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Drinfeld-Manin instanton and its noncommutative generalization*

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Abstract The Drinfeld-Manin construction of $U(N)$ instanton is reformulated in the ADHM formulism, which gives explicit general solutions of the ADHM constraints for $U(N)$ $(N \geqslant 2k-1)$ k-instantons. For the $N < 2k-1$ case, implicit results are given systematically as further constraints. We find that this formulism can easily be generalized to the noncommutative case, where the explicit solutions are also obtained.

Key words instanton, ADHM construction, noncommutative gauge field theory

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1 Introduction

Instanton solutions in gauge field theory are of great physical and mathematical interest [1–6]. Many significant achievements have been made in this area since their discovery in 1975 [7].

Because of the great significance of instanton solutions in various aspects of physics and mathematics, it is necessary to obtain all these solutions in gauge field theory. This task was almost accomplished in 1978, when Atiyah, Hitchin, Drinfeld and Manin (ADHM) established the famous construction of instantons for almost all gauge groups[2)] [8, 9]. This ADHM construction essentially reduces the problem of solving a set of nonlinear partial differential equations, which defines the instantons, to that of solving a set of quadratic algebraic equations, called the ADHM constraints. It gives the most general instanton configurations, and so provides the probability of learning the whole instanton moduli spaces.

But even algebraic equations are not always solvable, so the ADHM constraints remain a difficult problem. In other words, it is hard to attain satisfactory parametrization of instanton moduli spaces. For gauge group $U(N)$, or essentially $SU(N)$, during a rather long time since the presentation of ADHM construction, general solutions of the ADHM constraints are known only when $k = 1$ and N is arbitrary, or $k \leqslant 3$ and $N = 2$ [9–11] (except for the Drinfeld-Manin parametrization explained below), where k is the topological charge, or equivalently the instanton number [12], which is an integer classifying the instanton solutions. In 1999, Dorey et al. essentially rediscovered the Drinfeld-Manin parametrization for $N \geqslant 2k$ [13], of which they seemed unaware.

In recent years, the study of gauge field theory in noncommutative space time has become an active research area [14–16], mostly due to its relevance to string theory [17]. An interesting phenomenon in noncommutative gauge field theory is that instanton solutions survive the space-time noncommutativity, and the moduli spaces of them become even better behaved [18]. Correspondingly, the ADHM construction has been generalized to the noncommutative case [19, 20][3)]. The noncommutative ADHM constraints seem even more difficult to solve: for gauge group $U(N)$, up to now, only when $k = 1$ and N is arbitrary, or $k = 2$ and $N = 1$, have general solutions been known [24].

Drinfeld and Manin presented another construc-

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2) More precisely, the construction for exceptional groups is not known yet.

3) In fact, the ADHM constraints arise naturally as the D-flat condition of the worldvolume theory of the Dp-brane in Dp-brane-D(p+4)-brane bound systems [21, 22]. When a constant NS-NS B-field is present in the worldvolume of the D(p+4)-branes, the worldvolume theory of the D(p+4)-branes becomes noncommutative, and a Fayet-Iliopoulos D-term appears in the worldvolume theory of the Dp-branes [23]. Corresponding to this term, one must add a constant term to the ADHM constraints.

tion of instantons [25] shortly after the ADHM construction, from a slightly different point of view. This construction explicitly gives parametrization of the $U(2k)$ k-instanton moduli space. In addition, all $U(N)$ k-instanton configurations can be indirectly obtained. Their original description of this construction was in a vector-bundle language. In this article we will translate it into the more familiar ADHM language and see how they give explicit general solutions of the ADHM constraints with gauge group $U(N)$ ($N \geqslant 2k-1$) and topological charge k. For the $N<2k-1$ case, the further constraints are hard to solve explicitly, but our systematic discussion may be useful for the indirect way to the collective coordinate integral [13], in this case. Moreover, fortunately, a noncommutative generalization of this ADHM formulation of Drinfeld-Manin instanton is straightforward. In fact, translating other constructions of instantons into the ADHM construction has been proved an efficient way to generalize them to the noncommutative case[1)].

This paper is organized as follows. In Sec. 2 and Sec. 3, we recall the definition of instantons and the ADHM construction, in the commutative case and the noncommutative case, respectively. In Sec. 4, the Drinfeld-Manin construction is briefly reviewed and reformulated in the ADHM formulism. This construction is generalized to the noncommutative case in Sec. 6. Sec. 5 gives the explicit solution of the ADHM matrix, which can be applied to both the commutative and the noncommutative cases. In Appendix A, the conditions for a Hermitian matrix of restricted rank are given. These conditions are needed in the discussion of the $N<2k$ case.

2 Instantons and (ordinary) ADHM construction

Instanton solutions in (Euclidean) gauge field theory were discovered by Belavin, Polyakov, Schwartz and Tyupkin (BPST) in 1975 [7]. They are defined by the so-called (anti-)self-dual equations,

$$\tilde{F}_{mn} = \pm F_{mn}, \quad (m,n=1,2,3,4), \tag{1}$$

and the solutions are known as self-dual (SD, for "+" sign) and anti-self-dual (ASD, for "−" sign) instantons. The definition of dual field $\tilde{F}_{mn}$ is familiar in electrodynamics, which is

$$\tilde{F}_{mn} = \frac{1}{2}\epsilon_{mnpq}F_{pq}, \tag{2}$$

when the standard Euclidean metric $g_{mn}=\delta_{mn}$ is assumed. We note that the notions of SD and ASD are interchanged by a parity transformation. Without loss of generality, we will consider only the ASD instantons.

All the (ASD) instanton solutions can be obtained by the ADHM construction [8, 9], as follows. In this construction we introduce the following ingredients (for $U(N)$ gauge theory with instanton number k):

1) $k\times k$ matrix $B_{1,2}$, $k\times N$ matrix I and $N\times k$ matrix J, 2) the following quantities:

$$\mu_{\mathrm{r}} = [B_1,B_1^\dagger]+[B_2,B_2^\dagger]+I\,I^\dagger - J^\dagger J, \tag{3}$$

$$\mu_{\mathrm{c}} = [B_1,B_2]+I\,J. \tag{4}$$

The claim of ADHM is as follows:

1) Given $B_{1,2}$, I and J such that $\mu_{\mathrm{r}}=\mu_{\mathrm{c}}=0$, an ASD gauge field can be constructed.

2) All ASD gauge fields can be obtained in this way.

It is convenient to introduce a quaternionic notation for the 4-dimensional Euclidean space-time indices,

$$x \equiv x^n\sigma_n, \quad \bar{x} \equiv x^n\bar{\sigma}_n = x^\dagger, \tag{5}$$

where $\sigma_n \equiv (\mathrm{i}\vec{\tau},1)$ and τ^c, $c=1,2,3$ are the three Pauli matrices, and the conjugate matrices $\bar{\sigma}_n \equiv (-\mathrm{i}\vec{\tau},1) = \sigma_n^\dagger$. Then the basic object in the ADHM construction is the $(N+2k)\times 2k$ matrix Δ, which is linear in the space-time coordinates,

$$\Delta = a + b\bar{x} \equiv a + b(\bar{x}\otimes 1_k), \tag{6}$$

where the constant matrices

$$a = \begin{pmatrix} I^\dagger & J \\ B_2^\dagger & -B_1 \\ B_1^\dagger & B_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0_{N\times k} & 0_{N\times k} \\ 1_k & 0 \\ 0 & 1_k \end{pmatrix}. \tag{7}$$

It is easy to check that the ADHM constraints (3) and (4) are equivalent to the so-called factorization condition,

$$\Delta^\dagger\Delta = \begin{pmatrix} f^{-1} & 0 \\ 0 & f^{-1} \end{pmatrix}, \tag{8}$$

where $f(x)$ is a $k\times k$ Hermitian matrix. From the above condition, we can construct a Hermitian projection operator P,

$$P = \Delta f \Delta^\dagger. \tag{9}$$

Here and after we use the following abbreviation for expressions with f:

$$\Delta f \Delta^\dagger \equiv \Delta \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \Delta^\dagger = \Delta(1_2\otimes f)\Delta^\dagger.$$

1) See Appendix A of Ref. [20] for the generalization of 't Hooft construction to the noncommutative case through its ADHM data.

Obviously, the null space of $\Delta^\dagger(x)$ is of N dimensions for generic x. The basis vector for this null space can be assembled into an $(N+2k)\times N$ matrix $U(x)$,

$$\Delta^\dagger U=0, \tag{10}$$

which can be chosen to satisfy the following orthonormal condition,

$$U^\dagger U=1. \tag{11}$$

The above orthonormal condition guarantees that $UU^\dagger$ is also a Hermitian projection operator. Now it can be proved (see Ref. [20]) that the completeness relation

$$P+UU^\dagger=1 \tag{12}$$

holds if U contains the whole null space of $\Delta^\dagger$. In other words, this completeness relation requires that U consists of all the zero modes of $\Delta^\dagger$.

The (anti-Hermitian) gauge potential is constructed from U by the following formula,

$$A_m=U^\dagger\partial_m U. \tag{13}$$

Then we get the corresponding field strength,

$$\begin{aligned}F_{mn} &= \partial_{[m}A_{n]}+A_{[m}A_{n]}\equiv\partial_m A_n-\partial_n A_m+[A_m,A_n]\\ &= \partial_{[m}(U^\dagger\partial_{n]}U)+(U^\dagger\partial_{[m}U)(U^\dagger\partial_{n]}U)=\partial_{[m}U^\dagger(1-UU^\dagger)\partial_{n]}U\\ &= \partial_{[m}U^\dagger\Delta f\Delta^\dagger\partial_{n]}U=U^\dagger\partial_{[m}\Delta f\partial_{n]}\Delta^\dagger U=U^\dagger b\bar{\sigma}_{[m}\sigma_{n]}fb^\dagger U\\ &= 2\mathrm{i}\bar{\eta}^c_{mn}U^\dagger b(\tau^c f)b^\dagger U.\end{aligned} \tag{14}$$

Here, $\bar{\eta}^c_{mn}$ is the standard 't Hooft η-symbol, which is anti-self-dual,

$$\frac{1}{2}\epsilon_{mnpq}\bar{\eta}^c_{pq}=-\bar{\eta}^c_{mn}. \tag{15}$$

3 Noncommutative ADHM construction

First let us recall briefly the gauge field theory on noncommutative Euclidean space (time)[1)]. For a general noncommutative R^4, we mean a space with Hermitian-operator coordinates x^n, $n=1,\cdots,4$, which satisfy the following relations,

$$[x^m,x^n]=\mathrm{i}\theta^{mn}, \tag{16}$$

where θ^{mn} are real constants. If we assume the standard (Euclidean) metric for the noncommutative R^4, we can use the orthogonal transformation with positive determinant to change θ^{mn} into the following standard form,

$$(\theta^{mn})=\begin{pmatrix}0 & \theta^{12} & 0 & 0\\ -\theta^{12} & 0 & 0 & 0\\ 0 & 0 & 0 & \theta^{34}\\ 0 & 0 & -\theta^{34} & 0\end{pmatrix}. \tag{17}$$

By using this form of θ^{mn}, the only non-vanishing commutators are

$$[x^1,x^2]=\mathrm{i}\theta^{12},\quad [x^3,x^4]=\mathrm{i}\theta^{34}, \tag{18}$$

and the other two obtained by using the anti-symmetric property of commutators.

The full noncommutative gauge field theory demands most of the abstract notions from noncommutative geometry, such as differential forms and vector bundles on noncommutative spaces [27, 28]. But for the $U(N)$ gauge theory on noncommutative Euclidean space, things will be much simpler: in fact, the final effect is almost to replace all the coordinates in ordinary $U(N)$ gauge theory with the above operator coordinates. However, a definition of derivatives in the noncommutative case is necessary for any gauge field theory. We define

$$\partial_m f\equiv-\mathrm{i}\theta_{mn}[x^n,f], \tag{19}$$

where θ_{mn} is the matrix inverse of θ^{mn}. For our standard form (17) of θ^{mn} we have

$$\partial_1 f=\frac{\mathrm{i}}{\theta^{12}}[x^2,f],\quad \partial_2 f=-\frac{\mathrm{i}}{\theta^{12}}[x^1,f], \tag{20}$$

and similar relations for $\partial_{3,4}$.

Now we recall the noncommutative ADHM construction [19] briefly here. By introducing the same data as above but considering the coordinates as noncommutative, we see that the factorization condition (8) still gives $\mu_{\mathrm{c}}=0$, but μ_{r} no longer vanishes. It is easy to check that the following relation holds,

$$\mu_{\mathrm{r}}=\zeta\equiv 2\theta^{12}+2\theta^{34}. \tag{21}$$

The form (8) of ADHM constraints is invariant whether the space time is commutative or not.

The space-time noncommutativity brings nontrivial effects on the physics of gauge field theory. A remarkable example is the mixing between the infrared (IR) and the ultraviolet (UV) degrees of freedom [29]. Concerning the ADHM construction, in the noncommutative case the operator $\Delta^\dagger\Delta$ always has

1) For general reviews of noncommutative geometry and field theory, see, for example, [14–16, 26].

no zero mode (see Ref. [14]) and the moduli spaces of noncommutative instantons are better behaved than their commutative counterparts (see, for example, the lectures by H. Nakajima [18]). A related interesting fact is that noncommutative $U(1)$ gauge theory allows nonsingular instanton solutions [19, 30], while in the commutative case the simplest gauge group for which nonsingular instanton solutions exist is $SU(2)$.

Whether in the commutative case or in the noncommutative case, we find that the above ADHM construction with b in the canonic form (7) is unaffected by the following transformations,

$$\Delta \to \begin{pmatrix} 1_N & 0 \\ 0 & 1_2 \otimes u \end{pmatrix} \Delta (1_2 \otimes u^\dagger), \tag{22}$$

where $u \in U(k)$. This is called the auxiliary symmetry of the ADHM construction, which acts on a, f and U as

$$B_1 \to uB_1u^\dagger, \tag{23}$$

$$B_2 \to uB_2u^\dagger, \tag{24}$$

$$I \to uI, \tag{25}$$

$$J \to Ju^\dagger, \tag{26}$$

$$f \to ufu^\dagger, \tag{27}$$

$$U \to \begin{pmatrix} 1_N & 0 \\ 0 & 1_2 \otimes u \end{pmatrix} U. \tag{28}$$

Now we can do a parameter counting for the (commutative or noncommutative) ADHM $U(N)$ k-instanton. a in the form (7) contains $4k^2+4Nk$ real parameters. The ADHM constraints (3,4) impose $3k^2$ real conditions on them, and the auxiliary symmetry removes further k^2 real degrees of freedom. In total we have $4Nk$ real parameters left, which is expected according to physical analysis [9].

The above ADHM construction is also unaffected by the following transformations,

$$\Delta \to \begin{pmatrix} \mathcal{U} & 0 \\ 0 & 1_{2k} \end{pmatrix} \Delta, \quad \mathcal{U} \in SU(N), \tag{29}$$

which can be regarded as the global gauge rotations of the instanton configuration. This global gauge symmetry leaves $B_{1,2}$ and f unchanged and acts on I, J and U as

$$I \to I\mathcal{U}^\dagger, \tag{30}$$

$$J \to \mathcal{U}J, \tag{31}$$

$$U \to \begin{pmatrix} \mathcal{U} & 0 \\ 0 & 1_{2k} \end{pmatrix} U\mathcal{U}^\dagger. \tag{32}$$

If we wish to eliminate this global gauge symmetry from the $4Nk$ real parameters and retain the "purely" physical degrees of freedom, the number of independent real parameters will be $4Nk-N^2+1$ for $k \geqslant N/2$, and $4Nk-N^2+(N-2k)^2+1=4k^2+1$ for $K \leqslant N/2$, because in this case only $N^2-(N-2k)^2-1$ degrees of freedom in the $SU(N)$ group act nontrivially on I and J.

4 ADHM formulation of the Drinfeld-Manin construction

Shortly after the ADHM construction was established, Drinfeld and Manin successfully constructed all instanton solutions from a so-called "instanton bundle" point of view [25], which we call the Drinfeld-Manin construction. In this construction, the Euclidean space time is compactified by a point to S^4 and the instanton gauge potentials are considered as Levi-Civita connections on some nontrivial vector bundles, named instanton bundles, on this S^4. The instanton bundles are complex bundles (for the case of $U(N)$ gauge group) orthogonally complementary, under some metrics, to a trivial vector bundle M. The (anti-)self-duality of the Levi-Civita field strength imposes some conditions on the metric, which are actually the ADHM constraints.

We can always perform a complex linear transformation (on the basis vectors of the fibre space) to make the (Hermitian) metric standard. If we have done so, then the column vectors of Δ in the ADHM construction constitute a basis of the section space of M. So the matrix U consists of orthonormal basis vectors of the section space of the instanton bundle L, and $UU^\dagger$ is the projection operator corresponding to L. As is familiar to us, the gauge potential (13) is natural as the Levi-Civita connection on L. The above statements briefly explain how the instanton bundle can be related to the familiar ADHM objects.

To formulate the Drinfeld-Manin construction in the ADHM language, we first concentrate on the $U(2k)$ k-instanton case. Now

$$h = \begin{pmatrix} b & a \end{pmatrix} \tag{33}$$

is a $4k \times 4k$ square matrix, and

$$\Delta = hX, \tag{34}$$

where

$$X \equiv \begin{pmatrix} \bar{x} \otimes 1_k \\ 1_{2k} \end{pmatrix}. \tag{35}$$

Thus we have

$$\Delta^\dagger \Delta = X^\dagger h^\dagger h X = X^\dagger \begin{pmatrix} 1_{2k} & \underline{a} \\ \underline{a}^\dagger & a^\dagger a \end{pmatrix} X \equiv X^\dagger Q X, \tag{36}$$

where

$$\underline{a} \equiv \begin{pmatrix} B_2^\dagger & -B_1 \\ B_1^\dagger & B_2 \end{pmatrix} \tag{37}$$

is the lower blocks of a.

In fact, the column vectors of X constitute a basis of the section space of M (before we perform the complex linear transformation mentioned above) and Q is the corresponding metric. From the ADHM point of view now, to make $\Delta^\dagger\Delta$ of the factorized form (8), it is easy to see that Q must satisfy the following factorization condition,

$$a^\dagger a = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \tag{38}$$

where R is a $k\times k$ constant Hermitian matrix. Using the auxiliary symmetry transformation (22), we can make R diagonalized,

$$R = \mathrm{diag}(r_1, r_2, \cdots, r_k), \quad r_1 \leqslant r_2 \leqslant \cdots \leqslant r_k. \tag{39}$$

On the one hand, we can assume the above form of R to fix the auxiliary symmetry, which is nonphysical; on the other hand, even assuming this cannot completely fix the auxiliary symmetry: for generic R this residual symmetry is $U(1)^{\times k}/U(1)$, and if some of the r_i are equal, this residual symmetry is even larger. Further, for generic R, this residual symmetry can be completely fixed by requiring (k–1) of the off-diagonal elements of B_1 or B_2, say $(B_1)_{ik}$ $(i=1,2,\cdots,k-1)$, to be real; special cases of coincident r_i can be carefully treated as well.

To sum up, we can choose $\underline{a}$ and R of the form (39) as the collective coordinates of the $U(2k)$ k-instanton, while removing some of the degrees of freedom in $\underline{a}$. Obviously, the number of independent real parameters is $4k^2+k-(k-1)=4k^2+1$, which coincides with the parameter counting in last section. Noting that

$$a^\dagger a = \underline{a}^\dagger \underline{a} + K^\dagger K, \tag{40}$$

where

$$K \equiv \begin{pmatrix} I^\dagger & J \end{pmatrix} \tag{41}$$

is the upper blocks of a, $\underline{a}$ and R must satisfy the condition that

$$S \equiv 1_2 \otimes R - \underline{a}^\dagger \underline{a} \tag{42}$$

is a positive semidefinite matrix. This condition introduces a boundary to the span of the parameters in $\underline{a}$ and R. Thus we have obtained parametrization of the $U(2k)$ k-instanton moduli space, though the complicated boundary makes it a little imperfect, which is an inevitable consequence of the highly nontrivial topology of the instanton moduli space. This parametrization (also for the following $N>2k$ case) was, in fact, rediscovered by Dorey et al. in 1999 [13], but they did not point out the relation between their work and [25].

Now the matrix K can be expressed in terms of $\underline{a}$ and R due to

$$K^\dagger K = S. \tag{43}$$

Because in the present case K is a square matrix, one may naturally take $K=K^\dagger=S^{1/2}$, which automatically eliminates the global gauge degrees of freedom. This expression of K seems simple and explicit, but it includes three steps: diagonalizing, extracting the square root, and undoing the diagonalization. In fact, to diagonalize S we need to solve an equation of degree k, which we must avoid if we have better choices. Fortunately, a better choice does exist. We may have in mind the simplification of quadratic forms via congruent transformations in basic linear algebra,

$$B^{\mathrm{T}} E B = A, \tag{44}$$

where E is the canonical form of A. If A is nonsingular, E will be the identity matrix. Otherwise, E will have the form $\mathrm{diag}(1,\cdots,1,0,\cdots,0)$, which can be considered, from a different point of view, as E always being the identity while allowing B to be singular,

$$B^{\mathrm{T}} B = A. \tag{45}$$

The transformation matrix B can be easily obtained by completing squares or by simultaneous row and column transformations, without solving any nonlinear equations. Now S here is a Hermitian form, not a quadratic one, but the method is similar. By completing squares to simplify S, we will give the explicit solution to Eq. (43) in Sec. 5.

Next we can consider the $N\neq 2k$ cases. These are very simple. If $N>2k$, it is easy to find, as has been shown in much of the literature, a natural embedding of the above $U(2k)$ solution K in the $U(N)$ solution K',

$$K' = \begin{pmatrix} 0_{(N-2k)\times 2k} \\ K \end{pmatrix}. \tag{46}$$

This gives the $4k^2+1$ "purely" physical degrees of freedom of the $U(N)$ k-instanton. To get all the "ADHM" degrees of freedom, i.e., including the global gauge rotations, we only need to perform the following transformations,

$$K' \to \mathcal{U}K', \quad \mathcal{U} \in \frac{U(N)}{U(1)\times U(N-2k)}, \tag{47}$$

which add $N^2-(N-2k)^2-1$ more parameters to the "purely" physical degrees of freedom and make the total number of real parameters $4Nk$.

If $N < 2k$, we can simply restrict the rank of S not greater than N. Then from Eq. (43) it is easy to see that K can take the following form,

$$K=\begin{pmatrix} K' \\ 0_{(2k-N)\times 2k} \end{pmatrix}, \tag{48}$$

where the $N\times 2k$ matrix K' is the ADHM matrix for the $U(N)$ k-instanton. Linear algebra tells us that for an $l\times l$ Hermitian matrix H the condition $\mathrm{rank}(H)\leqslant l-r$ is equivalent to r^2 real conditions on the elements of H. So the number of "purely" physical parameters is $4k^2+1-(2k-N)^2=4Nk-N^2+1$, which again coincides with the parameter counting in last section. The global gauge rotations are introduced as

$$K'\to \mathcal{U}K', \quad \mathcal{U}\in SU(N), \tag{49}$$

which supply the other N^2-1 real parameters for all the "ADHM" degrees of freedom. So far, everything seems fine, but in fact the $(2k-N)^2$ real conditions cause another problem. The appendix A of this paper will show how to explicitly write down these conditions on elements of $\underline{a}$ and R. There we will see that for $N<2k$–1 they are too complicated to solve, so this formulism is not appropriate for giving explicit solutions for this case. However, these systematic conditions may be useful for an indirect method for the instanton collective coordinate integral [13], which is left for future work.

Only the $N=2k-1$ case is simple. In this case, there is only one condition,

$$\det(S)=0, \tag{50}$$

which from Eq.(42) can be regarded as a quadratic equation of one of the r_i, say r_k. So we can take the same free parameters as in the $N=2k$ case except r_k, and express r_k in terms of the other parameters. The quadratic equation (50) has two roots. A little thought will make it clear that one of the eigenvalues of S has been negative when we take the smaller root. Thus we can only take the greater one as r_k, which accomplishes parametrization of the $U(2k-1)$ k-instanton moduli space.

5 Explicit solution of the ADHM matrix

Let Z be the column vector

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2k} \end{pmatrix}. \tag{51}$$

Then the Hermitian form that we wish to simplify is

$$Z^\dagger S Z=\sum_{i,j=1}^{2k} z_i^* S_{ij} z_j. \tag{52}$$

The first step is to complete the square with respect to z_1,

$$\begin{aligned} Z^\dagger S Z \ =\ & S_{11}\left|z_1+\sum_{j=2}^{2k}\frac{S_{1j}}{S_{11}}z_j\right|^2-S_{11}^{-1}\sum_{i,j=2}^{2k} z_i^* S_{i1}S_{1j}z_j \\ & +\sum_{i,j=2}^{2k} z_i^* S_{ij} z_j. \end{aligned} \tag{53}$$

Now the combination of the last two terms in the above equation is a Hermitian form of $z_2, z_3, \cdots, z_{2k}$, which can be recast as

$$\begin{aligned} & S_{11}^{-1}\sum_{i,j=2}^{2k} z_i^*(S_{11}S_{ij}-S_{i1}S_{1j})z_j \\ & =S_{11}^{-1}\sum_{i,j=2}^{2k} z_i^* S_{11;ij} z_j, \end{aligned} \tag{54}$$

with $S_{11;ij}$ the second-order minor (with diagonal elements S_{11} and S_{ij}) of S. Then completing the square with respect to z_2 gives

$$\begin{aligned} Z^\dagger S Z\ =\ & S_{11}\left|z_1+\sum_{j=2}^{2k}\frac{S_{1j}}{S_{11}}z_j\right|^2+S_{11}^{-1}S_{11;22}\left|z_2+\sum_{j=3}^{2k}\frac{S_{11;2j}}{S_{11;22}}z_j\right|^2 \\ & +S_{11}^{-1}\left(-S_{11;22}^{-1}\sum_{i,j=3}^{2k} z_i^* S_{11;i2}S_{11;2j}z_j+\sum_{i,j=3}^{2k} z_i^* S_{11;ij} z_j\right). \end{aligned} \tag{55}$$

Further simplification of the last term in the above equation needs the following formula to hold for an arbitrary square matrix M,

$$M_{N;mn}M_{N;ij}-M_{N;in}M_{N;mj}=M_N M_{N;mn;ij}, \tag{56}$$

where N is a square sub-matrix of M, $M_{N;mn}$ the minor (with respect to N plus M_{mn}) of M and so on. In fact, we have the most general formula

$$\det\begin{pmatrix} M_{N;i_1j_1} & M_{N;i_1j_2} & \cdots & M_{N;i_1j_l} \\ M_{N;i_2j_1} & M_{N;i_2j_2} & \cdots & M_{N;i_2j_l} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N;i_lj_1} & M_{N;i_lj_2} & \cdots & M_{N;i_lj_l} \end{pmatrix} = M_N^{l-1} M_{N;i_1j_1;i_2j_2;\cdots;i_lj_l}, \tag{57}$$

which is proved in Appendix B. Applying Eq. (56) to Eq. (55) gives

$$\begin{aligned} Z^\dagger S Z \;=\; & S_{11}\left|z_1+\sum_{j=2}^{2k}\frac{S_{1j}}{S_{11}}z_j\right|^2 \\ & +S_{11}^{-1}S_{11;22}\left|z_2+\sum_{j=3}^{2k}\frac{S_{11;2j}}{S_{11;22}}z_j\right|^2 \\ & +S_{11;22}^{-1}\sum_{i,j=3}^{2k} z_i^* S_{11;22;ij} z_j. \end{aligned} \tag{58}$$

So we can iterate the above procedure of completing squares to the last term of the above equation, which finally leads to

$$Z^\dagger S Z = \sum_{i=1}^{2k}\frac{S_i}{S_{i-1}}\left|z_i+\sum_{j=i+1}^{2k}\frac{S_{i-1;ij}}{S_i}z_j\right|^2, \tag{59}$$

where we have defined

$$S_i = S_{11;22;\cdots;ii} \tag{60}$$

the i-th principal minor of S, and

$$S_{i-1;ij} = S_{11;22;\cdots;(i-1)(i-1);ij}, \quad S_0 = 1. \tag{61}$$

We know that the condition for the Hermitian matrix S to be positive semidefinite is

$$S_1 \geqslant 0, \quad S_2 \geqslant 0, \quad \cdots, \quad S_{2k} \geqslant 0. \tag{62}$$

In the above discussion, we have assumed that all the S_is are non-vanishing, which is not always the case. If we encounter a vanishing S_i at some step, we must adjust the ordering of the rest z_js, so that S_i is non-vanishing. Is it always possible for us to achieve a non-vanishing S_i in this way? We prove in Appendix C that a non-vanishing S_i can be so achieved for an arbitrary positive semidefinite Hermitian matrix S, unless it is of rank $i-1$. Furthermore, if S is of rank l, then the series in Eq. (59) will be truncated after the l-th term. After that, we should undo the reordering of z_js, and finish the simplification of the original Hermitian form (52). Anyway, in the generic (i.e., positive definite) case, Eq. (59) is the explicit result of this simplification, which means that

$$\begin{aligned} K_{ij} \;&=\; \frac{S_{i-1;ij}}{S_{i-1}^{1/2}S_i^{1/2}} \qquad (i\leqslant j) \\ K_{ij} \;&=\; 0 \qquad (i>j) \end{aligned} \tag{63}$$

is an explicit solution of Eq.(43).

6 Noncommutative Drinfeld-Manin instanton

How to establish the Drinfeld-Manin construction in the noncommutative case is an interesting problem. Appealing to the well-developed ADHM construction may be much easier than considering noncommutative instanton bundles. The commutative ADHM construction can be regarded as a special case ($\zeta=0$) of the noncommutative ADHM construction. So we can anticipate that it is straightforward to generalize the ADHM formulism of the Drinfeld-Manin construction to the noncommutative case.

In fact, like Eq. (38), the factorization condition (8) in the noncommutative case gives the following condition on a,

$$\begin{aligned} a^\dagger a \;&=\; \begin{pmatrix} R+\zeta & 0 \\ 0 & R \end{pmatrix} \\ &=\; \mathrm{diag}(r_1+\zeta,\cdots,r_k+\zeta,r_1,\cdots,r_k). \end{aligned} \tag{64}$$

So we can similarly choose $\underline{a}$ and r_i $(i=1,2,\cdots,k)$ as the collective coordinates of the noncommutative $U(2k)$ k-instanton (while removing some of the degrees of freedom in $\underline{a}$ as in the commutative case). Now Eq. (42) becomes

$$S \equiv \begin{pmatrix} R+\zeta & 0 \\ 0 & R \end{pmatrix} - \underline{a}^\dagger \underline{a}, \tag{65}$$

and the following things are the same as in the commutative case.

To be more clear, our solution of the noncommutative ADHM $U(2k)$ k-instanton is

$$a = \begin{pmatrix} S^{1/2} \\ \underline{a} \end{pmatrix}, \tag{66}$$

where S is defined in Eq. (65) and $\underline{a}$ defined in Eq. (37). And we must keep in mind that the square root here is understood in the sense of the simplification of Hermitian forms, as explained in the last section, with the explicit expression (63). It is easy to check that this solution does satisfy the corresponding ADHM constraints, and it has the correct number of free parameters, as we have mentioned above.

The techniques to deal with the $N \neq 2k$ cases in the noncommutative case and that in the commutative case are exactly the same. In fact, the global gauge rotations in gauge field theory are unaffected by the space-time noncommutativity. Again, the $N=2k-1$ case is simple enough to solve. So we also obtain parametrization of the noncommutative $U(N)$ $(N \geqslant 2k-1)$ k-instanton moduli space.

To end this paper, let us focus on the two-instanton case. For $k=2$, we essentially obtain explicit general solutions of the (commutative or noncommutative) ADHM constraints for $U(N)$ $(N \geqslant 3)$ gauge groups. Counting the $U(2)$ two-instanton solution already known [9, 11], we have general solutions of all the commutative $U(N)$ two-instantons. However, the general solution of the noncommutative $U(2)$ two-instanton, which may be of much interest, is yet to be found.

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Appendix A

Conditions for a Hermitian matrix of restricted rank

Consider an $l \times l$ Hermitian matrix H. We introduce the following decomposition of H,

$$H = \begin{pmatrix} F_{r\times r} & C \\ C^{\dagger} & \underline{H}_{(l-r)\times(l-r)} \end{pmatrix}, \tag{A1}$$

and define an $(l-r+1)\times(l-r+1)$ matrix

$$H'_{ij} = \begin{pmatrix} F_{ij} & C_i \\ C_j^{\dagger} & \underline{H} \end{pmatrix}, \tag{A2}$$

where C_i is the ith row of C. Assuming $\det(\underline{H}) \neq 0$, then the following two propositions are equivalent:

1) $\mathrm{rank}(H) = l-r$;

2) $\det(H'_{ij}) = 0$ for all $i,j=1,2,\cdots,r$.

It is apparent that the latter can be deduced from the former. Now we explain how the former can be deduced from the latter.

First, for a fixed j, the $(l-r)\times(l-r+1)$ matrix

$$\underline{H}' \equiv \begin{pmatrix} C_j^{\dagger} & \underline{H} \end{pmatrix} \tag{A3}$$

is obviously of rank $l-r$. Then $\det(H'_{ij}) = 0$ means that the rank will not increase when we append a row $C'_i \equiv \begin{pmatrix} F_{ij} & C_i \end{pmatrix}$ to $\underline{H}'$, so C'_i is a linear combination of the row vectors of $\underline{H}'$. This is the case for all i, so we can conclude that the following matrix,

$$H_j \equiv \begin{pmatrix} F_j & C \\ C_j^{\dagger} & \underline{H} \end{pmatrix}, \tag{A4}$$

is of rank $l-r$, where F_j is the jth column of F.

Next, the $l\times(l-r)$ matrix

$$H' \equiv \begin{pmatrix} C \\ \underline{H} \end{pmatrix} \tag{A5}$$

is again of rank $l-r$. Thus $\mathrm{rank}(H_j) = l-r$ means that the rank will not increase when we append a column

$$\hat{C}_j \equiv \begin{pmatrix} F_j \\ C_j^{\dagger} \end{pmatrix} \tag{A6}$$

to H', so $\hat{C}_j$ is a linear combination of the column vectors of H'. Again, this is the case for all j, so we attain the desired result $\mathrm{rank}(H) = l-r$.

Because H is Hermitian, $\det(H'_{ij}) = 0$ are in fact r^2 real conditions. The combination of $\det(\underline{H}) \neq 0$ and these conditions is a sufficient condition for $\mathrm{rank}(H) \leqslant l-r$. Of course, it is not necessary. If $\det(\underline{H}) = 0$ for the decomposition (A1), we must take another $(l-r)\times(l-r)$ submatrix of H as $\underline{H}$ and obtain another r^2 real conditions. If H has no nonsingular $(l-r)\times(l-r)$ submatrix, the rank of H is less than $l-r$. Altogether, the requirement $\mathrm{rank}(H) \leqslant l-r$ is achieved.

Appendix B

Proof of the determinant formula

In order to prove the determinant formula (57), we use the familiar formula of determinant decomposition,

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D-CA^{-1}B). \tag{B1}$$

We have

$$M_{N;ij} = M_N(M_{ij} - M_{i*}N^{-1}M_{*j}) \tag{B2}$$

according to Eq. (B1), where M_{i*} is the row vector corresponding to the block C in Eq. (B1) and M_{*j} the column vector. Thus,

$$\begin{pmatrix} M_{N;i_1j_1} & M_{N;i_1j_2} & \cdots & M_{N;i_1j_l} \\ M_{N;i_2j_1} & M_{N;i_2j_2} & \cdots & M_{N;i_2j_l} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N;i_lj_1} & M_{N;i_lj_2} & \cdots & M_{N;i_lj_l} \end{pmatrix} = M_N \begin{pmatrix} M_{i_1j_1} & M_{i_1j_2} & \cdots & M_{i_1j_l} \\ M_{i_2j_1} & M_{i_2j_2} & \cdots & M_{i_2j_l} \\ \vdots & \vdots & \ddots & \vdots \\ M_{i_lj_1} & M_{i_lj_2} & \cdots & M_{i_lj_l} \end{pmatrix} - M_N \begin{pmatrix} M_{i_1*} \\ M_{i_2*} \\ \vdots \\ M_{i_l*} \end{pmatrix} N^{-1} \begin{pmatrix} M_{*j_1} & M_{*j_2} & \cdots & M_{*j_l} \end{pmatrix}.$$

Taking determinants of both sides of the above equation then gives Eq. (57).

Appendix C

A property of positive semidefinite Hermitian matrices

Suppose that we have non-vanishing $S_1, S_2, \cdots, S_{i-1}$, as stated in Sec. 5, so the rank of S is at least $i-1$. If $S_i = S_{i-1;ii}$ is zero, then $S_{i-1;ij} = S^*_{i-1;ji}$ for any $i+1 \leqslant j \leqslant 2k$ must also be zero, since otherwise

$$S_{i-1;ii;jj} = S^{-1}_{i-1}(S_i S_{i-1;jj} - |S_{i-1;ij}|^2) < 0, \tag{C1}$$

contradicting the fact that S is positive semidefinite. Interchanging the index i with any other index l $(i < l \leqslant 2k)$ and applying again the above argument, if all the resulting S_is are zero, then all the $S_{i-1;mn}$ $(i \leqslant m, n \leqslant 2k)$ must vanish identically. From the discussion in the Appendix A, we have to conclude that the rank of S is exactly $i-1$.

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