0$, ξ_n reduces to the normalization factor $[L^0_n(-\lambda^2)]^{-\frac{1}{2}}$ in one-parameter states ($\lambda_1=\lambda=\lambda_1$) introduced in Ref. [1] ($L^0_n(x)$ are the Laguerre polynomials of n the order) and in the limit $\lambda_1,\lambda_2\to 0$ it reduces to unity.

The non-Hermitian Hamiltonian H_{λ_1,λ_2}, is isospectral with H_0 [11], i.e.,

$$H_{\lambda_1,\lambda_2}|n\rangle_{\lambda_1,\lambda_2}=E_{n,\lambda_1,\lambda_2}|n\rangle_{\lambda_1,\lambda_2}, \tag{11}$$

where $E_{n,\lambda_1,\lambda_2}=E_n=n+\dfrac{1}{2}$ and $n=0,1,2,\cdots$. Also the actions of a and $a^{\dagger}_{\lambda_1,\lambda_2}$ on λ_1, λ_2-bases are as follows

$$a|n\rangle_{\lambda_1,\lambda_2} = \frac{\xi_n}{\xi_{n-1}}\sqrt{n}|n-1\rangle_{\lambda_1,\lambda_2}, \quad (12)$$

$$a^\dagger_{\lambda_1,\lambda_2}|n\rangle_{\lambda_1,\lambda_2} = \frac{\xi_n}{\xi_{n+1}}\sqrt{n+1}|n+1\rangle_{\lambda_1,\lambda_2}. \quad (13)$$

As it may be expected, in the limit $\lambda_1 \to 0$, we recover the results of the λ-states in Ref. [1] and in the limit $\lambda_1,\ \lambda_2 \to 0$, we get the usual actions of ladder operators of a harmonic oscillator. The inner product of the states in Eq. (9), also reads as

$$ {}_{\lambda_1,\lambda_2}\langle m|n\rangle_{\lambda_1,\lambda_2} = \sqrt{m!n!}\,\xi_n\xi_m \sum_{r=0}^{\min(m,n)} \frac{\lambda_2^{m+n-2r}}{r!} \times \sum_{k=0}^{\left[\frac{m-r}{2}\right]}\sum_{j=0}^{\left[\frac{n-r}{2}\right]} \left(\frac{1}{2}\right)^{k+j} \times \frac{\lambda_1^{k+j}\lambda_2^{-2(k+j)}}{k!(m-r-2k)!j!(n-r-2j)!}. \quad (14)$$

Let us end this section by mentioning some of the interesting and important points we may further conclude.

a) It is important for our further work to establish that the non-orthogonal states $|n\rangle_{\lambda_1,\lambda_2}$ which can be regarded as the bases of a new Fock space. For this purpose, the necessary and sufficient conditions mentioned for a one-dimensional quantum Fock space in Ref. [12] will be investigated. In our case, these conditions are, briefly, (i) existence of a vacuum state, such that $a|0\rangle = 0$, note that we have considered $|0\rangle_{\lambda_1,\lambda_2} = |0\rangle$, (ii) $\langle 0|aa^\dagger_{\lambda_1,\lambda_2}|0\rangle > 0$, (iii) $[aa^\dagger_{\lambda_1,\lambda_2}, a^\dagger_{\lambda_1,\lambda_2}a] = 0$ (in Ref. [1] there is a typing error) and $aa^\dagger_{\lambda_1,\lambda_2} \neq a^\dagger_{\lambda_1,\lambda_2}a$.

b) $a^\dagger_{\lambda_1,\lambda_2}a|n\rangle_{\lambda_1,\lambda_2} \equiv N_{\lambda_1,\lambda_2}|n\rangle_{\lambda_1,\lambda_2} = n|n\rangle_{\lambda_1,\lambda_2}$, so $\hat{N}_{\lambda_1,\lambda_2}$ can be regarded as the number operator in the new Fock space. Moreover, from this equation, we see that $(n+\lambda_1 a^2+\lambda_2 a)|n\rangle_{\lambda_1,\lambda_2} = n|n\rangle_{\lambda_1,\lambda_2}$, which indicates simply the ladder operator formalism [13] of the state $|n\rangle_{\lambda_1,\lambda_2}$.

c) By Eqs. (12) and (13), we obtain respectively

$$a^n|n\rangle_{\lambda_1,\lambda_2} = \xi_n\sqrt{n!}|0\rangle_{\lambda_1,\lambda_2}, \quad (15)$$

$$|n\rangle_{\lambda_1,\lambda_2} = \frac{\xi_n}{\sqrt{n!}}(a^\dagger_{\lambda_1,\lambda_2})^n|0\rangle_{\lambda_1,\lambda_2}, \quad (16)$$

which will be helpful for our next calculations.

2 Construction of CSs in $|n\rangle_{\lambda_1,\lambda_2}$ basis

Using the algebraic definition of CSs as the eigenstates of the annihilation operator [14, 15]

$$a|\alpha,\lambda_1,\lambda_2\rangle = \alpha|\alpha,\lambda_1,\lambda_2\rangle, \quad (17)$$

we want to construct the CSs in the new deformed bases introduced in Eq. (9). As usual, expanding $|\alpha,\lambda_1,\lambda_2\rangle$ in terms of $|n\rangle_{\lambda_1,\lambda_2}$ bases as $|\alpha,\lambda_1,\lambda_2\rangle = \sum_{n=0}^{\infty} C_n|n\rangle_{\lambda_1,\lambda_2}$, setting in Eq. (17) and using Eq. (12), we finally get

$$\sum_{n=0}^{\infty} C_n\frac{\xi_n}{\xi_{n-1}}\sqrt{n}|n-1\rangle_{\lambda_1,\lambda_2} = \alpha\sum_{n=0}^{\infty} C_n|n\rangle_{\lambda_1,\lambda_2}. \quad (18)$$

The coefficients C_n will be obtained as

$$C_n = C_0\frac{\alpha^n}{\sqrt{n!}\xi_n}, \quad (19)$$

where we have used the fact that $\xi_0 = 1$. For the normalization condition of the state $|\alpha,\ \lambda_1,\ \lambda_2\rangle$ one yields

$$C_0 = \exp\left[-\frac{1}{2}\lambda_1\Re(\alpha^2) - \lambda_2\Re(\alpha) - \frac{|\alpha|^2}{2}\right], \quad (20)$$

where $\Re(\alpha)$ is the real part of α. Finally, the normalized $\lambda_1,\ \lambda_2$-CS takes the form

$$|\alpha,\lambda_1,\lambda_2\rangle = \exp\left[-\frac{1}{2}\lambda_1\Re(\alpha^2) - \lambda_2\Re(\alpha) - \frac{|\alpha|^2}{2}\right] \times \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}\xi_n}|n\rangle_{\lambda_1,\lambda_2}, \quad (21)$$

which is a new representation of CSs in $|n\rangle_{\lambda_1,\lambda_2}$ basis. Their inner product, which allows over-completeness, can be expressed as

$$\langle\alpha,\lambda_1,\lambda_2|\beta,\lambda_1,\lambda_2\rangle = N_{\alpha,\beta,\lambda_1,\lambda_2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha^{*m}\alpha^n \times \sum_{r=0}^{\min(m,n)}\frac{\lambda_2^{m+n-2r}}{r!}\sum_{k=0}^{\left[\frac{m-r}{2}\right]}\sum_{j=0}^{\left[\frac{n-r}{2}\right]}\left(\frac{1}{2}\right)^{k+j} \times \frac{\lambda_1^{k+j}\lambda_2^{-2(k+j)}}{k!(m-r-2k)!j!(n-r-2j)!}, \quad (22)$$

with

$$N_{\alpha,\beta,\lambda_1,\lambda_2} = \exp\left[-\frac{1}{2}\lambda_1(\Re(\alpha^2)+\Re(\beta^2)) - \lambda_2(\Re(\alpha) + \Re(\beta)) - \frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2}\right]. \quad (23)$$

Now, we imply that, by Eq. (16), the CSs in Eq. (21) can be expressed in terms of the lowest eigenstate of H_{λ_1,λ_2} as

$$|\alpha,\lambda_1,\lambda_2\rangle = \exp\left[-\frac{1}{2}\lambda_1\Re(\alpha^2)-\lambda_2\Re(\alpha) -\frac{|\alpha|^2}{2}\right]\mathrm{e}^{\alpha a^\dagger_{\lambda_1,\lambda_2}}|0\rangle. \tag{24}$$

By using the BCH lemma, Eq. (24) can be rewritten as

$$\begin{aligned}|\alpha,\lambda_1,\lambda_2\rangle &= C_0\mathrm{e}^{\frac{1}{2}\lambda_1\alpha^2+\lambda_2\alpha+\frac{|\alpha|^2}{2}}|\alpha\rangle=\mathrm{e}^{-\frac{1}{2}\lambda_1\Re(\alpha^2)-\lambda_2\Re(\alpha)-\frac{|\alpha|^2}{2}}\mathrm{e}^{\frac{1}{2}\lambda_1\left(\Re(\alpha^2)+\mathrm{i}\Im(\alpha^2)\right)+\lambda_2(\Re(\alpha)+\mathrm{i}\Im(\alpha))+\frac{|\alpha|^2}{2}}|\alpha\rangle\\ &= D_{\lambda_1,\lambda_2}(\alpha)|0\rangle,\end{aligned} \tag{25}$$

where we assumed that $D_{\lambda_1,\lambda_2}(\alpha) = \mathrm{e}^{\mathrm{i}\left(\frac{1}{2}\lambda_1\Im(\alpha^2)+\lambda_2\Im(\alpha)\right)}D(\alpha)$ and the imaginary part of x is denoted by $\Im(x)$. Since $D(\alpha)|0\rangle = |\alpha\rangle$, which is the standard CS, we conclude that $|\alpha,\lambda_1,\lambda_2\rangle$ is identical to $|\alpha\rangle$, up to a phase factor and therefore $|\alpha,\lambda_1,\lambda_2\rangle=|\alpha\rangle$ whenever $\alpha\in R$; a result that may be expected from the eigenvalue Eq. (17). So, obviously, there is no problem with resolution of the identity

$$\int_C \mathrm{d}\mu(\alpha)|\alpha,\lambda_1,\lambda_2\rangle\langle\alpha,\lambda_1,\lambda_2|=\hat{I} \tag{26}$$

with $\mathrm{d}\mu(\alpha)\doteq\frac{1}{\pi}\mathrm{d}^2\alpha$.

We would like to emphasize that all we have done in this section is the derivation of the explicit form of canonical CSs in a deformed Fock space $|n\rangle_{\lambda_1,\lambda_2}$, which are non-orthogonal, and we called them a new representation of canonical CSs. As another result, we conclude that by a particular superposition of $|n\rangle_{\lambda_1,\lambda_2}$ bases (which themselves exhibit squeezing [11]), we have obtained canonical CSs. Note that, both the orthogonal bases $|n\rangle$ which commonly have been used in the construction of CSs, and the (one- and two-parameter) non-orthogonal bases introduced (and applied) by us, are non-classical states. But, orthogonal bases have sub-Poissonian statistics (due to the fact that their Mandel parameters are equal to -1) without squeezing, while non-orthogonal bases exhibit squeezing.

The dynamical evolution of the λ_1,λ_2-CSs in $|n\rangle_{\lambda_1,\lambda_2}$ basis, may be simply obtained due to the linear spectrum feature of H_{λ_1,λ_2}

$$\begin{aligned}U(t)|\alpha,\lambda_1,\lambda_2\rangle &= \exp\left[-\frac{1}{2}\lambda_1\Re(\alpha^2)-\lambda_2\Re(\alpha)-\frac{|\alpha|^2}{2}\right]\\ &\quad\times\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}\xi_n}\mathrm{e}^{-\mathrm{i}tH_{\lambda_1,\lambda_2}}|n\rangle_{\lambda_1,\lambda_2}\\ &= \mathrm{e}^{-\frac{\mathrm{i}t}{2}}|\alpha(t),\lambda_1,\lambda_2\rangle,\end{aligned} \tag{27}$$

where we have used $\alpha(t)\equiv\alpha\mathrm{e}^{-\mathrm{i}t}$. This means the time evolution of λ_1, λ_2-CSs in this non-orthogonal bases remains coherent in the same bases for all time (temporal stability).

3 Construction of SSs in λ_1,λ_2 basis

According to the statement of Solomon and Katriel [16], conventional SSs are obtained by the action of a linear combination of creation and annihilation operators on an arbitrary state. Now, by generalizing this procedure to $a^\dagger_{\lambda_1,\lambda_2}$ introduced in Eq. (1) and $a_{\lambda_1,\lambda_2}=a$ of the deformed oscillator we have

$$(a-\eta a^\dagger_{\lambda_1,\lambda_2})|\eta,\lambda_1,\lambda_2\rangle=0 \quad \eta\in C. \tag{28}$$

This equation for $\lambda_1,\lambda_2 = 0$ leads to the squeezed vacuum states

$$|\eta\rangle=C_0\exp\left[\frac{\eta(a^\dagger)^2}{2}\right]|0\rangle,$$

where C_0 is a suitable normalization coefficient. But, in general, from Eq. (28) and by following a similar procedure to that in a previous section for $\lambda_1,\lambda_2\neq 0$, we will arrive at a new representation for λ_1,λ_2-SSs as

$$|\eta,\lambda_1,\lambda_2\rangle=C_0\sum_{n=0}^{\infty}\frac{\eta^n}{\xi_{2n}}\sqrt{\frac{(2n-1)!!}{(2n)!!}}|2n\rangle_{\lambda_1,\lambda_2}, \tag{29}$$

where ξ_{2n} may be obtained from Eq. (10). For the normalization factor, one may get

$$\begin{aligned}C_0 &= \left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\eta^n\eta^{*m}(2n-1)!!(2m-1)!!\right.\\ &\quad\times\sum_{r=0}^{\min(2m,2n)}\frac{\lambda_2^{2m+2n-2r}}{r!}\sum_{k=0}^{[\frac{2m-r}{2}]}\sum_{j=0}^{[\frac{2n-r}{2}]}\left(\frac{1}{2}\right)^{k+j}\\ &\quad\left.\times\frac{\lambda_1^{k+j}\lambda_2^{-2(k+j)}}{k!(2m-r-2k)!j!(2n-r-2j)!}\right]^{-\frac{1}{2}}.\end{aligned} \tag{30}$$

These λ_1, λ_2-SSs are normalizable, provided that the coefficients C_0 are nonzero and finite. From the above discussion, we immediately conclude that translating $|\eta,\lambda_1,\lambda_2\rangle$ (in $|n\rangle_{\lambda_1,\lambda_2}$ basis) to a state in the standard Fock space $|n\rangle$ does not coincide with the one we previously referred to in the beginning of this section. This is due to the fact that, unlike the introduced CSs in Eq. (21), in obtaining the λ_1, λ_2-

SSs both the annihilation and the deformed creation operators are contributed (see Eq. (28)).

4 Non-classical properties

In this section, we will introduce the non-classical criteria which we will consider in our numerical results.

4.1 Mandel parameter

The standard CSs possess the Poissonian distribution as $P(n)=|\langle n|\alpha\rangle|^2=\mathrm{e}^{-|\alpha|^2}|\alpha|^{2n}/n!$, whose mean and variance are equal to $|\alpha|^2$. Similarly, in the case of our λ_1, λ_2-CSs we can define:

$$P_{\lambda_1,\lambda_2}(n)=|_{\lambda_1,\lambda_2}\langle n|\alpha,\lambda_1,\lambda_2\rangle|^2, \tag{31}$$

which may be interpreted as the probability of finding the states $|\alpha,\,\lambda_1,\,\lambda_2\rangle$ in a non-orthogonal $|n\rangle_{\lambda_1,\lambda_2}$ basis. So,

$$\begin{aligned} {}_{\lambda_1,\lambda_2}\langle n\rangle_{\lambda_1,\lambda_2} &= \sum_{n=0}^{\infty} nP_{\lambda_1,\lambda_2}(n),\\ {}_{\lambda_1,\lambda_2}\langle n^2\rangle_{\lambda_1,\lambda_2} &= \sum_{n=0}^{\infty} n^2P_{\lambda_1,\lambda_2}(n). \end{aligned} \tag{32}$$

To examine the statistics of the states, Mandel's Q-parameter is widely used, which characterizes the quantum statistics of the states of the field. This parameter has been defined as $Q=\langle n^2\rangle-\langle n\rangle^2/\langle n\rangle-1$ [17]. To check Poissonian, sub-Poissonian or supper-Poissonian statistics, as in Ref. [1], we can introduce a further extension definition for Mandel parameter [17] as follows:

$$q_{\lambda_1,\lambda_2}=\frac{{}_{\lambda_1,\lambda_2}\langle n^2\rangle_{\lambda_1,\lambda_2}-{}_{\lambda_1,\lambda_2}\langle n\rangle^2_{\lambda_1,\lambda_2}}{{}_{\lambda_1,\lambda_2}\langle n\rangle_{\lambda_1,\lambda_2}}-1, \tag{33}$$

where $n=a^\dagger a$.

We can also introduce an alternative deformed Mandel parameter, using the deformed number operator $N_{\lambda_1,\lambda_2}=a^\dagger_{\lambda_1,\lambda_2}a$ as:

$$Q_{\lambda_1,\lambda_2}=\frac{{}_{\lambda_1,\lambda_2}\langle N^2_{\lambda_1,\lambda_2}\rangle_{\lambda_1,\lambda_2}-{}_{\lambda_1,\lambda_2}\langle N_{\lambda_1,\lambda_2}\rangle^2_{\lambda_1,\lambda_2}}{{}_{\lambda_1,\lambda_2}\langle N_{\lambda_1,\lambda_2}\rangle_{\lambda_1,\lambda_2}}-1. \tag{34}$$

When $Q_{\lambda_1,\lambda_2}=0$, the states exhibit Poissonian, $Q_{\lambda_1,\lambda_2}<0$ sub-Poissonian, and $Q_{\lambda_1,\lambda_2}>0$ super-Poissonian statistics in $|n\rangle_{\lambda_1,\lambda_2}$ basis. The same argument may be followed for q_{λ_1,λ_2}-parameter. It is interesting to notice that, while we have the lower bound $Q_{\lambda_1,\lambda_2}\geqslant-1$ there is no such boundary for q_{λ_1,λ_2}. Strictly speaking, $Q_{\lambda_1,\lambda_2}=0$ if the deformed number operator is considered, the situation that is the same as an ordinary Q-parameter in orthonormal basis $|n\rangle$ when $n=a^\dagger a$.

4.2 Quadrature squeezing

Quadrature squeezing is another property that the quantum states may possess. Firstly, we define

$$x=\frac{1}{\sqrt{2}}(a+a^\dagger),\quad p=\frac{1}{\mathrm{i}\sqrt{2}}(a-a^\dagger)$$

and $(\Delta x_i)^2=\langle x_i^2\rangle-\langle x_i\rangle^2$ where $x_i=x,p$. Uncertainty relation in x is obtained as

$$\begin{aligned}(\Delta x)^2 \;=\; &\frac{1}{2}[1+(1-2\lambda_1)\langle a^2\rangle+\langle a^{\dagger2}\rangle+2\langle a^\dagger_{\lambda_1,\lambda_2}a\rangle\\ &-2\lambda_2\langle a\rangle-\langle a\rangle^2-\langle a^\dagger\rangle^2-2\langle a\rangle\langle a^\dagger\rangle].\end{aligned} \tag{35}$$

Similarly, for p-quadrature, one has

$$\begin{aligned}(\Delta p)^2 \;=\; &\frac{1}{2}[1-(1+2\lambda_1)\langle a^2\rangle-\langle a^{\dagger2}\rangle+2\langle a^\dagger_{\lambda_1,\lambda_2}a\rangle\\ &-2\lambda_2\langle a\rangle+\langle a\rangle^2+\langle a^\dagger\rangle^2-2\langle a\rangle\langle a^\dagger\rangle],\end{aligned} \tag{36}$$

where all of the expectation values should be calculated with respect to the λ_1, λ_2 coherent and squeezed states.

5 Numerical results and conclusion

From Figs. 1 and 2, it is seen that the squeezing effect does not occur for the generalized CSs $|\alpha,\,\lambda_1,\lambda_2\rangle$. But, as it is observed, with increasing λ_1, for fixed λ_2-parameters the uncertainties in both quadratures tend to 0.5 (the uncertainty of vacuum or canonical CSs). The same criterion for the SSs $|\eta,\,\lambda_1,\,\lambda_2\rangle$ is investigated in Figs. 3 and 4. The squeezing effect may be seen in p-quadrature for some fixed values of λ_1.

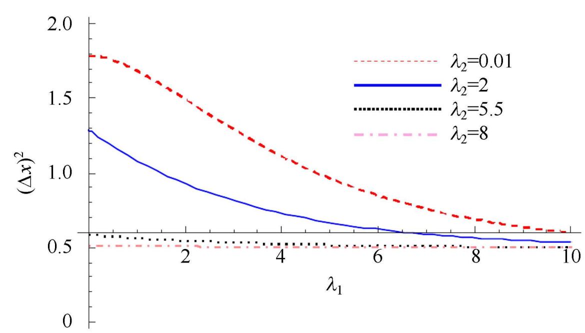

Fig. 1. The graph of $(\Delta x)^2$ for CSs $|\alpha,\,\lambda_1,\,\lambda_2\rangle$, as a function of λ_1, for λ_2=0.01, 2, 5.5, 8, $\alpha=0.8$.

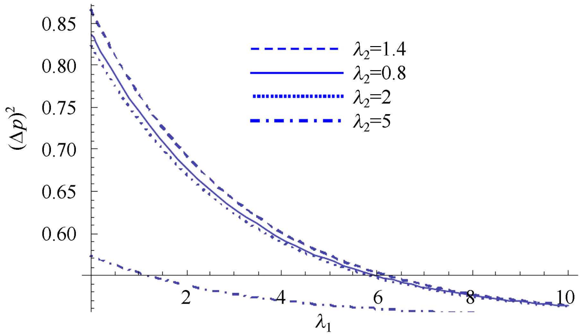

Fig. 2. The graph of $(\Delta p)^2$ for CSs $|\alpha,\lambda_1,\lambda_2\rangle$, as a function of λ_1, for λ_2 =0.8, 1.4, 2, 5, $\alpha=0.8$.

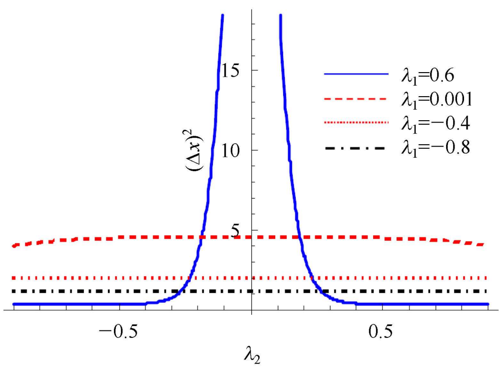

Fig. 3. The graph of $(\Delta x)^2$ for SSs $|\eta,\ \lambda_1,\ \lambda_2\rangle$, as a function of λ_2, for $\lambda_1=-0.8$, -0.4, 0.001, 0.6, $\eta=0.8$.

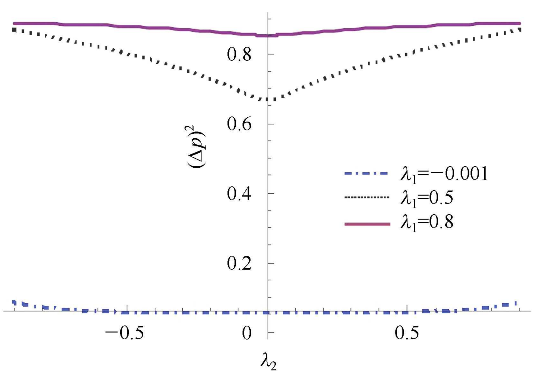

Fig. 4. The graph of $(\Delta p)^2$ for SSs $|\eta,\ \lambda_1,\ \lambda_2\rangle$, as a function of λ_2, for $\lambda_1=-0.001$, 0.5, 0.8, $\eta=0.8$.

In Figs. 5 and 6 we displayed the Mandel parameter Q_{λ_1,λ_2} as a function of λ_1 (for fixed values of λ_2) corresponding to, respectively, CSs $|\alpha,\lambda_1,\lambda_2\rangle$ and SSs $|\eta,\ \lambda_1,\ \lambda_2\rangle$. The negativity of this quantity in some regions is observed from the two figures, showing the non-classicality of the introduced states. Figs. 7 and 8 show the deformed Mandel parameter q_{λ_1,λ_2} defined in (33) for, again, CSs $|\alpha,\ \lambda_1,\ \lambda_2\rangle$ and SSs $|\eta,\ \lambda_1,\ \lambda_2\rangle$, respectively. The negativity of this parameter in some regions of λ_2 (for fixed values of λ_1) or in some regions of λ_1 (for fixed values of λ_2) clearly shows the non-classicality features of these states.

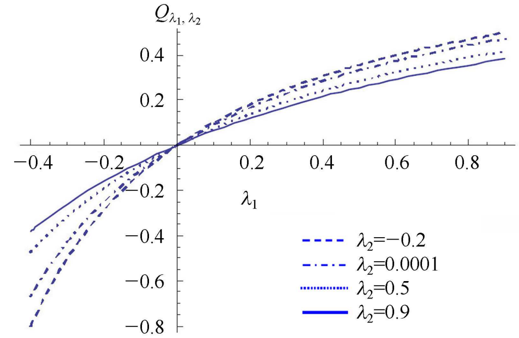

Fig. 5. The graph of Mandel parameter Q_{λ_1,λ_2} for CSs $|\alpha,\ \lambda_1,\ \lambda_2\rangle$ as a function of λ_1, for $\lambda_2=-0.2$, 0.0001, 0.5, 0.9, $\alpha=2$.

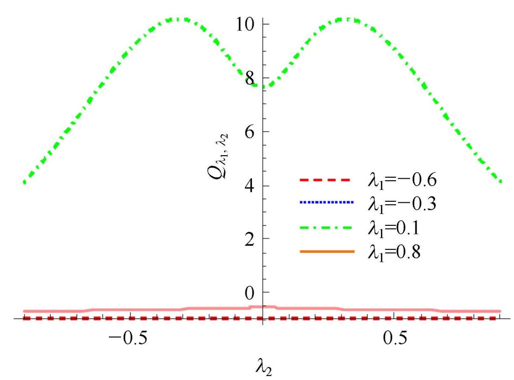

Fig. 6. The graph of Mandel parameter Q_{λ_1,λ_2} for SSs $|\eta,\ \lambda_1,\ \lambda_2\rangle$ as a function of λ_2, for $\lambda_1=-0.6$, -0.3, 0.1, 0.8, $\eta=0.8$.

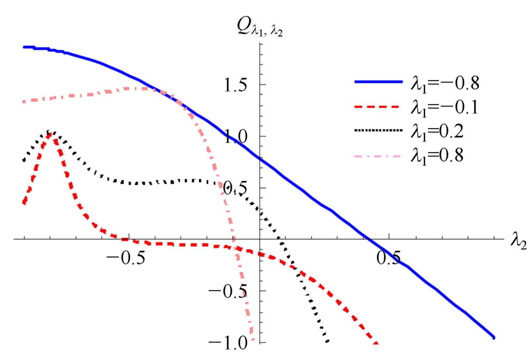

Fig. 7. The graph of Mandel parameter q_{λ_1,λ_2} for CSs $|\alpha,\ \lambda_1,\ \lambda_2\rangle$ as a function of λ_2, for $\lambda_1=-0.8$, -0.1, 0.2, 0.8, $\alpha=0.8$.

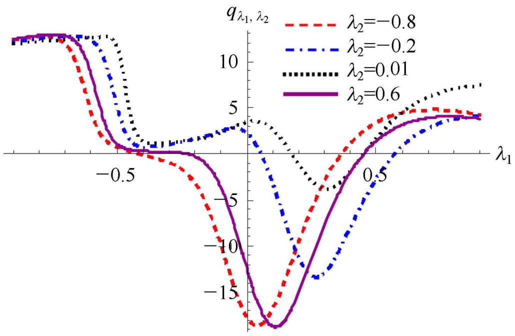

Fig. 8. The graph of Mandel parameter q_{λ_1,λ_2} for SSs $|\eta,\ \lambda_1,\ \lambda_2\rangle$ as a function of λ_1, for $\lambda_2=-0.8,\ -0.2,\ 0.01,\ 0.6,\ \eta=0.8$.

In conclusion, in this paper we have used the non-orthogonal squeezed states [11] $|n\rangle_{\lambda_1,\lambda_2}$, which are the eigenstates of the non-Hermitian Hamiltonian H_{λ_1,λ_2}, as the bases for the construction of our CSs and SSs. We illustrated that these states can be regarded as the bases of our infinite dimensional Hilbert space with a defined scalar product. Our motivation for this consideration is the increased generality and flexibility of the non-orthogonal basis: $\{|n\rangle_{\lambda_1,\lambda_2}, n\in N\}_{n=0}^{\infty}$ rather than orthogonal one $\{|n\rangle, n\in N\}_{n=0}^{\infty}$, and even the earlier one-parameter non-orthogonal bases $\{|n\rangle_{\lambda}, n\in N\}_{n=0}^{\infty}$, established by one of us [1]. Also, the place of the deformations which lead to the outlined bases in the general framework of the representation theory of CSs is deeply established. Then, we concluded that, by some special superposition of the deformed Fock space, we can obtain new representations of CSs $|\alpha,\ \lambda_1,\ \lambda_2\rangle$, as well as SSs $|\eta,\ \lambda_1,\ \lambda_2\rangle$ in the new two-parameters basis. Interestingly, in this Fock space, we obtained a set of new physical aspects; for instance, squeezing and sub-Poissonian statistics as some non-classical features. It is noticeable that in the canonical CSs, which is a composition of orthogonal bases, neither of these features may be observed. So, as it is implied in Ref. [8], in some classes of generalized CSs, in which the non-classicality signs are revealed (nonclassical states), one may find their root in the non-orthogonality of basis, mathematically. Indeed, transforming the orthogonal basis to a non-orthogonal one in the "canonical coherent states" results in the appearance of non-classicality features.

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