Deviations of the lepton mapping matrix from the Harrison-Perkins-Scott form

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Abstract   We propose a simple set of hypotheses governing the deviations of the leptonic mapping matrix from the Harrison-Perkins-Scott (HPS) form. These deviations are supposed to arise entirely from a perturbation of the mass matrix in the charged lepton sector. The perturbing matrix is assumed to be purely imaginary (thus maximally T-violating) and to have a strength in energy scale no greater (but perhaps smaller) than the muon mass. As we shall show, it then follows that the absolute value of the mapping matrix elements pertaining to the tau lepton deviate by no more than $O((m_\mu/m_\tau)^2) \approx 3.5 \times 10^{-3}$ from their HPS values.

Assuming that $(m_\mu/m_\tau)^2$ can be neglected, we derive two simple constraints on the four parameters θ_{12}, θ_{23}, θ_{31}, and δ of the mapping matrix. These constraints are independent of the details of the imaginary T-violating perturbation of the charged lepton mass matrix. We also show that the e and μ parts of the mapping matrix have a definite form governed by two parameters α and β; any deviation of order m_μ/m_τ can be accommodated by adjusting these two parameters.

Key words   lepton mapping matrix, CP and T violation, Jarlskog invariant, timeon

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1   Introduction

The last decade has seen a well-defined situation take form with respect to neutrino oscillations. The lepton mapping matrix is at least approximately described by the "tribimaximal" formula of Harrison, Perkins and Scott [1], and the differences of squared neutrino masses are known to order of magnitude. The data on the mapping angles are so far consistent with the HPS values, but best fits suggest some small deviations. There is as yet no information on the T-violating phase angle.

With respect to the mapping angles, the task of theoretical model construction has been sorting itself into two directions: one is to devise a natural way [2] in which the HPS formula can arise as a zeroth approximation, and the other is to propose a perturbative mechanism [3] that gives rise to deviations. This paper confines itself to the second task.

In a recent paper [4], we suggested that T-violation in both quarks and leptons could arise from the coupling of the Dirac matrix $i\gamma_4\gamma_5$ with an undiscovered particle (called timeon) of large mass. For

leptons, it was proposed that the coupling occurs only for the charged leptons, and without it the mapping matrix would be exactly of the Harrison-Perkins-Scott [HPS] form. Both are also assumed in this paper. As we shall see, many of the results of the timeon paper can be derived without the additional assumptions that the bare mass of the electron is zero and that the T-violating coupling acts only on one vector in the flavor space.

The hypotheses proposed in this paper are thus a weaker subset of those in Ref. [4]; these are
(i) The left-handed charged leptons are eigenstates of a hermitian matrix

$$L = L_0 + iL_1, \qquad (1.1)$$

where L_0 and L_1 are real.
(ii) The "bare" charged leptons (i.e., eigenstates of L_0) mix with neutrino precisely according to the Harrison-Perkins-Scott (HPS) matrix.
(iii) the strength of L_1 is of order of the muon mass m_μ or less.

In Section 2, we shall show that assumptions (i)-

(iii) lead to very small, of the order of

$$\left(\frac{m_\mu}{m_\tau}\right)^2 \cong 3.5 \times 10^{-3} \tag{1.2}$$

deviations from HPS in the absolute values of three of the mapping matrix elements

$$|U_{31}|^2 = \frac{1}{6}, \qquad |U_{32}|^2 = \frac{1}{3}$$

and

$$|U_{33}|^2 = \frac{1}{2}. \tag{1.3}$$

Thus, there are two relations, to be discussed in Section 3, between three mapping angles θ_{12}, θ_{23}, θ_{31} and the T violating phase $e^{i\delta}$ in the lepton mapping matrix. These relations are valid to the accuracy of order of (m_μ/m_τ), but not to that of $(m_\mu/m_\tau)^2$. Another consequence of (1.3) is that to the same accuracy, the entire lepton mapping matrix can be described by two real parameters, as will be summarized by the (α, β) theorem in Section 4. In Section 5, we shall discuss the experimental implications of these relationships.

Throughout the paper the mapping angles θ_{12}, θ_{23}, θ_{31} and the T-violating phase δ are related to the mapping matrix elements U_{ij} by

$$U_{11} = \cos\theta_{31}\cos\theta_{12}, \qquad U_{12} = \cos\theta_{31}\sin\theta_{12}$$
$$U_{13} = \sin\theta_{31}e^{-i\delta}, \qquad U_{23} = \sin\theta_{23}\cos\theta_{31}$$

and

$$U_{33} = \cos\theta_{23}\cos\theta_{31} . \tag{1.4}$$

We shall write the physical charged lepton states as $|e\rangle$, $|\mu\rangle$, $|\tau\rangle$ and the corresponding bare states as $|e_0\rangle$, $|\mu_0\rangle$, $|\tau_0\rangle$. The effect of the perturbation iL_1 is to cause the physical states to differ from the corresponding bare states by a unitary transformation $K = [K_{ll_0}]$ so that

$$|l\rangle = \sum_{l_0} K_{ll_0}|l_0\rangle, \tag{1.5}$$

with

$$K_{ll_0} = \langle l_0|l\rangle, \tag{1.6}$$

where l, l_0 refer to e, μ, τ and the corresponding e_0, μ_0, τ_0. The free neutrino eigenstates will be called $|\nu_1\rangle$, $|\nu_2\rangle$ and $|\nu_3\rangle$ in the usual way. In the present proposal, deviations from the HPS mapping matrix are due entirely to the perturbation on the charged lepton mass matrix. Thus, the masses of the free neutrinos do not affect these deviations. The neutrino masses do not play any role in this paper.

For convenience of notations, we shall introduce charged lepton stats $|1\rangle$, $|2\rangle$, $|3\rangle$ which are precisely related (without mixing) to the neutrino states $|\nu_1\rangle$, $|\nu_2\rangle$, $|\nu_3\rangle$ via the weak interaction. This enables us to write for example $\langle 1|e\rangle$ for what is usually called $\langle\nu_1|\nu_e\rangle$, and likewise $\langle 1|e_0\rangle$ for $\langle\nu_1|\nu_{e_0}\rangle$. The physical mapping matrix is then

$$U = [U_{1k}], \tag{1.7}$$

where

$$U_{1k} = \langle k|l\rangle, \tag{1.8}$$

with k being 1, 2, or 3. It then follows from (1.5) that

$$U = KU_0 \tag{1.9}$$

or for the example of $k = 1$ and $l = e$, the element U_{e1} is

$$\langle 1|e\rangle = \langle 1|e_0\rangle\langle e_0|e\rangle + \langle 1|\mu_0\rangle\langle\mu_0|e\rangle$$
$$+ \langle 1|\tau_0\rangle\langle\tau_0|e\rangle, \tag{1.10}$$

where the elements $\langle 1|e_0\rangle$, $\langle 1|\mu_0\rangle$ and $\langle 1|\tau_0\rangle$ refer to those of U_0, and are precisely the HPS matrix elements; i.e.,

$$\langle 1|e_0\rangle = \sqrt{\frac{2}{3}}, \qquad \langle 2|e_0\rangle = \sqrt{\frac{1}{3}}, \ \langle 3|e_0\rangle = 0,$$

$$\langle 1|\mu_0\rangle = -\sqrt{\frac{1}{6}}, \ \langle 2|\mu_0\rangle = \sqrt{\frac{1}{3}}, \ \langle 3|\mu_0\rangle = \sqrt{\frac{1}{2}},$$

$$\langle 1|\tau_0\rangle = \sqrt{\frac{1}{6}}, \qquad \langle 2|\tau_0\rangle = -\sqrt{\frac{1}{3}}, \ \langle 3|\tau_0\rangle = \sqrt{\frac{1}{2}}. \tag{1.11}$$

2 Effect of large tau mass

Consider the mapping element between the state

$$k = 1, 2, 3 \tag{2.1}$$

and the τ-state:

$$\langle k|\tau\rangle = \langle k|e_0\rangle\langle e_0|\tau\rangle + \langle k|\mu_0\rangle\langle\mu_0|\tau\rangle$$
$$+ \langle k|\tau_0\rangle\langle\tau_0|\tau\rangle. \tag{2.2}$$

We shall compute $|\langle k|\tau\rangle|^2$ to the accuracy of (m_μ/m_τ), but neglecting corrections of order $(m_\mu/m_\tau)^2$. By first-order perturbation theory, we have

$$\langle e_0|\tau\rangle \cong \langle e_0|iL_1|\tau_0\rangle/(m_{\tau_0} - m_{e_0})$$
$$\cong i\langle e_0|L_1|\tau_0\rangle/m_\tau. \tag{2.3}$$

Likewise,

$$\langle\mu_0|\tau\rangle \cong i\langle\mu_0|L_1|\tau_0\rangle/m_\tau . \tag{2.4}$$

By hypothesis (iii), both these elements are of order of (m_μ/m_τ). Therefore

$$1 - |\langle\tau_0|\tau\rangle|^2 = |\langle e_0|\tau\rangle|^2 + |\langle\mu_0|\tau\rangle|^2$$
$$\sim O[(m_\mu/m_\tau)^2]. \tag{2.5}$$

It follows that with neglect of $O[(m_\mu/m_\tau)^2]$,

$$\langle k|\tau\rangle = \mathrm{i}\langle k|e_0\rangle\langle e_0|L_1|\tau_0\rangle/m_\tau$$
$$+\mathrm{i}\langle k|\mu_0\rangle\langle\mu_0|L_1|\tau_0\rangle/m_\tau+\langle k|\tau_0\rangle. \tag{2.6}$$

By hypothesis (i), the elements of L_1 are real, and $\langle k|e_0\rangle$, $\langle k|\mu_0\rangle$ and $\langle k|\tau_0\rangle$ are also real since these are HPS matrix elements. Thus, from (2.6) we have

$$|\langle k|\tau\rangle|^2 = m_\tau^{-2}\big|\langle k|e_0\rangle\langle e_0|\mathrm{i}L_1|\tau_0\rangle$$
$$+\langle k|\mu_0\rangle\langle\mu_0|\mathrm{i}L_1|\tau_0\rangle\big|^2+(\langle k|\tau_0\rangle)^2$$
$$= \langle k|\tau_0\rangle^2+O((m_\mu/m_\tau)^2) ; \tag{2.7}$$

i.e., with the neglect of $O(m_\mu/m_\tau)^2$,

$$|\langle 1|\tau\rangle|^2 = \frac{1}{6} , \qquad |\langle 2|\tau\rangle|^2 = \frac{1}{3}$$

and

$$|\langle 3|\tau\rangle|^2 = \frac{1}{2} , \tag{2.8}$$

the same absolute values as HPS. (See also Eq. (12) of Xing [3].)

3   Consequences of the model

The standard form of the mapping matrix is

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{31} & 0 & s_{31}\mathrm{e}^{-\mathrm{i}\delta} \\ 0 & 1 & 0 \\ -s_{31}\mathrm{e}^{\mathrm{i}\delta} & 0 & c_{31} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{31}c_{12} & c_{31}s_{12} & s_{31}\mathrm{e}^{-\mathrm{i}\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{31}\mathrm{e}^{\mathrm{i}\delta} & c_{12}c_{23}-s_{12}s_{23}s_{31}\mathrm{e}^{\mathrm{i}\delta} & s_{23}c_{31} \\ s_{12}s_{23}-c_{12}c_{23}s_{31}\mathrm{e}^{\mathrm{i}\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{31}\mathrm{e}^{\mathrm{i}\delta} & c_{23}c_{31} \end{pmatrix}, \tag{3.1}$$

with

$$s_{ij}=\sin\theta_{ij} \quad \text{and} \quad c_{ij}=\cos\theta_{ij}. \tag{3.2}$$

Eq. (2.8) can then be written as

$$|s_{12}s_{23}-c_{12}c_{23}s_{31}\mathrm{e}^{\mathrm{i}\delta}|^2 = |U_{31}|^2 = \frac{1}{6}, \tag{3.3}$$

$$|-c_{12}s_{23}-s_{12}c_{23}s_{31}\mathrm{e}^{\mathrm{i}\delta}|^2 = |U_{32}|^2 = \frac{1}{3} \tag{3.4}$$

and

$$c_{23}^2c_{31}^2 = |U_{33}|^2 = \frac{1}{2}. \tag{3.5}$$

(Here, U_{3j} is the same $U_{\tau j}=\langle j|\tau\rangle$ of previous sections, and likewise for other U_{ij}.)

It is convenient to express relations in terms of quantities that vanish in the HPS limit. From (3.5), we find

$$c_{23}^2s_{31}^2 = \frac{1}{2}\tan^2\theta_{31} \tag{3.6}$$

and

$$(2c_{23}^2-1)c_{31}^2 = 1-c_{31}^2 = s_{31}^2 , \tag{3.7}$$

which on division by c_{31}^2 gives

$$\cos 2\theta_{23} = \tan^2\theta_{31} . \tag{3.8}$$

Note that both sides of (3.6)–(3.8) vanish at the HPS point.

Next, the difference of (3.3) and (3.4) gives

$$\frac{1}{6} = |U_{32}|^2-|U_{31}|^2$$
$$= |-c_{12}s_{23}-s_{12}c_{23}s_{31}\mathrm{e}^{\mathrm{i}\delta}|^2-|s_{12}s_{23}-c_{12}c_{23}s_{31}\mathrm{e}^{\mathrm{i}\delta}|^2$$

$$= (s_{23}^2-c_{23}^2s_{31}^2)(c_{12}^2-s_{12}^2)+4c_{12}s_{12}c_{23}s_{23}s_{31}\cos\delta. \tag{3.9}$$

From (3.5), we have

$$s_{23}^2-c_{23}^2s_{31}^2 = s_{23}^2-c_{23}^2(1-c_{31}^2)$$
$$= \frac{1}{2}-\cos 2\theta_{23} \tag{3.10}$$

and

$$(c_{23}s_{23}s_{31})^2 = (c_{23}s_{23}c_{31}\tan\theta_{31})^2 = \frac{1}{2}(s_{23}\tan\theta_{31})^2, \tag{3.11}$$

which on account of (3.8) can also be written as

$$(c_{23}s_{23}s_{31})^2 = \frac{1}{2}s_{23}^2\cos 2\theta_{23}$$
$$= \frac{1}{4}(1-\cos 2\theta_{23})\cos 2\theta_{23}. \tag{3.12}$$

Using (3.10)–(3.12), we may write (3.9) as an equation of θ_{12}, θ_{23} and δ,

$$\frac{1}{6} = \left(\frac{1}{2}-\cos 2\theta_{23}\right)\cos 2\theta_{12}$$

$$+\big[(1-\cos 2\theta_{23})\cos 2\theta_{23}\big]^{\frac{1}{2}}\sin 2\theta_{12}\cos\delta. \tag{3.13}$$

To obtain a relation free of square roots, we may shift the term containing $\cos 2\theta_{12}$ to the left-hand side, multiply the equation by 2 and then square both sides. This yields

$$\left[\frac{1}{3}-(1-2\cos 2\theta_{23})\cos 2\theta_{12}\right]^2$$

$$= 4(1-\cos 2\theta_{23})\cos 2\theta_{23}\sin^2 2\theta_{12}\cos^2\delta, \tag{3.14}$$

which can be solved as a quadratic equation in either $\cos 2\theta_{12}$ or $\cos 2\theta_{23}$, supposing that the other is given as well as $\cos^2 \delta$. Eqs. (3.8) and (3.14) may be taken as two useful equations relating θ_{23} to θ_{31}, as well as θ_{12} to θ_{23} and δ. Both relations follow from hypotheses (i)–(iii) stated in Section 1, and are accurate to the accuracy of (m_μ/m_τ).

For certain purposes, a further simplification can be achieved. Define $\theta_{12}^{\mathrm{HPS}}$ to be the HPS value of θ_{12}, so that

$$\cos 2\theta_{12}^{\mathrm{HPS}} = \frac{1}{3} \ . \tag{3.15}$$

Introduce a positive angle ϕ such that

$$\sin^2 \phi = \cos 2\theta_{23} \ . \tag{3.16}$$

Then the square root of (3.12) can be written as

$$s_{23} c_{23} s_{31} = \frac{1}{2} \sin \phi \cos \phi \tag{3.17}$$

and (3.13) gives, after being multiplied by 2,

$$\frac{1}{3} = \cos 2\phi \cos 2\theta_{12} + \sin 2\phi \sin 2\theta_{12} \cos \delta \ . \tag{3.18}$$

On account of (3.15), we may write this as

$$\cos 2\theta_{12}^{\mathrm{HPS}} = \cos 2\phi \cos 2\theta_{12} + \sin 2\phi \sin 2\theta_{12} \cos \delta \ , \tag{3.19}$$

which is precisely the law of cosines for a spherical triangle, as shown in Fig. 1. The sides of the triangle are

$$2\theta_{12}, \quad 2\phi \quad \text{and} \quad 2\theta_{12}^{\mathrm{HPS}} \ , \tag{3.20}$$

and δ is the angle between $2\theta_{12}$ and 2ϕ. (Note that θ_{23} and θ_{31} are explicit functions of ϕ through (3.16) and (3.8)).

In the absence of T violation, we have

$$\cos \delta = \pm 1 \tag{3.21}$$

and correspondingly,

$$\theta_{12} = \theta_{12}^{\mathrm{HPS}} \pm \phi \ . \tag{3.22}$$

In the presence of T violation, we may write (3.19) as

$$\left[\frac{1}{2}(1+\cos\delta) + \frac{1}{2}(1-\cos\delta) \right] \cos 2\theta_{12}^{\mathrm{HPS}}$$
$$= \frac{1}{2}(1+\cos\delta) \cos 2(\theta_{12}-\phi)$$
$$+ \frac{1}{2}(1-\cos\delta) \cos 2(\theta_{12}+\phi) \tag{3.23}$$

which, in turn, leads to

$$\frac{1+\cos\delta}{1-\cos\delta} = \frac{\cos 2\theta_{12}^{\mathrm{HPS}} - \cos 2(\theta_{12}+\phi)}{\cos 2(\theta_{12}-\phi) - \cos 2\theta_{12}^{\mathrm{HPS}}} \tag{3.24}$$

and, on account of (3.15)

$$\frac{1+\cos\delta}{1-\cos\delta} = \frac{1-3\cos 2(\theta_{12}+\phi)}{3\cos 2(\theta_{12}-\phi)-1}. \tag{3.25}$$

The above LHS is an increasing function of $\cos\delta$, and its RHS at fixed ϕ is an increasing function of θ_{12}. Thus,

$$\left(\frac{\partial \theta_{12}}{\partial \cos\delta} \right)_{\theta_{23}} = \left(\frac{\partial \theta_{12}}{\partial \cos\delta} \right)_{\theta_{31}} = \left(\frac{\partial \theta_{12}}{\partial \cos\delta} \right)_{\phi} > 0 \ . \tag{3.26}$$

From (3.8), (3.16), (3.22), (3.26) and by eliminating δ, we obtain the following statement relating the three mapping angles.

$$\cos 2\theta_{23} = \tan^2 \theta_{31} = \sin^2 \phi \geqslant \sin^2(\theta_{12}-\theta_{12}^{\mathrm{HPS}}), \tag{3.27}$$

where in the last relation, the inequality holds for $\cos^2 \delta < 1$, and the equality when $\cos^2 \delta = 1$.

The Jarlskog invariant J [5] is given by

$$J = s_{12} c_{12} s_{23} c_{23} s_{31} c_{31}^2 \sin\delta \ . \tag{3.28}$$

From (3.17), we can also write

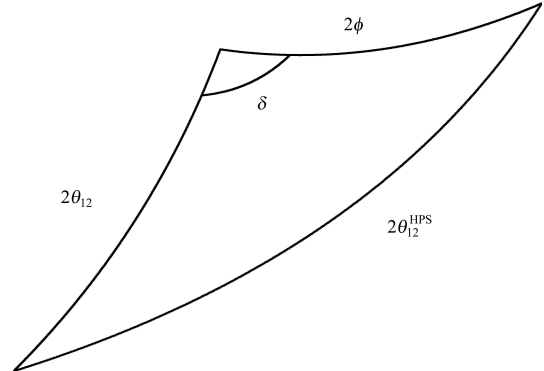$$J = \frac{1}{8} \sin 2\theta_{12} \sin 2\phi \cos^2\theta_{31} \sin\delta \ . \tag{3.29}$$

Fig. 1. The spherical triangle described by (3.19), with $\sin^2 \phi = \cos 2\theta_{23} = \tan^2 \theta_{31}$.

4 The alpha-beta theorem

This section is devoted to establishing a theorem that shall be called the alpha-beta theorem.

Theorem: Suppose the mapping matrix U to have its τ-elements given (apart from their phases) by the HPS values

$$|U_{\tau 1}|^2 = \frac{1}{6} \ , \quad |U_{\tau 2}|^2 = \frac{1}{3} \ , \quad |U_{\tau 3}|^2 = \frac{1}{2}, \tag{4.1}$$

as in (3.3)–(3.5), and that the third τ-element is real and positive with

$$U_{\tau 3} = \frac{1}{\sqrt{2}} \ . \tag{4.2}$$

Then there exist real numbers α and β, such that

$$U = S_1^{-1} V S_2, \tag{4.3}$$

where S_1 and S_2 are both diagonal unitary matrices, and

$$V = \begin{pmatrix} \sqrt{\frac{2}{3}}\sin\frac{\alpha}{2} + \sqrt{\frac{1}{6}}\cos\frac{\alpha}{2}e^{i\beta} & \sqrt{\frac{1}{3}}\sin\frac{\alpha}{2} - \sqrt{\frac{1}{3}}\cos\frac{\alpha}{2}e^{i\beta} & -\sqrt{\frac{1}{2}}\cos\frac{\alpha}{2}e^{i\beta} \\ -\sqrt{\frac{1}{6}}\sin\frac{\alpha}{2} + \sqrt{\frac{2}{3}}\cos\frac{\alpha}{2}e^{-i\beta} & \sqrt{\frac{1}{3}}\sin\frac{\alpha}{2} + \sqrt{\frac{1}{3}}\cos\frac{\alpha}{2}e^{-i\beta} & \sqrt{\frac{1}{2}}\sin\frac{\alpha}{2} \\ \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \tag{4.4}$$

To prove the theorem, we make use of the following lemma, proved in Appendix B.

<u>Lemma</u>: Let W be a 3×3 unitary matrix of the form

$$W = \begin{pmatrix} t & \xi \\ \tilde{\eta} & d \end{pmatrix}, \tag{4.5}$$

where t is a 2×2 matrix, ξ and η are both real 2×1 column matrices and d a real number. Then t can be written in terms of ξ, η, d and an extra real parameter β by the formula

$$t = (1-d^2)^{-1}(-d\xi\tilde{\eta} + \xi'\tilde{\eta}'e^{-i\beta}), \tag{4.6}$$

where ξ' and η' are both real 2×1 column matrices satisfying

$$\tilde{\xi}'\xi' = \tilde{\xi}\xi , \quad \tilde{\eta}'\eta' = \tilde{\eta}\eta$$

and

$$\tilde{\xi}'\xi = \tilde{\eta}'\eta = 0 . \tag{4.7}$$

Supposing the Lemma to be established, we prove the alpha-beta theorem as follows:

The five matrix elements in the third row and the third column of U can all be made real by introducing an extra phase factor into each of these elements. This task can be achieved by introducing unitary diagonal matrices S_1' and S_2' such that

$$W = S_1' U S_2'^{-1} \tag{4.8}$$

has the form (4.5) required by the lemma. Moreover, for our applications,

$$\eta = \begin{pmatrix} \sqrt{\frac{1}{6}} \\ -\sqrt{\frac{1}{3}} \end{pmatrix} \tag{4.9}$$

and

$$d = \sqrt{\frac{1}{2}} . \tag{4.10}$$

The corresponding vector η' is, in accordance with (4.7),

$$\eta' = \begin{pmatrix} -\sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{6}} \end{pmatrix}, \tag{4.11}$$

with the signs in η and η' being chosen for later convenience; the ambiguity will be subsumed in the arbitrariness of β in (4.6).

Since W is unitary, we have

$$t^\dagger t + \eta\tilde{\eta} = 1, \tag{4.12}$$

$$\tilde{\xi}t + d\tilde{\eta} = 0 \tag{4.13}$$

and

$$\tilde{\xi}\xi = 1 - d^2 = \frac{1}{2} . \tag{4.14}$$

Hence, we may define

$$\xi = \sqrt{\frac{1}{2}} \begin{pmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} \end{pmatrix} \tag{4.15}$$

and on account of (4.7)

$$\xi' = \sqrt{\frac{1}{2}} \begin{pmatrix} \sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2} \end{pmatrix}. \tag{4.16}$$

Substituting these expressions into (4.5)–(4.6), we find

$$t = -\begin{pmatrix} \sqrt{\frac{1}{6}}\cos\frac{\alpha}{2} + \sqrt{\frac{2}{3}}\sin\frac{\alpha}{2}e^{-i\beta} & -\sqrt{\frac{1}{3}}\cos\frac{\alpha}{2} + \sqrt{\frac{1}{3}}\sin\frac{\alpha}{2}e^{-i\beta} \\ -\sqrt{\frac{1}{6}}\cos\frac{\alpha}{2} + \sqrt{\frac{2}{3}}\cos\frac{\alpha}{2}e^{-i\beta} & \sqrt{\frac{1}{3}}\sin\frac{\alpha}{2} + \sqrt{\frac{1}{3}}\cos\frac{\alpha}{2}e^{-i\beta} \end{pmatrix}. \tag{4.17}$$

Assembling W according to (4.5), we find that the matrix

$$V \equiv \begin{pmatrix} -e^{i\beta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} W = S_1'' W \tag{4.18}$$

is given by (4.4), and that establishes the alpha-beta theorem, with

$$S_1 = S_1'' S_1' \tag{4.19}$$

and

$$S_2 = S_2' . \tag{4.20}$$

By using the alpha-beta theorem, we can derive several interesting relations between the four parameters θ_{12}, θ_{23}, θ_{31} and δ of the mapping matrix U. These will be discussed in Appendix C.

Remark It will be seen that the above expression for V is identical to the matrix $V_{1-\mathrm{map}}$ shown in Table 1 of Ref. [4], when correction terms in the quantities

$$\chi_{\mathrm{e}}, \ \chi_{\mathrm{p}}, \ c_{\mathrm{e}}, \ c_{\mathrm{p}} \tag{4.21}$$

are neglected; the "$\chi_{\mathrm{e}}, \ \chi_{\mathrm{p}}$" quantities will be shown in Appendix A of this paper to leave the absolute values of the matrix elements in (4.1) unchanged except by an amount of $O((m_{\mu}/m_{\tau})^2)$. It follows therefore, from the alpha-beta theorem just established, that the "$c_{\mathrm{e}}, \ c_{\mathrm{p}}$" correction terms in the upper two rows of $V_{1-\mathrm{map}}$ in Ref. [4], which are admittedly of first order in (m_{μ}/m_{τ}), can be taken into account (to that order) by adjusting the values of α and β, which in Ref. [4] were restricted to be certain given expressions in terms of the detailed matrices G and F.

The outcome is that any experimental predictions made from using Table 1 of Ref. [4], plus the knowledge that its "$c_{\mathrm{e}}, \ c_{\mathrm{p}}$"-corrections are of first order and its χ-corrections of second order in m_{μ}/m_{τ}, can just as well be made on the basis of the weaker hypotheses (i)–(iii) stated in Section 1 of this paper.

5   Discussion

1) In the HPS limit, from (1.11) and (3.1)

$$\sin\theta_{12}^{\mathrm{HPS}} = \sqrt{\frac{1}{3}}, \quad \sin\theta_{31}^{\mathrm{HPS}} = 0$$

and

$$\sin\theta_{23}^{\mathrm{HPS}} = \sqrt{\frac{1}{2}}. \tag{5.1}$$

From (3.8), we have

$$1 - \sin^2 2\theta_{23} = \tan^4\theta_{31} . \tag{5.2}$$

A striking feature of our model is that it predicts a much smaller deviation from HPS in θ_{23} than in θ_{31}. Since θ_{31} is known to be small, from (5.2) we expect θ_{23} even closer to its HPS value of $45°$, as a linear deviation in θ_{23} would be quadratic in θ_{31}.

At present, current data [6–10] are compatible (within 1σ) with the HPS values of θ_{23} and θ_{31}, but there is a suggestion that $\sin^2\theta_{31}$ may be about 0.015. If we take this value, then

$$\sin^2 2\theta_{31} = 0.0591 \tag{5.3}$$

and from (5.2)

$$\sin^2 2\theta_{23} = 0.9998. \tag{5.4}$$

These data seem not yet precise enough to say whether the deviation of $\sin^2 2\theta_{23}$ from 1 is as small as given by (5.4).

2) Next, we turn to our second relation, (3.16) and (3.18) relating θ_{12} to θ_{23} and δ. We may replace $\cos 2\theta_{23}$ with $\tan^2\theta_{31}$ in accordance with (3.8). At any fixed δ, these equations define a curve describing the variations of

$$x = \sin^2\theta_{12} \quad \mathrm{vs} \quad y = \sin^2\theta_{31} . \tag{5.5}$$

The envelope of the family of such curves is shown in Fig. 2, and corresponds to

$$\cos\delta = \pm 1 . \tag{5.6}$$

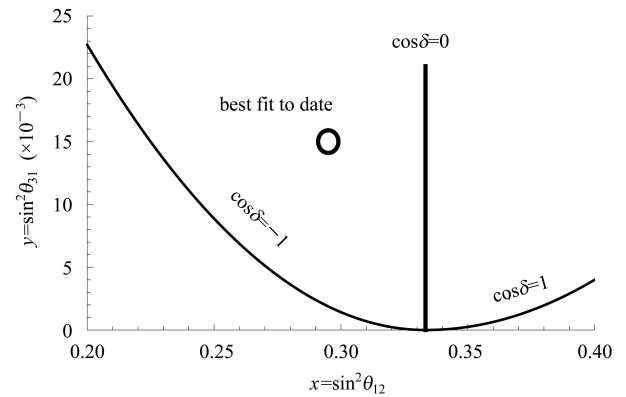The region below the envelope corresponds to $\cos^2\delta > 1$ and is therefore forbidden.

Fig. 2.   The curve described by (5.5) and (5.6). Points below the curve are forbidden and the HPS limit is $(x,y) = \left(\dfrac{1}{3}, 0\right)$.

An examination of current data [10–13] indicates that points on the outermost curve (no T violation) are far from the best fit, and that the forbidden region below the curve is improbable. As the best fit (represented by the circle) shown in Fig. 2 already prefers large T violation, a measurement of δ, combined with improved precisions in θ_{12} and θ_{31}, would give a sensitive test to our model.

3) It is of interest to compare the assumptions and results of Ge, He and Yin [GHY, Ref. [14]] and those of this paper. Both papers regard the HPS mapping matrix as correct to 0th order, and concentrate on the 1st-order deviations from it. In GHY, these deviations are attributed to a perturbation in the neutrino sector, whereas in the present paper the perturbation arises in the charged lepton sector.

In the notations of this paper, a perturbation in the charged lepton sector leads to a mapping matrix U given by (1.9)

$$U = KU_0,$$

whereas a perturbation in the neutrino sector would yield an equivalent form

$$U = U_0 K'. \tag{5.7}$$

A difference appears only when different physical approximations are made in K and K'. As a result, the constraints arrived at on the four parameters θ_{12}, θ_{23}, θ_{31} and δ can be quite different. Their result [GHY(4.87a)] in our notation is

$$\theta_{23} - 45° \cong -\theta_{31} \cot \theta_{12} \cos \delta. \tag{5.8}$$

or (to leading order in deviation from HPS)

$$\cos 2\theta_{23} \cong 2\sqrt{2} \tan \theta_{31} \cos \delta. \tag{5.9}$$

This differs from our (3.8) in two important ways: our model gives a relation between θ_{23} and θ_{31} independent of δ, and it makes $\cos 2\theta_{23}$ quadratic in $\tan\theta_{31}$ instead of linear. Thus, unless $\tan\theta_{31} \cong 2\sqrt{2}\cos\delta$, experiments now under way [15–18] could lead to a resolution between the hypothesis of charged lepton perturbation (this paper) and that of neutrino-perturbation (GHY).

Appendix A

In the timeon model [4], the three left-handed physical leptons are eigenvectors of a hermitian matrix

$$(G + iF)(G - iF) = G^2 + i[F, G] + F^2, \tag{A1}$$

where G and F are both real and symmetric. The eigenvalues of $(G+iF)(G-iF)$ are the squares of the physical masses, with the corresponding bare charged leptons the eigenvectors of G (or G^2). This is a slightly more involved situation than the one described in the present paper, but it leads to the same results.

Set

$$L_0 = G^2, \quad L_1 = i[F, G]$$

and

$$L_2 = F^2. \tag{A2}$$

The bare states are eigenvectors of L_0, and the corresponding physical ones, those of

$$L = L_0 + L_1 + L_2. \tag{A3}$$

The perturbation caused by L_2 on the state $|\tau\rangle$ is of the order

$$\left(\frac{F}{m_\tau}\right)^2 \tag{A4}$$

and therefore negligible. The perturbation caused by L_1 is of the order

$$\frac{F}{m_\tau} \tag{A5}$$

and purely imaginary. Thus, by following the discussion given in Section 2, we can readily arrive at the inference that the deviation of $|\langle k|\tau\rangle|$ from $|\langle k|\tau_0\rangle|$ is of order $(F/m_\tau)^2$.

Appendix B

Here we prove the lemma stated in Sec. 4. Let W be the unitary matrix given by (4.5); it follows then

$$1 = W^\dagger W = \begin{pmatrix} t^\dagger t + \eta\tilde\eta & t^\dagger \xi + d\eta \\ \tilde\xi t + d\tilde\eta & \tilde\xi\xi + d^2 \end{pmatrix} \tag{B1}$$

and

$$1 = WW^\dagger = \begin{pmatrix} tt^\dagger + \xi\tilde\xi & t\eta + d\xi \\ \tilde\eta t^\dagger + d\tilde\xi & \tilde\eta\eta + d^2 \end{pmatrix} \tag{B2}$$

in which t is a 2×2 matrix, ξ and η are both real 2×1 column matrices and d a real number. From the above equations, we have from the lower diagonal elements

$$\tilde\xi\xi = \tilde\eta\eta = 1 - d^2 \tag{B3}$$

and from the off-diagonal elements

$$\tilde\xi t + d\tilde\eta = 0, \\ t\eta + d\xi = 0. \tag{B4}$$

Let ξ' and η' be the two real column matrices that satisfy (4.7). We observe that the four products

$$\xi\tilde\eta, \quad \xi\tilde\eta', \quad \xi'\tilde\eta \quad \text{and} \quad \xi'\tilde\eta' \tag{B5}$$

form a complete basis for 2×2 matrices. Thus, we can express

$$t = t_{11}\xi\tilde\eta + t_{12}\xi\tilde\eta' + t_{21}\xi'\tilde\eta + t_{22}\xi'\tilde\eta' \tag{B6}$$

in which $t_{11}, \cdots, t_{22}$ are four coefficients.

Combining (B6) with (B3) and (4.7), we have

$$\tilde\xi t = t_{11}(\tilde\xi\xi)\tilde\eta + t_{12}(\tilde\xi\xi)\tilde\eta'$$

and

$$t\eta = t_{11}\xi(\tilde\eta\eta) + t_{21}\xi'(\tilde\eta\eta). \tag{B7}$$

Combining these two equations with (B4), we find

$$t_{12} = t_{21} = 0$$

and

$$t_{11} = -\frac{d}{1 - d^2}. \tag{B8}$$

It will be convenient to write the coefficient t_{22} as

$$t_{22} = \lambda t_{11}, \qquad (B9)$$

with λ an unknown complex number. Thus, we can write

$$t = -\frac{d}{1-d^2}(\xi\tilde{\eta} + \lambda\xi'\tilde{\eta}') . \qquad (B10)$$

Turn now to the upper left part of $W^\dagger W$; it gives

$$t^\dagger t + \eta\tilde{\eta} = I, \qquad (B11)$$

where I is the 2×2 unit matrix. From (B10), we find

$$t^\dagger t = \left(\frac{d}{1-d^2}\right)^2 (1-d^2)(\eta\tilde{\eta} + |\lambda|^2\eta'\tilde{\eta}'), \qquad (B12)$$

where we have used (B3) and (4.7) to eliminate the inner products in ξ and ξ'. Using (B11) and (B12) and after

some rearrangement, we have

$$\eta\tilde{\eta} + d^2|\lambda|^2\eta'\tilde{\eta}' = (1-d^2)I. \qquad (B13)$$

On the other hand, we can also verify that

$$\eta\tilde{\eta} + \eta'\tilde{\eta}' = (1-d^2)I. \qquad (B14)$$

by multiplying both sides on the right alternatively by η and by η'.

Thus, (B13) and (B14) lead to

$$d^2|\lambda|^2 = 1. \qquad (B15)$$

This enables us to introduce a phase factor

$$\mathrm{e}^{-i\beta} = -\lambda d, \qquad (B16)$$

so that (B10) becomes (4.6), and the lemma is established.

Appendix C

In this Appendix, we derive certain relations between the angles θ_{12}, θ_{23}, θ_{31}, and δ by a different route, making use of the alpha-beta theorem.

Let

$$U = U(\theta_{12},\ \theta_{23},\ \theta_{31},\ \delta) \qquad (C1)$$

and

$$V = V(\alpha,\ \beta) \qquad (C2)$$

be the matrices given by (3.1) and (4.4), and J_U and J_V, their respective Jarlskog invariants. From (3.28) and (4.4), we find

$$J_U = s_{12}c_{12}s_{23}c_{23}s_{31}c_{31}^2 \sin\delta, \qquad (C3)$$

$$J_V = \frac{1}{12}\sin\alpha\sin\beta. \qquad (C4)$$

Here, we assume (4.1) and therefore (by the alpha-beta theorem) also (4.3). It follows then

$$J_U = J_V \qquad (C5)$$

and for all (i,j),

$$|U_{ij}| = |V_{ij}| . \qquad (C6)$$

Denote

$$a = \theta_{23}, \qquad b = \theta_{31}, \qquad c = \theta_{12},$$
$$s_a = \sin\theta_{23}, \quad s_b = \sin\theta_{31}, \quad s_c = \sin\theta_{12} ,$$
$$c_a = \cos\theta_{23}, \quad c_b = \cos\theta_{31}, \quad c_c = \cos\theta_{12} ,$$

$$\Gamma = \cos 2\theta_{12} = c_c^2 - s_c^2 \qquad (C7)$$

and therefore

$$1 - \Gamma^2 = \sin^2 2\theta_{12} = 4s_c^2 c_c^2. \qquad (C8)$$

We shall explore the consequences of eliminating successively α, δ and β from (C5)–(C6).

(i) Determinations of $\cos\delta$ and $\cos\beta$

By equating

$$|U_{32}|^2 - |U_{31}|^2 = |V_{32}|^2 - |V_{31}|^2 = \frac{1}{6}, \qquad (C9)$$

we find $\cos\delta$ given by

$$[1 - 6\Gamma(s_a^2 - c_a^2 s_b^2)]^2 = (1-\Gamma^2)(12s_a c_a s_b \cos\delta)^2. \qquad (C10)$$

Likewise, from

$$|U_{13}|^2 = |V_{13}|^2 \qquad \text{and} \qquad |U_{23}|^2 = |V_{23}|^2, \qquad (C11)$$

it follows then

$$\sin^2\alpha = (4s_a s_b c_b)^2, \qquad (C12)$$

and from $|U_{12}|^2 = |V_{12}|^2$, we find

$$\sin^2\alpha\cos^2\beta = (1 - 3c_b^2 s_c^2)^2. \qquad (C13)$$

Thus,

$$\cos^2\beta = [(1 - 3c_b^2 s_c^2)/4s_a s_b c_b]^2. \qquad (C14)$$

(ii) Relation between $\sin\beta$ and $\sin\delta$

From $|U_{33}|^2 = |V_{33}|^2$,

$$c_a^2 c_b^2 = \frac{1}{2}, \qquad (C15)$$

which together with the equality of Jarlskog invariants (C3) and (C4) yield

$$2\sin^2\beta = 9s_c^2 c_c^2 \sin^2\delta . \qquad (C16)$$

(iii) Elimination of δ

Multiplying (C16) by $64s_a^2 c_a^2 s_b^2$, we have

$$128 s_a^2 c_a^2 s_b^2 \sin^2\beta = 144(4s_c^2 c_c^2)s_a^2 c_a^2 s_b^2 \sin^2\delta$$
$$= 144(1-\Gamma^2)s_a^2 c_a^2 s_b^2 \sin^2\delta, \qquad (C17)$$

with Γ given by (C7). Combining the above equation with (C10), we derive

$$[1-6\Gamma(s_a^2-c_a^2s_b^2)]^2+128s_a^2c_a^2s_b^2\sin^2\beta = 144(1-\Gamma^2)s_a^2c_a^2s_b^2. \tag{C18}$$

(iv) Elimination of β

Multiplying (C14) by $128s_a^2c_a^2s_b^2$, we find

$$
\begin{aligned}
128s_a^2c_a^2s_b^2\cos^2\beta &= 8(1-3c_b^2s_c^2)(c_a/c_b)^2 \\
&= 16(1-3c_b^2s_c^2)c_a^4 \tag{C19}
\end{aligned}
$$

on account of (C15). The sum of (C18) and (C19) gives

$$
\begin{aligned}
&[1-6\Gamma(s_a^2-c_a^2s_b^2)]^2+128s_a^2c_a^2s_b^2 \\
&=144(1-\Gamma^2)s_a^2c_a^2s_b^2+(4c_a^2-6s_c^2)^2 \tag{C20}
\end{aligned}
$$

and therefore a relation between the angles a, b and c:

$$[1-6\Gamma(s_a^2-c_a^2s_b^2)]^2-(4c_a^2-6s_c^2)^2 = 16(1-9\Gamma^2)s_a^2c_a^2s_b^2. \tag{C21}$$

It may appear that by combining (C21) with (3.8) one could arrive at a determination of a and b in terms of c (i.e., of θ_{23} and θ_{31} in terms of θ_{12}), without fixing δ. But as we shall show, (C21) and (3.8) are actually redundant.

Define

$$X \equiv 1-6\Gamma(s_a^2-c_a^2s_b^2), \tag{C22}$$

$$Y \equiv 4c_a^2-6s_c^2 \tag{C23}$$

and

$$Z \equiv 16(1-9\Gamma^2)s_a^2c_a^2s_b^2 . \tag{C24}$$

with $\Gamma = \cos 2c$ given by (C7). Thus (C21) becomes

$$X^2-Y^2=Z. \tag{C25}$$

From (3.8), we have $\cos 2a = \tan^2 b$ and therefore

$$2c_a^2c_b^2=1. \tag{C26}$$

Hence, we can express b in terms of a, c in terms of Γ and write (C22)–(C24) as

$$X+Y = 2(2c_a^2-1)(1+3\Gamma), \tag{C27}$$

$$X-Y = -4(c_a^2-1)(1-3\Gamma), \tag{C28}$$

and

$$Z=-8(2c_a^2-1)(c_a^2-1)(1-9\Gamma^2) . \tag{C29}$$

It follows then (C25) is an identity.

References

1 Harrison P F, Perkins D H, Scott W G. Phys. Lett. B, 2002, **530**: 167; HE X G, Zee A. Phys. Lett. B, 2003, **560**: 87

2 MA E, Rajasekaran G. Phys. Rev. D, 2001, **64**: 113012 [arXiv: hep-ph/010629]; MA E. Mod. Phys. Lett. A, 2002, **17**: 627 [arXiv: hep-ph/0203238]; Altarelli G, Ferruglio F. Nucl. Phys. B, 2006, **741**: 215 [arXiv: hep-ph/0512103]

3 XING Z Z. Phys. Lett. B, 2002, **533**: 85; Harrison P F, Scott W G. Phys. Lett. B, 2002, **535**: 163

4 Friedberg R, Lee T D. Ann. Phys., 2009, **324**: 2196; Araki T, Geng C Q. Phys. Rev. D, 2009, **79**: 077301 [arXiv:0901.4820v2]

5 Jarlskog C. Phys. Rev. D, 1987, **35**: 1685

6 Ashie Y et al (Super Kam collaboration). Phys. Rev. Lett., 2004, **93**: 101801 [arXiv:0808.2016]; Hosaka J et al. Phys. Rev. D, 2006, **74**: 032002; Abe K et al. Phys. Rev. Lett., 2006, **97**: 171801

7 Apollonio M et al (Chooz collaboration). Eur. Phys. J. C, 2003, **27**: 331 [arXiv:hep-ex/0301017]

8 Araki T et al (KamLAND collaboration). Phys. Rev. Lett., 2005, **94**: 081801 [arXiv:hep-ex/0406035]

9 Adamson P et al (MINOS collaboration). Phys. Rev. Lett., 2009, **103**: 261802 [arXiv:0909.4996]

10 Fogli G L et al. Phys. Rev. D, 2008, **78**: 033010; Schwetz T et al. New J. Phys., 2008, **10**: 113011; XING Z Z. Int. J. Mod. Phys. A, 2008, **23**: 4355

11 Abdurashitov J N et al (SAGE collaboration). Phys. Rev. C, 2009, **80**: 015907 [arXiv:0901.2200]

12 Aharmin B et al (SNO collaboration). Phys. Rev. Lett., 2008, **101**: 111501

13 Hosaka J et al (Super Kam collaboration). Phys. Rev. D, 2006, **73**: 112001 [arXiv:hep-ex/0508053]

14 GE S F, HE H J, YIN F R. JCAP 05 (2010) 017 [arXiv: 1001.0940]

15 Palomares C (Double-Chooz collaboration). [arXiv: 0911.3227]; Ardellier F et al. [arXiv:hep-ex/0405032]; Ardellier F et al. [arXiv:hep-ex/0606025]

16 Gue X et al (Daya Bay collaboration). [arXiv:hep-ex/070120]; CHEN S. J. Phys.: Conference Series, 2008, **120**: 052024; White C. J. Phys.: Conference Series, 2008, **136**: 022012

17 Ichikawa A K (J Parc collaboration). Lect. Notes Phys., 2009, **781**: 17

18 Ayres D S et al (Nova collaboration). [arXiv:hep-ex/0503053]