A study of transverse charge density of pions in relativistic quantum mechanics*

DONG Yu-Bing(董宇兵)[1,2;1)] WANG Yi-Zhan(王翼展)[1]

[1] Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China
[2] Theoretical Physics Center for Science Facilities (TPCSF), CAS, China

Abstract: The transverse charge density of pions is calculated based on relativistic quantum mechanics, where the pion is regarded as a quark-antiquark bound state. Corrections from the two spin-1/2 constituents and from the wave function of a quark and antiquark inside the bound system are discussed. The calculated results are compared to the results with a realistic effective Lagrangian approach as well as to that with a simple covariant model where the pion is regarded as a composite system with two scalar particles.

Key words: transverse charge density, pion, compositeness condition, relativistic quantum mechanics

PACS: 13.40.Gp, 12.39.Ki, 14.40.Aq **DOI:** 10.1088/1674-1137/35/8/003

1 Introduction

It is well known that the properties of the light pseudo-scalar mesons are essential to understand the phenomena of low-energy QCD. The pion meson, as the simplest elementary particle, has been widely investigated before. In particular, its electromagnetic (EM) form factor of F_1 has been studied in much work, since it can test both the physical ingredients and the theoretical models simultaneously. So far, some accurate data for F_1 at low Q^2 and also for a charge radius of r_π^2 have been determined. For example, several experiments have been carried out in order to measure the charge pion form factor in the past [1–3]. The experiments at JLab to measure the pion form factor in the space-like and low Q^2 region with higher accuracy are being planned [4]. In addition, experiments for the pion EM form factor in the time-like region are also in progress [5, 6]. It is expected that those measurements with high precision may provide good discrimination among the various theoretical approaches.

Apart from the EM form factor of the pion, the study of pion transverse density $\rho(b)$ has also been of great interest recently. This transverse density stands for the two-dimensional Fourier transform of the EM form factor F_1 and represents the charge density located at the transverse separation b from the center of transverse momentum in the infinite momentum frame [7–11]. It is pointed out that this two-dimensional density can directly relate to the matrix element of a density operator. However, the conventional three-dimensional Fourier transforms of the form factors cannot because the initial and final momenta are different and one cannot boost the initial and final states to the rest frame simultaneously [11]. In a recent paper by Miller [12], a simple covariant toy model is applied to study the transverse charge density of a bound state. This simple model is based on the idea of Weinberg and others [13]. However, only the scalar constituents, other than the spin-1/2 ones, are considered. Moreover, in Ref. [12], no correlation function, which stands for the distribution of the constituents inside the bound system, is taken into account. There is an effective Lagrangian approach that can consider the above two physical ingredients. It is based on the same idea of Ref. [13]. It should be mentioned that the approach has been extensively applied to the study of the new resonances in the hadronic molecular scenario as well as the study of the structures of the nucleon, deuteron and pion [14, 15]. In those papers, the correlation functions as well as

Received 29 November 2010, Revised 18 February 2011
* Supported by National Natural Science Foundation of China (10775148, 10975146, 11035006)
1) E-mail: dongyb@ihep.ac.cn

the properties of the spin-1/2 constituents of the system are considered. In Ref. [16], the effective Lagrangian approach is employed in the calculation of the transverse charge density of the pion.

In this work, we will study the transverse density of the charged pion based on an approach of relativistic quantum mechanics (RQM). RQM has been studied for a long time [17]. Three forms RQM, front form, instant form and point form, have already been applied to the study of the charged pion EM form factors in the literature. The recent analyses of Ref. [18] show that the divergence among the three forms of RQM in the description of the pion EM form factor is due to the problem of the space-time translation invariance. If this is restored by hand, the three forms of RQM will give a very similar result for the pion EM form factor and also a result similar to that from RQM with a dispersion relation. It should be mentioned that in RQM, the internal quark line is always set to be on the mass shell and this is not the case in field theory. In this work, RQM is employed to study transverse density and the estimated result is compared with the calculations in the effective field theory of Refs. [12] and [16].

2 Calculation of the pion and its transverse charge density based on RQM

To calculate the form factor of the charged pion ($I^G(J^P)=1^-(0^-)$), we have (see Fig. 1)

$$F(Q^2)(P_\mathrm{f}+P_\mathrm{i})^\mu=\langle P_\mathrm{f}\,|\,J^\mu\,|\,P_\mathrm{i}\rangle. \tag{1}$$

Then, the expression of the pion form factor in the front form of the relativistic quantum mechanics with $q^+=0$ is [18]

$$F_1(Q^2)=\frac{1}{\pi N}\int\mathrm{d}^2R\int_0^1\frac{\mathrm{d}x}{x(1-x)}\frac{I^0_\omega}{\tilde{I}^0_\omega}\psi(s_\mathrm{i})\psi^*_\mathrm{f}(s_\mathrm{f}), \tag{2}$$

where the invariant variables

$$\begin{aligned}s_\mathrm{i}\;&=\;(p+p_\mathrm{i})^2=\frac{m^2+p^2_{\mathrm{i}\perp}}{1-x}+\frac{m^2+p^2_\perp}{x}-P^2_{\mathrm{i}\perp}\\&=\;\frac{m^2+\vec{R}^2+x\vec{R}\cdot\vec{Q}_\perp+\dfrac{x^2}{4}\vec{Q}^2}{x(1-x)},\end{aligned} \tag{3}$$

$$\begin{aligned}s_\mathrm{f}\;&=\;(p+p_\mathrm{f})^2=\frac{m^2+p^2_{\mathrm{f}\perp}}{1-x}+\frac{m^2+p^2_\perp}{x}-P^2_{\mathrm{f}\perp}\\&=\;\frac{m^2+\vec{R}^2-x\vec{R}\cdot\vec{Q}_\perp+\dfrac{x^2}{4}\vec{Q}^2}{x(1-x)},\end{aligned} \tag{4}$$

with m being the quark mass. Moreover,

$$\bar{s}=\frac{s_\mathrm{i}+s_\mathrm{f}}{2}=\frac{1}{x(1-x)}\left(m^2+\vec{R}^2+\frac{x^2}{4}\vec{Q}^2\right). \tag{5}$$

The factor in Eq. (2)

$$\frac{I^0_\omega}{\tilde{I}^0_\omega}=\frac{2(1-x)\bar{s}+xq^2}{2(1-x)\sqrt{s_\mathrm{i}s_\mathrm{f}}} \tag{6}$$

takes into account the fact that we are dealing with spin-1/2 constituents instead of scalar ones. This factor plays the same role as the trace term in the calculation of field theory for spin-1/2 particles. Let's assume that the wave function is exponential-like with a free scale parameter Λ, then the explicit expression of the wave function is

$$\psi(s_{\mathrm{i,f}})=\exp\left(-\frac{s_{\mathrm{i,f}}}{\Lambda^2}\right). \tag{7}$$

After some algebra, the form factor of the charged pion in the relativistic quantum mechanics is

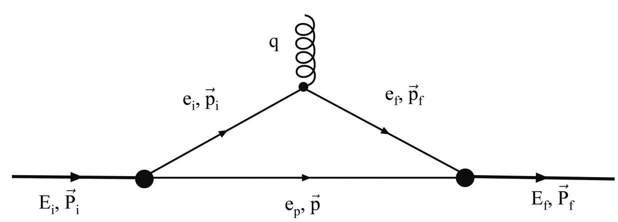

Fig. 1. Interaction

$$F(Q^2)=\frac{(2\pi)^2}{\pi N}\int_0^1\mathrm{d}x\int\mathrm{d}^2b\frac{\mathrm{e}^{-\mathrm{i}qb}}{(1-x)^2}\left\{\left[\frac{(1-x)^2}{2}Q^2+\frac{x(1-x)}{2}\Lambda^4\frac{\partial}{\partial\Lambda^2}\right\}\left|\Phi\left(x,\frac{b}{1-x}\right)\right|^2, \tag{8}$$

where

$$\begin{aligned}\Phi\left(\frac{b}{1-x}\right)\;&=\;\frac{1}{x(1-x)}\int\frac{\mathrm{d}^2l}{(2\pi)^2}\mathrm{e}^{\mathrm{i}\frac{\vec{l}\cdot\vec{b}}{1-x}}\frac{\psi(\bar{s}(l))}{\sqrt{\bar{s}(l)}}=\frac{1}{x(1-x)}\int\frac{\mathrm{d}^2l}{(2\pi)^2}\mathrm{e}^{\mathrm{i}\frac{\vec{b}\cdot\vec{l}}{1-x}}\sqrt{\frac{x(1-x)}{\vec{l}^2+m^2}}\mathrm{e}^{-\frac{\vec{l}^2+m^2}{\Lambda^2x(1-x)}}\\&=\;\frac{1}{(2\pi)^2\sqrt{x(1-x)}}\int\mathrm{d}l\mathrm{e}^{\mathrm{i}\frac{bl}{1-x}}K_0\left(\frac{l^2+m^2}{2\Lambda^2x(1-x)}\right)\mathrm{e}^{-\frac{l^2+m^2}{2\Lambda^2x(1-x)}},\end{aligned} \tag{9}$$

with K_0 being the Bessel function of the imaginary argument

$$K_\nu(zx) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)(2z)^\nu}{\Gamma\left(\frac{1}{2}\right)x^\nu}\int_0^\infty \frac{\cos xt\mathrm{d}t}{(t^2+z^2)^{\nu+\frac{1}{2}}}. \quad (10)$$

According to Miller [12], the transverse charge density $\rho(b)$ can be written in terms of the EM form factor as

$$F(Q^2) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2 b\rho(b)\mathrm{e}^{-\mathrm{i}qb}. \quad (11)$$

Then the transverse charged density of the pion in RQM is

$$\rho(b)_\mathrm{R} = \frac{16\pi^4}{\pi N}\int_0^1 \frac{\mathrm{d}x}{1-x}\left\{\frac{1-x}{2}Q^2\Phi + x\Lambda^4\left(\frac{\partial}{\partial \Lambda^2}\Phi\right)\right\}\Phi. \quad (12)$$

In the above equations, the normalization of N is defined as [18]

$$F(0) = \frac{1}{N}\int \mathrm{d}\bar{s}\psi^2(\bar{s})\frac{\sqrt{\bar{s}^2-4m^2\bar{s}}}{\bar{s}} = 1. \quad (13)$$

The transverse charged density of the pion in Eq. (12) can be compared with the result of the simple covariant approach in field theory by Miller

$$\rho(b)_\mathrm{M} = \frac{N_0^2}{4\pi}\frac{1}{(2\pi)^2}\int_0^1 \mathrm{d}x\frac{x}{1-x} \times K_0^2\left(\sqrt{m^2-M_\pi^2 x(1-x)}\frac{b}{1-x}\right), \quad (14)$$

where M_π is the mass of the pion meson. Moreover, the density in Eq. (12) can also be compared with the calculation based on the effective Lagrangian approach where the pion is considered to be a composed system by the two spin-1/2 particles and with the correlation function being taken into account. The obtained density from this effective Lagrangian approach is in the covariant form. It is [16]

$$\rho(b)_\mathrm{D} = \frac{N_1^2}{2\pi}\int_0^1 \frac{\mathrm{d}x}{1-x}\mathrm{e}^{\frac{2}{\Lambda^2}\left(m^2-\frac{M_\pi^2}{4}\right)}\left\{\left(\frac{Q^2}{2}(1-x)+xM_\pi^2\right)\left[K_0\left(\frac{b}{1-x}C\right)-f(\Lambda^2 C)\right]^2 + \Lambda^2\left[K_0\left(\frac{b}{1-x}C\right)-f(\Lambda^2 C)\right]\mathrm{e}^{-2\Lambda^2\frac{b^2}{(1-x)}}\right\}, \quad (15)$$

where

$$f(\Lambda^2 C) = \int_0^\infty \frac{\cos\left(\frac{b}{1-x}t\right)}{\sqrt{C^2+t^2}}\tilde{\Phi}\left[\sqrt{\frac{C^2+t^2}{2\Lambda^2(1-x)}}\right]\mathrm{d}t, \quad (16)$$

and $\tilde{\Phi}$ stands for an error function and $C^2 = m^2 - x(1-x)M_\pi^2$. In Eqs. (12, 14, 15), the constants N, N_0 and N_1 are determined by the normalization condition of the form factor $F(Q^2)$ at $Q^2 = 0$ (see Eq. (13)).

It should be reiterated that our approach with RQM has considered the correlation function of the quark-antiquark inside the pion and also considered the fact that the quark and antiquark are spin-1/2 particles. The two physical ingredients are taken into account explicitly by the wave functions of $\psi(s_{\mathrm{i,f}})$ and by the factor of $\frac{I_\omega^0}{\tilde{I}_\omega^0}$ in Eq. (6). To proceed with a numerical calculation of the transverse density in the relativistic quantum mechanics, two parameters are needed. They are the quark mass and the parameter Λ in the wave function of Eq. (7). According to Ref. [19], the string tensor of $\sigma_{\mathrm{st}} = \Lambda^2/2$ is 0.2 GeV^2 $\sim$ 1 GeV/fm. Moreover, the quark mass is fixed by requiring that the pion decay constant is f_π=0.0924 GeV. Then, the mass of the u or d quark is m=0.25 GeV.

Comparing the three versions of the transverse density of pion of $\rho_{\mathrm{R,M,D}}$, one clearly sees that the $\rho_{\mathrm{R,D}}$ are Q^2-dependent. The dependence results from

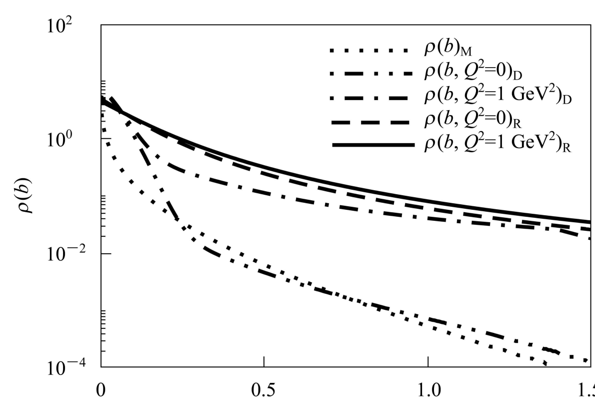

Fig. 2. $\rho(b)$, the dotted line stands for the results of Ref. [12]; the double-dotted-dashed and dotted-dashed curves represent the results of Ref. [16] with $Q^2 = 0$ and 1 GeV^2; the dashed and solid lines are the ones of the present work with RQM and $Q^2 = 0$ and 1 GeV^2, respectively.

the fact that the quark and antiquark are spin-1/2 particles and therefore results from the factor of $\frac{I^0_\omega}{\tilde{I}^0_\omega}$ or from the trace term of the Dirac spinors. This feature is neglected in the case of $\rho_{\rm M}$, where the quark and antiquark are considered as two spin-less particles. Moreover, the effect of the wave functions in the present RQM and in the effective Lagrangian approach [16] are displayed in the exponential term in Eqs. (9) and (15). Fig. 2 displays the calculated $\rho_{\rm R}(b)$ in the two cases of $Q^2=0$ and 1 GeV^2. Comparing the present $\rho_{\rm R}(b)$ with $\rho_{\rm D}(b)$, we find that the pronounced Q^2-dependence in $\rho_{\rm D}(b)$ changes to be a weak Q^2-dependence in $\rho_{\rm R}(b)$ for $Q^2=0$. Moreover, our $\rho_{\rm R}(b)$ is similar to $\rho_{\rm D}(b)$ with Q^2=1 GeV^2. It should be mentioned that the discrepancy between $\rho_{\rm R}$ and $\rho_{\rm D}$ results from the different treatment of the two fermion propagators in the calculation of the loop integral in RQM. Moreover, the discrepancy between $\rho_{\rm M}$ and the other two results from the considerations of the two spin-1/2 particles and of the wave function or correlation function. The remarkable differences among the three versions of the transverse density of the charged pion can be tested in future experiment measurements with high precision.

Discussions with Bertrand Desplanques and Ping Wang are appreciated.

References

1 Amendolia S R et al. Nucl. Phys. B, 1986, **277**: 168

2 Bekek C J et al. Phys. Rev. D, 1978, **17**: 1793

3 Brauel P et al. Z. Phys, 1979, **3**: 101

4 Volmer J et al. (The Jefferson Lab. F_π collaboration). Phys. Rev. Lett., 2001, **86**: 1713; Horn T et al. (The Jefferson Lab. F_π collaboration). Phys. Rev. Lett., 2006, **97**: 192001; Tadevosyan V (The Jefferson Lab. F_π collaboration). Phys. Rev. C, 2007, **75**: 055205; Huber G M et al. Phys. Rev. C, 2008, **78**, 045203; Blok H P et al. Phys. Rev. C, 2008, **78**: 045202

5 Aloisio A et al. (The KLOE collaboration). Phys. Lett. B, 2005, **606**: 12; 2006, **634**: 148

6 Pedlaa T K et al. (The CLEO collaboration). Phys. Rev. Lett., 2005, **95**: 261803

7 Soper D E. Phys. Rev. D, 1972, **5**: 1956

8 Burkardt M. Int. J. Mod. Phys. A, 2003, **18**: 173

9 Diehl M. Eur. Phys. J. C., 2002, **25**: 223; 2003, **31**: 223

10 Carlson C E, Vanderhaeghen M. Phys. Rev. Lett., 2008, **100**: 032004

11 Miller G A. Phys. Rev. Lett., 2007, **99**: 112001; Miller G A. Phys. Rev. C, 2009, **79**: 055204

12 Miller G A. Phys. Rev. C, 2009, **80**: 045210

13 Weinberg S. Phys. Rev. Lett., 1967, **18**: 188; Phys. Rev., 1968, **166**: 1568; SALAM A. Nuoco Cim., 1962, **25**: 224; Hayashi K et al. Fortschr. Phys., 1967, **15**: 625; Efimov G V, Ivanov M A. The Quark Confinement Model of Hadrons, IOP, Bristol, 1993

14 Faessler A, Gutsche T, Ivanov M A, Lyubovitskij V E, WANG P. Phys. Rev. D, 2003, **68**: 014011

15 Faessler A, Gutsche T, Holstein B R, Lyubovitskij V E, Nicmorus D, Pumsd-ard K. Phys. Rev. D, 2006, **74**: 074010; Faessler A, Gutsche T, Kovalenko S, Lyubovitskij V E. Phys. Rev. D, 2007, **76**: 014003; Faessler A, Gutsche T, Lyubovitskij V E, MA Y L. Phys. Rev. D, 2007, **76**: 114008; Faessler A, Gutsche T, Lyubovitskij V E, MA Y L. Phys. Rev. D, 2007, **76**: 014005; Faessler A, Gutsche T, Lyubovitskij V E, MA Y L. Phys. Rev. D, 2008, **77**: 114013; DONG Y B, Faessler A, Gutsche T, Lyubovitskij V E. Phys. Rev. D, 2008, **77**: 094013; DONG Y B, Faessler A, Gutsche T, Kovalenko S, Lyubovitskij V E. Phys. Rev. D, 2009, **79**: 094013; DONG Y B, Faessler A, Gutsche T, Lyubovitskij V E. Phys. Rev. D, 2010, **81**: 014006; DONG Y B, Faessler A, Gutsche T, Lyubovitskij V E. Phys. Rev. D, 2010, **81**: 074011

16 DONG Y B. Phys. Rev. C., 2010, **81**: 018201

17 Dirac P M. Rev. Mod. Phys., 1949, **21**: 392

18 Amghar A, Desplanques B, Theuβl L. Nucl. Phys. A, 2003, **714**: 213; HE J, Julia-Diaz B, DONG Y B. Phys. Lett. B, 2004, **602**: 212; Eur. Phys. A., 2005, **24**: 411; Despalnques B, DONG Y B. Eur Phys. J A, 2008, **37**: 33; Despalnques B, DONG Y B. arXiv: 0907.2835; Despalnques B, DONG Y B. arXiv: 1006.0578

19 Basdevant J L, Boukraa S Z. Phys. C, 1986, **30**: 103