On Hořava-Lifshitz cosmology *

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Abstract: We give a brief overview of the Hořava-Lifshitz-gravity theory, its modifications and its implications in cosmology. In particular, we discuss the various issues on the gravitational scalar mode, including its decoupling, its role as inflaton and its stability. Our analysis shows that the scalar mode could decouple naturally at $\lambda=1$ due to the extra gauge symmetry. On the other hand, the fact that the scalar mode becomes ghost when $1/3<\lambda<1$ is a real challenge to the theory. We try to overcome this problem by modifying the action such that the RG flow lies outside the problematic region. We discuss the cosmological implications of the theory.

Key words: Hořava-Lifshitz-gravity, cosmology, ghost

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1 Introduction

In the past few years, there have been very interesting interplays between high energy physics and condensed matter physics. On the one hand, the applications of AdS/CFT correspondence to various many-body strong coupling systems, ranging from quark-gluon-plasma in RHIC, superfluids, ultra-cold atoms to superconductor physics, are very fruitful. The applications are based on two essential features of AdS/CFT correspondence. One feature is that the correspondence is a strong/weak duality, which means that one can use weakly coupled gravity theory to study the strongly coupled field theory problem and vice versa. Another feature is that the AdS/CFT correspondence allows us to study the real-time process, which is otherwise inaccessible for other traditional field theory approaches.

On the other hand, condensed matter physics provides new concepts for high energy physics as well. One typical example is the introduction of anisotropic scaling. It turns out that in the quantum critical phenomena, space and time may take different scaling behavior,

$$\boldsymbol{x}\rightarrow b\boldsymbol{x},\quad t\rightarrow b^{z}t, \tag{1}$$

where z is the dynamical critical exponent characterizing the anisotropy between space and time. In the generic case that $z\neq 1$, the Lorentz symmetry is broken. One may assign a scaling dimension to the coordinates

$$[t]_{\rm s}=-z,\qquad [x_i]_{\rm s}=-1. \tag{2}$$

The first field theory model exhibiting the above anisotropic scaling is the Lifshitz scalar field theory with the critical exponent $z=2$ [1], which has the action

$$\mathcal{L}=\int{\rm d}^2x{\rm d}t\,((\partial_t\phi)^2-\lambda(\nabla^2\phi)^2)\,. \tag{3}$$

It has a line of fixed points parameterized by λ. Such fixed points with anisotropic scale invariance are usually called Lifshitz points. One interesting feature of the above action is that it satisfies the so-called detailed balance condition, which allows the (2+1)-dimensional theory to be related to a 2D massless scalar conformal theory directly. In fact, in the Schrodinger picture, the ground wave functional and even the correlation functions of (3) could be calculated from 2D field theory. Moreover, after being turned on the relevant perturbations

$$\sim\int{\rm d}^2x{\rm d}t(-\mu^2(\partial_i\phi)^2+m^4\phi^2), \tag{4}$$

the theory could be RG flowed to a theory with $z=1$ at IR. At IR, the Lorentz symmetry appears as an accidental symmetry. Another remarkable fact is that

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in this model, the speed of light μ is emergent and is not a constant anymore.

The Lifshitz scalar field theory and its generalizations have been used to study the quantum phase transitions in various strongly correlated electron systems. Moreover, in Ref. [2], a construction on the non-Abelian gauge theories with $z=2$ in arbitrary dimensions was presented. In Ref. [3], a general construction of renormalizable scalar and gauge field theories with arbitrary critical exponent z has been presented. For the field theories at the Lifshitz point, the Lorentz symmetry is broken explicitly by construction, and consequently the dispersion relation of the physical mode gets modified. This provides a natural framework to address the issues related to Lorentz symmetry breaking; for example, the time delays in Gamma-Ray bursts [3].

Another important feature of Lifshitz-like field theories is that they have better UV behaviors, due to the anisotropic scaling. For example, for ordinary Yang-Mills field, it is only renormalizable when the spacetime dimension is not greater than four, while for a Yang-Mills field with $z \geqslant 2$, it could still be renormalizable in five or even higher dimensions. This fact indicates that a gravity theory with anisotropic symmetry may also be renormalizable. This possibility has been investigated in Refs. [4, 5] and the following works. It was found that the gravity theory with anisotropic scaling has better UV behavior, and even though the theory involves higher derivative terms, it is still unitary.

In this article, we would like to give a brief review of the Hořava-Lifshitz-like gravity. We do not discuss all of the issues on the Hořava-Lifshitz gravity. Instead, we pay more attention to its cosmological implications. In the next section, we outline the construction of the Hořava-Lifshitz gravity and its generalization. In Section 3, we discuss the cosmological implications of the Hořava-Lifshitz-like gravity theories. We end with some discussions about the open issues on Hořava-Lifshitz gravity[1)].

2 Hořava-Lifshitz gravity and its modifications

Diffeomorphism is essential to Einstein's relativity theory of gravity. It has been widely believed to be exact in any theory of gravity. However, in the recent proposal by Hořava [4, 5] on gravity theory, it is no longer an exact symmetry. Due to the anisotropy, instead of diffeomorphism, we have the so-called foliation-preserving diffeomorphism. The transformation is now just

$$\begin{aligned} t &\rightarrow \tilde{t}(t), \\ x^i &\rightarrow \tilde{x^i}(x^j,t), \end{aligned} \tag{5}$$

which is generated by infinitesimal transformation,

$$\delta t=\xi^0(t), \quad \delta x^i=\xi^i(t,\vec{x}). \tag{6}$$

Since time direction plays a privileged role in the whole construction, it is more convenient to work with ADM metric,

$$\mathrm{d}s^2=-N^2\mathrm{d}t^2+g_{ij}(\mathrm{d}x^i+N^i\mathrm{d}t)(\mathrm{d}x^j+N^j\mathrm{d}t). \tag{7}$$

Here, N is the lapse function and N^i is called the shift variable. Both N and N_i are not dynamical variables. In fact, in Einstein's general relativity, the variations with respect to N and N_i give the super-Hamiltonian and super-momentum constraint, respectively. The physical degrees of freedom resides in the spatial metric components, modulo the gauge transformations to be discussed below. Due to the scaling (2), the scaling dimensions of the metric components are

$$[g_{ij}]_{\mathrm{s}}=0, \quad [N_i]_{\mathrm{s}}=z-1, \quad [N]_{\mathrm{s}}=0.$$

It would be interesting to study the case with a generic value of z. However, in this paper, we only focus on $z=3$.

The transformation (5) leads to the following gauge transformations on the metric components,

$$\begin{aligned} \delta g_{ij} &= \partial_i\xi^k g_{jk}+\partial_j\xi^k g_{ik}+\xi^k\partial_k g_{ij}+\xi^0\dot{g}_{ij}, \\ \delta N_i &= \partial_i\xi^j N_j+\xi^j\partial_j N_i+\dot{\xi}^j g_{ij}+\dot{\xi}^0 N_i+\xi^0\dot{N}_i, \\ \delta N &= \xi^j\partial_j N+\dot{\xi}^0 N+\xi^0\dot{N}. \end{aligned} \tag{8}$$

The above transformations could be obtained by taking a nonrelativistic limit of usual relativistic diffeomorphisms. It is more convenient and natural to choose N being projectable, just the function of t. There are a few advantages in working with this choice. With this choice, the gauge symmetry is simpler and transparent. Furthermore, in the Hamiltonian formulation, the constraints could form a closed algebra since the momentum conjugate to N does not lead to a local constraint [4]. As a result of fewer constraints than standard GR, the physical degrees of freedom in the theory includes not only the massless gravitons but also another propagating scalar. The existence of extra scalar field has profound meaning in cosmology. On the other hand, if one abandons the

1) There are a large number of research papers on the Hořava-Lifshitz-like gravity. Due to the limited space, we will not include all of them in the paper. Interested reader may search them easily from the papers listed in the References.

projectability condition and lets N be the function of both t and x^i, one will find that the theory will be ill-defined, as shown in Refs. [4, 6].

At the special value $\lambda = 1$, the theory develops an enhanced time-independent $U(1)$ gauge symmetry acting via

$$\delta N_i = \partial_i \epsilon, \quad \delta g_{ij} = 0. \tag{9}$$

Due to the existence of extra gauge symmetry, the scalar mode is not physical anymore. It is remarkable that even with this extra gauge symmetry, the total gauge symmetries are different from the usual diffeomorphisms in general relativity. In other words, the diffeomorphisms have not been recovered at $\lambda = 1$. This fact is essential to understand why at $\lambda = 1$ the extra scalar degree of freedom could be decoupled without trouble.

In terms of the ADM metric, the action of original Hořava-Lifshitz gravity theory can be written as [5]

$$\begin{aligned} S_{\mathrm{g}} = & \int \mathrm{d}t\mathrm{d}^3\boldsymbol{x}\sqrt{g}N\Bigg\{\frac{2}{\kappa^2}K_{ij}G^{ijkl}K_{kl} - \frac{\kappa^2}{2}\Bigg[\frac{1}{\omega^2}C_{ij} \\ & -\frac{\mu}{2}\left(R_{ij}-\frac{1}{2}Rg_{ij}+\Lambda_{\mathrm{W}}g_{ij}\right)\Bigg]\cdot G^{ijkl}\Bigg[\frac{1}{\omega^2}C_{ij} \\ & -\frac{\mu}{2}\left(R_{ij}-\frac{1}{2}Rg_{ij}+\Lambda_{\mathrm{W}}g_{ij}\right)\Bigg]\Bigg\}, \end{aligned} \tag{10}$$

where K_{ij} is the extrinsic curvature of the spatial hypersurface; C_{ij} is the Cotton tensor, which can be used to preserve the detailed-balanced condition in constructing the action; G^{ijkl} is the De Witt metric on the space of metrics that preserve the anisotropic diffeomorphism, and R_{ij} is the Ricci tensor in spatial hypersurface. Their definitions are

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \tag{11}$$

$$C_{ij} = \epsilon^{ikl}\nabla_k\left(R^j{}_l - \frac{1}{4}R\delta^j_l\right), \tag{12}$$

$$G^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl}+g^{il}g^{kl}\right) - \lambda g^{ij}g^{kl}. \tag{13}$$

Here and throughout the paper, a dot over the quantity means taking the derivative with respect to the cosmic time t, while a prime denotes that to comoving time η. The first term in (10) involving only the extrinsic curvature is the kinetic term, while the others are potential terms. λ is the coupling constant in the kinetic term, and runs expectedly to $\lambda = 1$ at IR regime at which the kinetic term goes back to the one in the general relativity. This specific form of the action is governed by the detailed-balance condition, which is just applied by Hořava for convenience to decrease the number of arbitrary parameters. The expansion of the action gives

$$\begin{aligned} S_{\mathrm{g}} = & \int \mathrm{d}t\mathrm{d}^3\boldsymbol{x}\sqrt{g}N\Bigg\{\frac{2}{\kappa^2}(K_{ij}K^{ij}-\lambda K^2) - \frac{\kappa^2}{2\omega^4}C_{ij}C^{ij} \\ & +\frac{\kappa^2\mu}{2\omega^2}\epsilon^{ijk}R_{il}\nabla_j R^l{}_k - \frac{\kappa^2\mu^2}{8}R_{ij}R^{ij} \\ & +\frac{\kappa^2\mu^2}{8(1-3\lambda)}\left(\frac{1-4\lambda}{4}R^2+\Lambda_{\mathrm{W}}R-3\Lambda_{\mathrm{W}}^2\right)\Bigg\}. \end{aligned} \tag{14}$$

Comparing this action with the Einstein-Hilbert action in the IR limit,

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{g}N\{(K_{ij}K^{ij}-K^2)+R-2\Lambda\}, \tag{15}$$

with $x^0 \equiv ct$, we can recover the speed of light, Newton constant and the cosmological constant by the parameters introduced before,

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda_{\mathrm{W}}}{1-3\lambda}}, \quad G = \frac{\kappa^2}{32\pi c}, \quad \Lambda = \frac{3}{2}\Lambda_{\mathrm{W}}. \tag{16}$$

Thus at IR the theory recovers nearly the usual general relativity, with the higher derivative terms of spatial metric components as the modifications. Even though the higher derivative terms are highly suppressed at IR, strictly speaking, the theory always breaks the diffeomorphism, and therefore the locally Lorentz invariance.

In Hořava's original paper [5], the coupling constant λ is expected to run to 1 in the IR limit. And at UV, because of the anisotropy between space and time, the speed of light is not a constant and may be extremely large, which could be used to explain the horizon and flatness problem. But from (16) we know that this can only occur in the case $\lambda < 1/3$ if we take Λ to be positive, taking into account the fact that Λ is directly related to the cosmological constant. However, this raises the worry that the marginal coupling constant λ can never run to its infrared value $\lambda = 1$, which is directly in contrast with our former description. To solve this problem, it was proposed that one should carry out analytical continuation on the parameters [7],

$$\mu \to \mathrm{i}\mu, \quad \omega \to -\mathrm{i}\omega, \tag{17}$$

which leaves the action real. And under this continuation, we see from (16) that

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda_{\mathrm{W}}}{3\lambda-1}}, \tag{18}$$

and there is no conflict between $\Lambda > 0$ and $\lambda > 1/3$. And when $\lambda \to 1/3$ proposed by Hořava as the ultraviolet value of this coupling constant, we have a very large speed of light, which can naturally solve the casuality problem in cosmology without inflation.

The action given above was constructed with respect to the detailed balance condition. However, the detailed balance condition may not be essential to the theory. This is because even with the detailed balance condition, the ground state eigenfunctional is not normalizable. As pointed out in Ref. [5], imposing such a condition is just pragmatic to simplify the action. In principle, one may relax this condition and consider a more general form of the action. For the theory to be power-counting renormalizable, the allowed terms are limited. The kinetic term is completely fixed, with the scaling dimension six. The potential terms should be gauge invariant, with the scaling dimension not greater than six. The marginal ones could be a combination of

$$\nabla_k R_{ij}\nabla^k R^{ij},\quad \nabla_k R_{ij}\nabla^i R^{jk},\quad R\Delta R, R^{ij}\Delta R_{ij},$$
$$R^3,\quad R^i_j R^j_k R^k_i,\quad RR_{ij}R^{ij},$$

and the relevant ones could be quadratic in the Ricci tensor and Ricci scalar, or linear in the Ricci scalar. The most general form of the action without the detailed balance condition could be found in Ref. [8]. In this paper, we do not want to consider the most general form of the action. Instead, we just consider the marginal spatial kinetic part and most relevant deformations, besides the time kinetic terms. The action we start with is of the form

$$S_{\rm g} = \int \mathrm{d}t\mathrm{d}^3\boldsymbol{x}\sqrt{g}N\left\{\alpha(K_{ij}K^{ij}-\lambda K^2)+\xi(\lambda)R\right.$$
$$+\sigma(\lambda)-\beta\left(\beta_1\nabla_i R_{jk}\nabla^i R^{jk}+\beta_2\nabla_i R_{jk}\nabla^j R^{ik}\right.$$
$$\left.\left.+\beta_3\nabla_i R\nabla^i R\right)\right\}. \tag{19}$$

Here we only keep the marginal terms that are power-counting renormalizable and dominant in the UV limit, besides the lower-dimensional terms to recover IR behaviors.

It is remarkable that the gravitational scalar mode is not always physical. Actually, from the perturbations around the flat spacetime, it was found that the scalar mode became a ghost when the parameter l lies between $\frac{1}{3}$ and 1 [5]. This is also true for the perturbations around the FRW universe [9]. The existence of the ghost is fatal to the theory. It means that the theory is not well defined, not mentioning UV completeness. One may expect that we can always work in the region outside $l\in[1/3,1]$. However, this cannot be guaranteed, considering our ignorance of the details of RG flow. On the other hand, in the practical application in cosmology, one wishes the RG flow to be from $l\sim 1/3$ to $l=1$ in original Hořava-Lifshitz gravity. We take a modest attitude and try to modify the Hořava-Lifshitz gravity such that the RG flow may happen always with $l>1$.

Because of the breakdown of the detailed balance condition, the coupling constants before each term are independent. The couplings could be connected to the speed of light, the Newtonian coupling constant and the cosmological constant of Einstein's general relativity in the IR limit,

$$c^2 = \frac{\xi}{\alpha}, \tag{20}$$

$$16\pi G = \frac{1}{c\alpha}, \tag{21}$$

$$\Lambda = -\frac{\sigma}{2\xi}. \tag{22}$$

Here we see that c^2 can be positive, if we choose a proper form of the function $\xi(\lambda)$. Furthermore, we can require c to be very large when λ is near its ultraviolet value. In Hořava's original paper, he suggested $\lambda\to 1/3$ at the UV limit, which gives a large speed of light in (16) or (18). Here we only take this condition as a constraint on the function $\xi(\lambda)$. For instance if the theory requires λ to be larger than unity at UV as we will propose as a condition to exclude the ghost field, the function $\xi(\lambda)$ may be divergent when λ tends to infinity.

3 Cosmological implication

For our use, let us have a glance at the classical dynamics of the universe under such an action. In a homogenous and isotropic universe,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2 h_{ij}\mathrm{d}x^i\mathrm{d}x^j,\quad h_{ij} = \delta_{ij} + \frac{\mathcal{K}x^ix^j}{1-\mathcal{K}\boldsymbol{x}^2}, \tag{23}$$

where $\mathcal{K}$ is the parameter to describe the spatial curvature. Under this metric, the universe is homogeneous and isotropic, which will greatly simplify the following discussions. To apply our foliated diffeomorphism, we need to use the ADM formalism of this Robertson-Walker metric, with the extrinsic curvature and the Ricci tensor to be

$$K_{ij} = H(t)g_{ij},\ K=3H(t), \tag{24}$$

$$R_{ij} = \frac{2\mathcal{K}}{a^2}g_{ij},\quad R=\frac{6\mathcal{K}}{a^2}, \tag{25}$$

where $H(t)=\dot{a}/a$ is the Hubble parameter.

We take the variation of the action (19) with respect to N, and have our first equation of constraint,

$$\int\sqrt{g}\left[-\frac{2}{\kappa^2}(K^{ij}K_{ij}-\lambda K^2)+\zeta R+\sigma\right]\mathrm{d}^3\boldsymbol{x}=\int\sqrt{g}\rho\mathrm{d}^3\boldsymbol{x}. \tag{26}$$

Here, ρ is the energy density of the Lifshitz matter in the universe, and can be written

$$\rho=-\frac{1}{\sqrt{g}}\frac{\delta S_{\mathrm{m}}}{\delta N}, \tag{27}$$

where S_{m} is the action of matter field, which can be a Lifshitz scalar, gauge field or something else. Because of the projectability of the lapse function $N(t)$, we only have a spatial-integral constraint here. This is generic for all of the Hořava-like models with a projectable lapse function $N(t)$ [8]. But, for a homogeneous and isotropic Friedman universe, this constraint equation is valid at every point, and the integral can be removed legally. Thus we have the first Friedman's equation [7],

$$H^2=\frac{c^2}{3\alpha(1-3\lambda)}\left[-\rho+\frac{6\mathcal{K}}{a^2}\xi(\lambda)+\sigma(\lambda)\right]. \tag{28}$$

Since ρ is the energy density of matter and radiation, $\sigma(\lambda)$ plays the role of "cosmological constant". Here, it is a function of λ and evolves when λ varies as the energy scale changes. This implicant dependence may be treated carefully when we are facing problems like the evolution of dark energy or the tilt of the power spectrum. But because the dependence of λ on the cosmic time is unknown, we will neglect this dependence and suppose that in the process we are interested in, the change in $\sigma(\lambda)$ is so small that it will not have any significant physical effect, and so is $H(\lambda)$. We see from (28) that if the universe is flat and dominated by the cosmological constant, for some λ greater than 1/3, we must have $\sigma(\lambda>1/3)<0$, which means that we have a positive cosmological constant $\Lambda>0$ at IR, since from (20), $\xi(\lambda>1/3)$ is always positive. These two conditions guaranteed the positivity of the cosmological constant and H^2. If the matter/radiation contribution could be ignored safely, the homogenous and isotropic solution is a pure de-Sitter spacetime, with an exponentially expanding scale factor $a(t)\propto\exp(Ht)$.

The second equation of constraint is obtained by taking the variation in the action with respect to the shift vector N_i,

$$\nabla_i(K^{ij}-\lambda Kg^{ij})=0. \tag{29}$$

Because the extrinsic curvature is homogeneous in a Friedman universe, as in (24), $K_{ij}\propto g_{ij}$, this equation is trivially satisfied for the background evolution. But it will supply a perturbative constraint equation up to the first order if the perturbations to the background metric are under consideration.

Finally, taking the variation of action (19) with respect to g_{ij}, we have the equation of motion of dynamical degree of freedom. The explicit expression is rather lengthy and has little to do with our following discussion, so we would like not to write it here.

In the remaining part of this section, we would like to discuss an interesting physical implication of the gravitational scalar mode. We will show that there is no need to introduce a scalar inflaton to seed the large scale structure. Instead, the gravitational scalar may play the role of inflaton. At the first step, let us review briefly the cosmological perturbation around flat FRW universe in Hořava-Lifshitz-like gravity. The perturbed metric is of the form

$$\begin{aligned}\mathrm{d}s^2 &= -\mathrm{d}t^2+a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\\ &= a^2(\eta)(-\mathrm{d}\eta^2+\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j).\end{aligned} \tag{30}$$

Here we use the co-moving time $\eta=\int\mathrm{d}t/a$ as a time variable. The fluctuations around the ADM metric could be

$$\mathcal{N} = a(\eta)(1+A), \tag{31}$$

$$\mathcal{N}_i = a(\eta)(\partial_i B+V_i), \tag{32}$$

$$g_{ij} = a^2(\eta)\{(1-2\psi)\delta_{ij}-\partial_i\partial_j E-2\partial_{(i}F_{j)}+h_{ij}\}, \tag{33}$$

where A,B,ψ,E are the scalar perturbations, V_i and F_j are the vector perturbations, and t_{ij} is the gauge-invariant tensor perturbation describing the gravitational wave. Let us focus on the scalar perturbation, and work with the gauge[1)]

$$A=0,\quad E=0. \tag{34}$$

Then the perturbed action of the gravitational field up to the second order can be written as

$$\begin{aligned}S^{(2)} = \int\mathrm{d}t\mathrm{d}^3\boldsymbol{x}\Bigg\{&3\alpha a^3(1-3\lambda)\left[\frac{2}{3}\frac{\dot{\psi}^2}{1-\lambda}+6H\psi\dot{\psi}\right.\\ &\left.+9H^2\psi^2\right]-\frac{2\beta}{a^3}(3\beta_1+2\beta_2+8\beta_3)\psi\partial^6\psi\\ &-2a\xi(\lambda)\psi\partial^2\psi\Bigg\}.\end{aligned} \tag{35}$$

Several remarks are in order:

1) From the action, it is obvious that the scalar mode ψ is physical when $\lambda<1/3$ and $\lambda>1$, while

1) For a detailed discussion on the gauge transformations of the cosmological perturbations, please see Refs. [9, 10].

when $1/3 < \lambda < 1$ the mode is a ghost, indicating that the theory is not well-defined. At the special value $\lambda = 1$, the mode is decoupled, as we will clarify below. And at $\lambda = 1/3$, the theory has extra symmetry, as discussed carefully in Ref. [5]. This fact is the same as the one found in Refs. [4, 5] where the perturbations around the flat spacetime were studied.

2) More interestingly, the equation of motion of ψ takes the following form,

$$\frac{1-3l}{1-l}\ddot{\psi}+\cdots. \tag{36}$$

This indicates that when $l \to 1$, the scalar field ψ could be decoupled naturally, in contrast with the claim in Ref. [11]. It seems that the strong coupling problem does not exist in our case.

3) The absence of the strong coupling problem may stem from the fact that we take different points of view on gauge transformations. In our case, we stick to the requirement that the lapse function should be projectable, as originally advocated in Ref. [5]. As a result, we do not expect that the diffeomorphism is recovered at $\lambda = 1$. Instead, the decoupling of the extra scalar mode comes from the fact that there is extra gauge symmetry at $l = 1$. This is conceptually different from the case studied in Ref. [11] and Fierz-Pauli massive gravity.

4) Technically, it is remarkable that the equation of motion of ψ has a prefactor proportional to $1/(1-l)$ rather than $(1-l)$. This difference has significant physical implications. In our case, this means that the scalar mode could be decoupled without trouble. Another way to see this is to cast the scalar mode into canonical form such that the mode becomes non-physical at $l = 1$. It is remarkable that in Refs. [4, 5], the equation of motion of the scalar mode around the flat spacetime background has the prefactor $(1-l)$. However, this is due to different gauge choice. It has been shown by rescaling the field that one has the same equation of motion. In fact, no matter what kind of gauge choice, the physical dispersion relation is exactly the same. This suggests that for the cosmological perturbations, the different gauge choice would not lead to a different dispersion relation. Namely, the extra scalar mode may decouple naturally as $l \to 1$.

From the above discussion, we know that the classical evolution of the scale factor in the Hořava era is determined by (28). In particular, when the cosmological constant is dominant and the universe is flat, the evolution is the exponential expansion like in a de Sitter phase. Now the Hubble parameter is a constant. For convenience, we define a conformal time η with $dt = a d\eta$ and introduce an auxiliary field $\chi = a\psi$. After taking the variation with respect to ψ, and changing to the momentum space, we have

$$\chi''(\eta)+\left(k^6H^4\bar{L}^4\eta^4+c_s^2k^2-\frac{2}{\eta^2}\right)\chi(\eta)=0, \tag{37}$$

where

$$c_s^2=\frac{1-\lambda}{1-3\lambda}c^2 \tag{38}$$

is the speed of sound, and

$$\bar{L}=\frac{L}{2\pi},\quad L=2\pi\left[\frac{\beta}{\alpha}\frac{1-\lambda}{1-3\lambda}(3\beta_1+2\beta_2+8\beta_3)\right]^{\frac{1}{4}} \tag{39}$$

is the characteristic length, which denotes the scale where the trans-Planckian effects become significant.

Either from the WKB approximation or from the method introduced to study the trans-Planckian physics, one can calculate the power spectrum of the gravitational scalar and find it to be scale invariant. This is not surprising since the classical evolution is a de-Sitter space, which is time translationally invariant. The interesting point is that even the classical evolution is not purely in de-Sitter phase due to the presence of matter, the scalar power spectrum is still scale invariant. This is because the gravitational scalar is dimensionless and so is insensitive to the scale transformation [3, 12]. However, it should be noted that this is not true for the original Hořava-Lifshitz cosmology due to the detailed balance condition [13]. For the details of calculation, please see Refs. [9, 13].

4 Conclusion and discussion

The Hořava-Lifshitz gravity is a theory with anisotropic scaling. In this theory, time plays a privileged role. The usual diffeomorphism is replaced by the foliation-preserving diffeomorphism. As a result, there is an extra degree of freedom. Such a degree of freedom is consumed by the extra gauge degree of freedom when $l = 1$. One attractive feature of the theory is that it is power-counting renormalizable and is expected to have better UV behaviors. Another interesting feature is that the gravitational scalar may play the role of inflaton and seed the large scale structure of our universe, as discussed before.

One concern of the Hořava-Lifshitz-like gravity is whether the theory is well- defined. The debate in the literature focused on if one should choose the lapse function to be the only the function of time, or in other words, if the lapse function should be projectable. The different choice seems to lead to completely different physics. For example, it was found that without the projectability condition, there were new static spherically symmetric solutions to the Hořava-Lifshitz gravity and its modifications [7, 14]. These new solutions may have profound physical implications in solar system tests. However, it is proved in Ref. [15] that these new solutions do not respect the projectability condition, and actually the only nontrivial solution besides the vacuum is the Schwarzschild-de-Sitter solution. From our point of view, the theory with the projectability condition gets rid of the pathologies found in Refs. [4, 6] and is simply well defined.

Another concern of the Hořava-Lifshitz-like gravity is with regard to the possible strong coupling of the gravitational scalar to the matter at IR [11]. Actually, this would not happen. The key point is that the diffeomorphism is only an approximate symmetry even at IR. The breakdown of full diffeomorphism at the IR fixed point suggests that the usual Stuckelberg trick used in the analysis in Ref. [11] could not be used directly, especially taking into account the projectability condition. We have shown in Section 3 that the gravitational scalar could be decoupled naturally.

However, the theory may suffer from other pathologies. One concern is the existence of the ghost excitation. We showed that as the perturbations around the flat spacetime, the scalar perturbation around the flat FRW universe could be a ghost in the parameter region $\frac{1}{3} < l < 1$. The presence of the ghost mode is a serious challenge to the theory.

The nature of the power spectra studied is purely gravitational. In particular, in the language of orthodox cosmology, the scalar perturbation is expected to set up the initial conditions and seed the anisotropy of large scalar structure in our universe. Some work has been done to reveal the evolution of perturbations after inflation in the Hořava-Lifshitz gravity [16]. After inflation ends, this gravitational perturbation must be converted into CMB anisotropy and matter inhomogeneity through some post-inflation evolotions. But still we do not know yet how to couple the gravitational scalar mode with, for instance, the radiation. This is an interesting issue, which we would like to study in the future.

One essential issue in the Hořava-Lifshitz gravity is its RG flow. In Ref. [17], it has been shown that in the Lifshitz-like scalar field theory, the RG flow may not lead the theory to the fixed point we want. Considering the numbers of the parameters in modified Hořava-Lifshitz gravity, this raises the concern if the theory can flow to IR fixed point $\lambda = 1$. Moreover, the details of RG flow can tell us if we can avoid the dangerous region, where the ghost excitation appears, even we start from a safe region. Furthermore, RG flow may be closely related to the physics in the inflationary era. It is not clear whether RG flow of the theory runs to its IR fixed point before the inflationary era. If it does, then the gravitational scalar is not dynamical and has nothing to do with inflation. Even if the energy scale to reach the IR limit is lower than the inflation era, there is an important question to answer: Did λ vary significantly in the inflationary era? The variation in l may tilt the power spectra and has interesting physical implications. In any case, the behavior of the Hořava-Lifshitz gravity theory under RG flow deserves careful investigation.

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