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Towards a gravitation theory in Berwald-Finsler space*

LI Xin(李昕)[1)] CHANG Zhe(常哲)[2)]

Institute of High Energy Physics, Chinese Academy of Sciences, P. O. Box 918(4), Beijing 100049, China

Abstract Finsler geometry is a natural and fundamental generalization of Riemann geometry. The Finsler structure depends on both coordinates and velocities. It is defined as a function on tangent bundle of a manifold. We use the Bianchi identities satisfied by the Chern curvature to set up a gravitation theory in Berwald-Finsler space. The geometric part of the gravitational field equation is nonsymmetric in general. This indicates that the local Lorentz invariance is violated.

Key words Finsler geometry, Berwald space, field equation

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1 Introduction

The possible violation of Lorentz invariance has been proposed within several models of quantum gravity (QG) as well as Very Special Relativity (VSR) [1]. A succinct list of QG includes tensor VEVs originating from string field theory [2], cosmologically varying moduli scenarios [3], spacetime foam models [4], semiclassical spin–network calculations in Loop QG [5, 6], noncommutative geometry gravity [7–10] and brane–world scenarios [11]. A common feature of these phenomenological studies on Planck scale physics is introducing the modified dispersion relations (MDR) for elementary particles. Girelli et al. [12] proposed a possible relation between MDR and Finsler geometry. Gibbons et al. [13] pointed out that VSR is Finsler geometry. In the VSR, CPT symmetry is preserved. VSR has radical consequences for neutrino mass mechanism. Lepton-number conserving neutrino masses are VSR invariant. The mere observation of ultra-high energy cosmic rays and analysis of neutrino data give an upper bound of 10^{-25} on the Lorentz violation [14].

The above facts imply that new physics may be connected with Finsler geometry. In fact, in 1941 Randers [15] published his work on possible application of Finsler geometry in physics. Properties of Randers space have been investigated exhaustively by both mathematicians and physicists [16–20].

In a recent paper [21], Kostelecky studied the effect of gravitation in the Lorentz- and CPT-violating Standard Model Extension (SME). The incorporation of Lorentz and CPT violation into general relativity based on Riemann-Cartan geometry was discussed. It provided dominant terms in the effective low-energy action for the gravitational sector, thereby completing the formulation of the leading-order terms in the SME with gravity. It shows that a generalized geometric framework is helpful in constructing a unification theory of gravity and electromagnetism, weak and strong interaction.

Finsler geometry is a natural and fundamental generalization of Riemann geometry. The Finsler structure depends on both coordinates and velocities. It is defined as a mapping function from tangent bundle of a manifold to R^1. S. S. Chern [22] proved that there is a unique connection in the Finsler manifold that is torsion free and almost g-compatible. We use the Bianchi identities satisfied by Chern curvature to set up a gravitation theory in Berwald-Finsler space. The geometric part of the gravitational field equation is nonsymmetric in general. This indicates that the local Lorentz invariance is violated. Nontrivial solutions of the gravitational field equation are presented.

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1) E-mail: lixn@mail.ihep.ac.cn

2) E-mail: changz@mail.ihep.ac.cn

This paper is organized as follows. In Sec. 2, we briefly review the basic concept and notations of Finsler geometry [23]. The torsion free Chern connection and corresponding curvature are introduced. The first and second Bianchi identities for curvature are presented. Sec. 3 is devoted to constructing a gravitation theory in Berwald-Finsler space. Finally, we give the conclusion and remarks.

2 Finsler geometry

2.1 Finsler manifold

Denote by T_xM the tangent space at $x\in M$, and by TM the tangent bundle of M. Each element of TM has the form (x,y), where $x\in M$ and $y\in T_xM$. The natural projection $\pi: TM\to M$ is given by $\pi(x,y)\equiv x$.

A Finsler structure of M is a function

$$F{:}TM\to[0,\infty)$$

with the following properties:

(i) Regularity: F is C^∞ on the entire slit tangent bundle $TM\backslash 0$.

(ii) Positive homogeneity : $F(x,\lambda y)=\lambda F(x,y)$ for all $\lambda>0$.

(iii) Strong convexity: The $n\times n$ Hessian matrix

$$g_{ij}\equiv\left(\frac{1}{2}F^2\right)_{y^iy^j}$$

is positive-definite at every point of $TM\backslash 0$, where we have used the notation $(\ )_{y^i}=\dfrac{\partial}{\partial y^i}(\ )$, and the symbol $TM\backslash 0$ means the tangent vector y is nonzero in the tangent bundle TM.

Finsler geometry has its genesis in integrals of the form

$$\int_s^r F(x^1,\cdots,x^n;\frac{\mathrm{d}x^1}{\mathrm{d}t},\cdots,\frac{\mathrm{d}x^n}{\mathrm{d}t})\mathrm{d}t\,. \tag{1}$$

Throughout the paper, the lowering and raising of indices are carried out by the fundamental tensor g_{ij} defined above, and its matrix inverse g^{ij}. Given a manifold M and a Finsler structure F on TM, the pair (M,F) is called a Finsler manifold. It is obvious that the Finsler structure F is a function of (x^i,y^i). In the case of F depending on x^i only , the Finsler manifold reduces to a Riemannian manifold.

The symmetric Cartan tensor can be defined as

$$A_{ijk}\equiv\frac{F}{2}\frac{\partial g_{ij}}{\partial y^k}=\frac{F}{4}(F^2)_{y^iy^jy^k}\ , \tag{2}$$

The Cartan tensor vanishes if and only if g_{ij} has no y-dependence. So the Cartan tensor is a measurement of the deviation from the Riemannian manifold.

Using Euler's theorem on homogeneous function, we can get useful properties of the fundamental tensor g_{ij} and Cartan tensor A_{ijk}

$$g_{ij}l^i = F_{y^j}, \tag{3}$$

$$g_{ij}l^il^j = 1, \tag{4}$$

$$y^i\frac{\partial g_{ij}}{\partial y^k} = 0,\quad y^j\frac{\partial g_{ij}}{\partial y^k}=0,\quad y^k\frac{\partial g_{ij}}{\partial y^k}=0, \tag{5}$$

$$y^iA_{ijk} = y^jA_{ijk}=y^kA_{ijk}=0, \tag{6}$$

where $l^i\equiv\dfrac{y^i}{F}$.

2.2 Chern connection

The nonlinear connection N^i_j on $TM\backslash 0$ is defined as

$$N^i_j\equiv\gamma^i_{jk}y^k-\frac{A^i_{jk}}{F}\gamma^k_{rs}y^ry^s\,, \tag{7}$$

where γ^i_{jk} is the formal Christoffel symbols of the second kind

$$\gamma^i_{jk}\equiv\frac{g^{is}}{2}\left(\frac{\partial g_{sj}}{\partial x^k}+\frac{\partial g_{sk}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^s}\right). \tag{8}$$

The invariant connection under the transform $y\longrightarrow\lambda y$ is of the form

$$\frac{N^i_j}{F}\equiv\gamma^i_{jk}l^k-A^i_{jk}\gamma^k_{rs}l^rl^s. \tag{9}$$

Usually, we define the covariant derivatives $\nabla\dfrac{\partial}{\partial x^i}$ and $\nabla\mathrm{d}x^i$ as

$$\nabla\frac{\partial}{\partial x^i} \equiv \omega^j_i\frac{\partial}{\partial x^j}, \tag{10}$$

$$\nabla\mathrm{d}x^i \equiv -\omega^i_j\mathrm{d}x^j, \tag{11}$$

where ω^i_j is the connection 1-forms. The operator ∇ has the same linear property as the covariant derivatives defined on the Riemannian manifold.

Here, we introduce the Chern connection that is torsion freeness

$$\mathrm{d}x^j\wedge\omega^i_j=0 \tag{12}$$

and almost g-compatibility

$$\mathrm{d}g_{ij}-g_{kj}\omega^k_i-g_{ik}\omega^k_j=2A_{ijs}\frac{\delta y^s}{F}, \tag{13}$$

where

$$\delta y^i\equiv\mathrm{d}y^i+N^i_j\mathrm{d}x^j. \tag{14}$$

A theorem given by S. S. Chern [22] guarantees the uniqueness of the Chern connection. Theorem (Chern): Let (M,F) be a Finsler manifold. The pulled-back bundle π^*TM admits a unique linear connection, called the Chern connection. Its connection forms are characterized by the structural equations of (12) and (13).

We ignore the proof of the theorem, and just give some consequences of it directly. Torsion freeness is equivalent to the absence of $\mathrm{d}y^i$ terms in ω^i_j; namely,

$$\omega^i_j = \Gamma^i_{jk}\mathrm{d}x^k, \tag{15}$$

together with the symmetry

$$\Gamma^i_{jk} = \Gamma^i_{kj}, \tag{16}$$

and almost g-compatibility implies that

$$\Gamma^i_{jk} = \frac{g^{is}}{2}\left(\frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s}\right), \tag{17}$$

where

$$\frac{\delta}{\delta x^i} \equiv \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial x^j}. \tag{18}$$

The dual basis of $\dfrac{\partial}{\partial y^i}$ is δy^i. As before, we prefer to work with

$$\frac{\delta y^i}{F} = \frac{1}{F}(\mathrm{d}y^i + N^i_j\mathrm{d}x^j), \tag{19}$$

which is invariant under the rescaling of y. Here, we give the relation between N^i_j and Γ^i_{jk}

$$\Gamma^i_{jk}l^j = \frac{N^i_k}{F}, \tag{20}$$

which will be useful in this paper.

We will work on two new natural local bases that are dual to each other: $\left\{\dfrac{\delta}{\delta x^i}, F\dfrac{\partial}{\partial y^i}\right\}$ for the tangent bundle of $TM\backslash 0$, $\left\{\mathrm{d}x^i, \dfrac{\delta y^i}{F}\right\}$ for the cotangent bundle of $TM\backslash 0$.

One can check that the transformation law of the Chern connection on the Finsler manifold is the same as the Riemannian connection on the Riemannian manifold. This fact is useful to guide us in defining the covariant derivative of a tensor.

Let $V \equiv V^j_i\dfrac{\partial}{\partial x^j}\otimes \mathrm{d}x^i$ be an arbitrary smooth local section of $\pi^*TM\otimes\pi^*T^*M$. The definitions of (10) and (11) and the property of operator ∇ imply that the covariant derivatives of V are

$$\nabla V \equiv (\nabla V)^j_i\frac{\partial}{\partial x^j}\otimes \mathrm{d}x^i, \tag{21}$$

where

$$(\nabla V)^j_i \equiv \mathrm{d}V^j_i + V^k_i\omega^j_k - V^j_k\omega^k_i. \tag{22}$$

∇V is a 1-form on $TM\backslash 0$. Thus, it can be expressed in terms of the natural basis $\left\{\mathrm{d}x^i, \dfrac{\delta y^i}{F}\right\}$,

$$(\nabla V)^j_i = V_i{}^j{}_{|s}\mathrm{d}x^s + V_i{}^j{}_{;s}\frac{\delta y^s}{F}. \tag{23}$$

Using the relation between the Chern connection and the connection 1-forms ω^i_j (15), we obtain the horizontal covariant derivative $V_i{}^j{}_{|s}$

$$V_i{}^j{}_{|s} = \frac{\delta V^j_i}{\delta x^s} + V^k_i\Gamma^j_{ik} - V^j_k\Gamma^k_{is}, \tag{24}$$

and the vertical covariant derivative $V_i{}^j{}_{;s}$

$$V_i{}^j{}_{;s} = F\frac{\partial V^j_i}{\partial y^s}. \tag{25}$$

The treatment for the tensor fields of higher rank is similar to the methods used on the Riemannian manifold. Here, we give the results of covariant derivatives of the fundamental tensor g and the norm 1 vector l:

$$g_{ij|s} = g^{ij}{}_{|s} = 0, \tag{26}$$

$$g_{ij;s} = 2A_{ijs} \quad\text{and}\quad g^{ij}{}_{|s} = -2A^{ij}{}_s, \tag{27}$$

$$l^i{}_{|s} = l_{i|s} = 0, \tag{28}$$

$$l^i{}_{;s} = \delta^i_s - l^i l_s \quad\text{and}\quad l_{i;s} = g_{is} - l_i l_s. \tag{29}$$

2.3 Curvature

The curvature 2-forms of the Chern connection are

$$\Omega^i_j \equiv \mathrm{d}\omega^i_j - \omega^k_j\wedge\omega^i_k. \tag{30}$$

The expression of Ω^i_j in terms of the natural basis $\{\mathrm{d}x^i, \dfrac{\delta y^i}{F}\}$ is of the form

$$\Omega^i_j \equiv \frac{1}{2}R^i_{jkl}\mathrm{d}x^k\wedge\mathrm{d}x^l + P^i_{jkl}\mathrm{d}x^k\wedge\frac{\delta y^l}{F} + \frac{1}{2}Q^i_{jkl}\frac{\delta y^k}{F}\wedge\frac{\delta y^l}{F}, \tag{31}$$

where R, P and Q are the hh-, hv-, vv-curvature tensors of the Chern connection, respectively. The following property is manifest

$$R_j{}^i{}_{kl} = -R_j{}^i{}_{lk}, \tag{32}$$

$$Q_j{}^i{}_{kl} = -Q_j{}^i{}_{lk}. \tag{33}$$

We are now in the position to demonstrate the Bianchi identities for the curvature.

The exterior differential of the structural equation (12) gives

$$\mathrm{d}x^j\wedge\mathrm{d}\omega^i_j = 0. \tag{34}$$

The combination of equations (34) and (12) shows that

$$\mathrm{d}x^j\wedge\Omega^i_j = 0. \tag{35}$$

Substituting Eq. (35) into (31), we get

$$\begin{aligned}&\frac{1}{2}R^i_{jkl}\mathrm{d}x^j\wedge\mathrm{d}x^k\wedge\mathrm{d}x^l + \\ &P^i_{jkl}\mathrm{d}x^j\wedge\mathrm{d}x^k\wedge\frac{\delta y^l}{F} + \\ &\frac{1}{2}Q^i_{jkl}\mathrm{d}x^j\wedge\frac{\delta y^k}{F}\wedge\frac{\delta y^l}{F} = 0.\end{aligned} \tag{36}$$

The three terms on the left side are completely independent. Thus, all of them should vanish. This gives identities

$$R_j{}^i{}_{kl}+R_k{}^i{}_{lj}+R_l{}^i{}_{jk} = 0, \tag{37}$$

$$P_j{}^i{}_{kl} = P_k{}^i{}_{jl}, \tag{38}$$

$$Q_j{}^i{}_{kl} = 0. \tag{39}$$

Then, the curvature 2-forms can be simplified as

$$\Omega^i_j \equiv \frac{1}{2}R^i_{jkl}\mathrm{d}x^k\wedge\mathrm{d}x^l+P^i_{jkl}\mathrm{d}x^k\wedge\frac{\delta y^l}{F}. \tag{40}$$

Tedious but straightforward manipulation of the exterior differential on the structural equation (13) gives

$$\Omega_{ij}+\Omega_{ji} = -2(\nabla A)_{ijk}\wedge\frac{\delta y^k}{F}-2A_{ijk}\left[\mathrm{d}\left(\frac{\delta y^k}{F}\right)+\omega^k_l\wedge\frac{\delta y^l}{F}\right]. \tag{41}$$

This can be rewritten as

$$\frac{1}{2}(R_{ijkl}+R_{jikl})\mathrm{d}x^k\wedge\mathrm{d}x^l+(P_{ijkl}+P_{jikl})\mathrm{d}x^k\wedge\frac{\delta y^l}{F}= -A_{iju}R^u_{kl}\mathrm{d}x^k\wedge\mathrm{d}x^l-2(A_{iju}P^u_{kl}+A_{ijl|k})\mathrm{d}x^k\wedge\frac{\delta y^l}{F}+2(A_{ijk;l}-A_{ijk}l_l)\frac{\delta y^k}{F}\wedge\frac{\delta y^l}{F}, \tag{42}$$

where we have used the abbreviations

$$R^i{}_{kl} \equiv l^jR_j{}^i{}_{kl} \tag{43}$$

$$P^i{}_{kl} \equiv l^jP_j{}^i{}_{kl}. \tag{44}$$

Equalization of three different types of terms of two sides of equation (42) shows identities

$$R_{ijkl}+R_{jikl} = -2A_{iju}R^u{}_{kl}, \tag{45}$$

$$P_{ijkl}+P_{jikl} = -2(A_{iju}P^u{}_{kl}+A_{ijl|k}), \tag{46}$$

$$A_{ijk;l}-A_{ijk}l_l = A_{ijl;k}-A_{ijl}l_k. \tag{47}$$

Formula (32) and Identity (37),(45) enable us to get the fourth property of hh-curvature,

$$R_{klji}-R_{jikl} = (B_{klji}-B_{jikl})+(B_{kilj}+B_{ljki})+(B_{iljk}-B_{jkil}), \tag{48}$$

where, for convenience, we have used the notation $B_{ijkl}\equiv -A_{iju}R^u_{kl}$. On the Riemannian manifold, the Cartan tensor vanishes. This means that $B_{ijkl}=0$ on the Riemannian manifold. The familiar properties of the Riemannian curvature

$$\tilde{R}_{ijkl}+\tilde{R}_{ijlk} = 0,$$
$$\tilde{R}_{ijkl}+\tilde{R}_{kjli}+\tilde{R}_{ljik} = 0,$$
$$\tilde{R}_{ijkl}+\tilde{R}_{jikl} = 0,$$
$$\tilde{R}_{ijkl}-\tilde{R}_{klij} = 0,$$

can be deduced directly from the four properties of hh-curvature (32), (37), (45) and (48). Making use of Identity (46) and Eqs. (6), (28), we may get a constitutive relation for P_{ijkl},

$$P_{jikl} = -(A_{ijl|k}-A_{jkl|i}+A_{kil|j})+A^u_{ij}\dot{A}_{ukl}-A_{jk}{}^u\dot{A}_{uil}+A_{ki}{}^u\dot{A}_{ujl}, \tag{49}$$

where

$$\dot{A}_{ijk}\equiv A_{ijk|s}l^s\,. \tag{50}$$

Contracting P_{ijkl} with l^i in Eq. (49), we obtain an important relation

$$P_{jkl}\equiv l^iP_{ijkl}=-\dot{A}_{jkl}. \tag{51}$$

The expression of R and P can be got by substituting Formula (30) into (40),

$$R_j{}^i{}_{kl} = \frac{\delta\Gamma^i_{jl}}{\delta x^k}-\frac{\delta\Gamma^i_{jk}}{\delta x^l}+\Gamma^i_{hk}\Gamma^h_{jl}-\Gamma^i_{hl}\Gamma^h_{jk}, \tag{52}$$

$$P_j{}^i{}_{kl} = -F\frac{\partial\Gamma^i_{jk}}{\partial y^l}. \tag{53}$$

These are the counterparts of the Riemannian curvature expessed in terms of the Christoffel symbols $\tilde{\Gamma}^i_{jk}$

$$\tilde{R}_j{}^i{}_{kl} = \frac{\partial\tilde{\Gamma}^i_{jl}}{\partial x^k}-\frac{\partial\tilde{\Gamma}^i_{jk}}{\partial x^l}+\tilde{\Gamma}^i_{hk}\tilde{\Gamma}^h_{jl}-\tilde{\Gamma}^i_{hl}\tilde{\Gamma}^h_{jk}. \tag{54}$$

Before ending the section, we present the second Bianchi identity. Exterior differential of the Chern connection (30) gives

$$\mathrm{d}\Omega^i_j-\omega^k_j\wedge\Omega^i_k+\omega^i_k\wedge\Omega^k_j=0. \tag{55}$$

Substituting (40) into the above equation, we obtain

$$\frac{1}{2}\mathrm{d}R_j{}^i{}_{kl}\wedge\mathrm{d}x^k\wedge\mathrm{d}x^l+\mathrm{d}P_j{}^i{}_{kl}\wedge\mathrm{d}x^k\wedge\frac{\delta y^l}{F}-P_j{}^i{}_{kl}\mathrm{d}x^k\wedge\mathrm{d}\left(\frac{\delta y^l}{F}\right)=\frac{1}{2}R_r{}^i{}_{kl}\omega^r_j\wedge\mathrm{d}x^k\wedge\mathrm{d}x^l-\frac{1}{2}R_j{}^r{}_{kl}\omega^i_r\wedge\mathrm{d}x^k\wedge\mathrm{d}x^l+P_r{}^i{}_{kl}\omega^r_j\wedge\mathrm{d}x^k\wedge\frac{\delta y^l}{F}-P_j{}^r{}_{kl}\omega^i_r\wedge\mathrm{d}x^k\wedge\frac{\delta y^l}{F}. \tag{56}$$

To evaluate $\mathrm{d}\left(\frac{\delta y^l}{F}\right)$, by making use of formula (28) and the definition of the covariant derivative, we first rewrite $\frac{\delta y^l}{F}$ as

$$\frac{\delta y^l}{F}=\mathrm{d}l^l+\Gamma^l_{jk}l^k\mathrm{d}x^j+\frac{\mathrm{d}F}{F}l^l. \tag{57}$$

Then, one has

$$\begin{aligned}\mathrm{d}\left(\frac{\delta y^l}{F}\right) &= \mathrm{d}l^j\wedge\omega^l_j+l^j\mathrm{d}\omega^l_j+\mathrm{d}l^l\wedge\frac{\mathrm{d}F}{F}= \\ & l^j\Omega^l_j+l^j\omega^k_j\wedge\omega^l_k+ \\ & \left(\frac{\delta y^j}{F}-\omega^j_k l^k-l^j\frac{\mathrm{d}F}{F}\right)\wedge\omega^l_j+ \\ & \left(\frac{\delta y^l}{F}-\omega^l_k l^k\right)\wedge\frac{\mathrm{d}F}{F}= \\ & l^j\Omega^l_j+\frac{\delta y^j}{F}\wedge\left(\omega^l_j-l_j\frac{\delta y^l}{F}\right),\end{aligned} \tag{58}$$

here we have used the identity

$$l_i\frac{\delta y^i}{F}=\frac{\mathrm{d}F}{F} \tag{59}$$

to get the third equality.

Substituting Formula (58) into (56) and noticing the torsion freeness of the Chern connection, we obtain

$$\begin{aligned}&\frac{1}{2}\nabla R_j{}^i{}_{kl}\wedge\mathrm{d}x^k\wedge\mathrm{d}x^l+\nabla P_j{}^i{}_{kl}\wedge\mathrm{d}x^k\wedge\frac{\delta y^l}{F}= \\ &P_j{}^i{}_{kl}l^t\mathrm{d}x^k\wedge\left(\frac{1}{2}R_t{}^l{}_{rs}\mathrm{d}x^r\wedge\mathrm{d}x^s+P_t{}^l{}_{rs}\mathrm{d}x^r\wedge\frac{\delta y^s}{F}\right)- \\ &P_j{}^i{}_{kl}l_r\mathrm{d}x^k\wedge\frac{\delta y^r}{F}\wedge\frac{\delta y^l}{F}.\end{aligned} \tag{60}$$

In a natural basis, we can rewrite Eq. (60) into the form

$$\begin{aligned}&\frac{1}{2}(R_j{}^i{}_{kl|t}-P_j{}^i{}_{ku}R^u{}_{lt})\mathrm{d}x^k\wedge\mathrm{d}x^l\wedge\mathrm{d}x^t+ \\ &\frac{1}{2}(R_j{}^i{}_{kl;t}-2P_j{}^i{}_{kt|l}+2P_j{}^i{}_{ku}\dot{A}^u{}_{lt})\mathrm{d}x^k\wedge\mathrm{d}x^l\wedge\frac{\delta y^t}{F}+ \\ &(P_j{}^i{}_{kl;t}-P_j{}^i{}_{kl}l_t)\mathrm{d}x^k\wedge\frac{\delta y^l}{F}\wedge\frac{\delta y^t}{F}=0.\end{aligned} \tag{61}$$

The three terms on the left side are completely independent. Then, we get the following identities

$$\begin{aligned}&R_j{}^i{}_{kl|t}+R_j{}^i{}_{lt|k}+R_j{}^i{}_{tk|l}= \\ &P_j{}^i{}_{ku}R^u{}_{lt}+P_j{}^i{}_{lu}R^u{}_{tk}+P_j{}^i{}_{tu}R^u{}_{kl},\end{aligned} \tag{62}$$

$$R_j{}^i{}_{kl;t}=P_j{}^i{}_{kt|l}-P_j{}^i{}_{lt|k}-(P_j{}^i{}_{ku}\dot{A}^u{}_{lt}-P_j{}^i{}_{lu}\dot{A}^u{}_{kt}), \tag{63}$$

$$P_j{}^i{}_{kl;t}-P_j{}^i{}_{kt;l} = P_j{}^i{}_{kl}l_t-P_j{}^i{}_{kt}l_l. \tag{64}$$

3 Gravitation theory in Berwald space

Einstein successfully proposed his general relativity in Riemannian space to describe gravity. It is interesting to investigate the behaviors of gravitation in more general Finsler spaces. Let us briefly recall the setup method of the Einstein field equation on the Riemannian manifold. One starts from the second Bianchi identities on the Riemannian manifold

$$\tilde{R}_j{}^i{}_{kl|t}+\tilde{R}_j{}^i{}_{lt|k}+\tilde{R}_j{}^i{}_{tk|l}=0. \tag{65}$$

The metric-compatibility

$$\tilde{g}_{ij|k}=0 \quad \text{and} \quad \tilde{g}^{ij}{}_{|k}=0, \tag{66}$$

and contraction of (65) with $\tilde{g}^{jt}$ gives that

$$\tilde{R}^{ji}{}_{kl|j}+\tilde{R}^i{}_{l|k}-\tilde{R}^i{}_{k|l}=0, \tag{67}$$

where $\tilde{R}^i{}_l\equiv\tilde{R}^{ij}{}_{jl}$ is the Ricci tensor. Lowering the index i and contracting with $\tilde{g}^{ik}$, we obtain

$$\tilde{R}^j{}_{l|j}+\tilde{R}^j{}_{l|j}-\tilde{S}_{|l}=0, \tag{68}$$

where $\tilde{S}=\tilde{g}^{ij}\tilde{R}_{ij}$ is the scalar curvature. An equivalent but more familiar form is

$$\left(\tilde{R}^{jl}-\frac{1}{2}\tilde{g}^{jl}\tilde{S}\right)_{|j}=0. \tag{69}$$

In the weak field limit, the gravitation theory should reduce to the Newtonian theory. Einstein suggested his gravitational field equation of the form

$$\tilde{R}_{jl}-\frac{1}{2}\tilde{g}_{jl}\tilde{S}=8\pi GT_{jl}, \tag{70}$$

where T_{jl} is the energy–momentum tensor and G is Newton's constant.

In the paper, we use a similar approach to discuss gravitation on the Finsler manifold. Let us introduce first two notions for Ricci curvature: the Ricci scalar Ric and the Ricci tensor Ric_{ij}.

The Ricci scalar is defined as

$$Ric=g^{ik}R_{ik}, \tag{71}$$

where $R_{ik}\equiv l^jR_{jikl}l^l$ is symmetric. The Ricci tensor on the Finsler manifold was first introduced by Akbar-Zadeh [24]

$$Ric_{ik}\equiv\left(\frac{1}{2}F^2Ric\right)_{y^iy^k}, \tag{72}$$

which is manifestly symmetric and covariant. Expanding y derivatives in the defining formula for the Ricci tensor Ric_{ik}, we get

$$\begin{aligned}Ric_{ik} &= \frac{1}{4}(Ric_{;i;k}+Ric_{;k;i})+ \\ &\frac{3}{4}(l_iRic_{;k}+l_kRic_{;i})+g_{ik}Ric.\end{aligned} \tag{73}$$

Substituting the defining formula for the Ricci scalar

Ric into the above equation, we obtain

$$\begin{aligned} Ric_{ik} = & \frac{1}{2}(R_{k\ \ si}^{\ s}+R_{i\ \ sk}^{\ s})+ \\ & \frac{1}{4}l^j l^l(R_{j\ \ sl;k;i}^{\ s}+R_{j\ \ sl;i;k}^{\ s})- \\ & \frac{1}{4}l^j l^l(l_i R_{j\ \ sl;k}^{\ s}+l_k R_{j\ \ sl;i}^{\ s})+ \\ & \frac{1}{2}l^j(R_{i\ \ sj;k}^{\ s}+R_{j\ \ si;k}^{\ s}+R_{k\ \ sj;i}^{\ s}+R_{j\ \ sk;i}^{\ s})= \\ & \frac{1}{2}(R_{k\ \ si}^{\ s}+R_{i\ \ sk}^{\ s})+E_{ik}\ , \end{aligned} \tag{74}$$

where we introduced the notation

$$\begin{aligned} E_{ik} \equiv & \frac{1}{4}l^j l^l(R_{j\ \ sl;k;i}^{\ s}+R_{j\ \ sl;i;k}^{\ s})- \\ & \frac{1}{4}l^j l^l(l_i R_{j\ \ sl;k}^{\ s}+l_k R_{j\ \ sl;i}^{\ s})+ \\ & \frac{1}{2}l^j(R_{i\ \ sj;k}^{\ s}+R_{j\ \ si;k}^{\ s}+R_{k\ \ sj;i}^{\ s}+R_{j\ \ sk;i}^{\ s}). \end{aligned} \tag{75}$$

Following the same setup process for the gravitational field equation in Riemannian space, we start from the second Bianchi identities (62). Contracting it with g^{jt}, lowering the index i, and contracting again with g^{ik}, we get

$$\begin{aligned} & R^{ji}_{\ \ il|j}+R^{ji}_{\ \ lj|i}+R^{ji}_{\ \ ji|l}= \\ & g^{jt}g^{ik}(P_{jiku}R^{u}_{\ lt}+P_{jilu}R^{u}_{\ tk}+P_{jitu}R^{u}_{\ kl}). \end{aligned} \tag{76}$$

Using the first Bianchi identity (45) and Formula (48), we can divide the left side of the above equation into a symmetric part labelled by [ ] and a nonsymmetric part labelled by { }

$$\begin{aligned} & R^{ji}_{\ \ il|j}+R^{ji}_{\ \ lj|i}+R^{ji}_{\ \ ji|l}=\left(Ric^{j}_{\ l}+\frac{1}{2}B_{k\ \ l}^{\ kj}-E^{j}_{\ l}\right)_{|j}+ \\ & \left(2B^{jk}_{\ \ lk}+Ric^{j}_{\ l}+\frac{1}{2}B_{k\ \ l}^{\ kj}-E^{j}_{\ l}\right)_{|j}-\delta^j_l(S-E)_{|j}= \\ & \left[(2Ric^{j}_{\ l}-\delta^j_l S)-(2E^{j}_{\ l}-\delta^j_l E)\right]_{|j}+\{B_{k\ \ l}^{\ kj}+2B^{jk}_{\ \ lk}\}_{|j}, \end{aligned} \tag{77}$$

where $E\equiv g^{ij}E_{ij}$ and $S=g^{ij}Ric_{ij}$. Using the constituent relation of the hv-curvature tensor (49), we rewrite the right side of Identity (76) as

$$\begin{aligned} & g^{jt}g^{ik}(P_{jiku}R^{u}_{\ lt}+P_{jilu}R^{u}_{\ tk}+P_{jitu}R^{u}_{\ kl})= \\ & 2(A^{j}_{\ lu|i}-A^{jr}_{\ \ l}\dot{A}_{riu})R^{u\ \ i}_{\ j}+ \\ & 2(A^{j}_{\ iu|j}-A_{u|i}+A^{r}\dot{A}_{riu}-A^{jr}_{\ \ i}\dot{A}_{rju})R^{u\ \ i}_{\ l}, \end{aligned} \tag{78}$$

where $A_r\equiv g^{ij}A_{ijr}$. Finally, we get the equivalent form of the identity (76)

$$\begin{aligned} & \left[\left(Ric^{jl}-\frac{1}{2}g^{jl}S\right)-\left(E^{jl}-\frac{1}{2}g^{jl}E\right)\right]_{|j}+ \\ & \left\{\frac{1}{2}B_{k}^{\ kjl}+B^{jkl}_{\ \ \ k}\right\}_{|j}=(A^{jl}_{\ \ u|i}-A^{jrl}\dot{A}_{riu})R^{u\ \ i}_{\ j}+ \\ & (A^{j}_{\ iu|j}-A_{u|i}+A^{r}\dot{A}_{riu}-A^{jr}_{\ \ i}\dot{A}_{rju})R^{uli}. \end{aligned} \tag{79}$$

A Finsler structure F is said to be of Berwald type if the Chern connection coefficients Γ^i_{jk} in natural coordinates have no y dependence. A direct proposition on Berwald space is that the hv–part of the Chern curvature vanishes identically

$$P_{j\ \ kl}^{\ i}=0, \tag{80}$$

and the hh–part of the Chern connection reduces to

$$R_{j\ \ kl}^{\ i} = \frac{\partial\Gamma^i_{jl}}{\partial x^k}-\frac{\partial\Gamma^i_{jk}}{\partial x^l}+\Gamma^i_{hk}\Gamma^h_{jl}-\Gamma^i_{hl}\Gamma^h_{jk}. \tag{81}$$

So, in the Berwald space the identity of (79) reduces as

$$\left[Ric^{jl}-\frac{1}{2}g^{jl}S\right]_{|j}+\left\{\frac{1}{2}B_{k}^{\ kjl}+B^{jkl}_{\ \ \ k}\right\}_{|j}=0. \tag{82}$$

Thus, the counterpart of Einstein's field equation on Berwald space takes the form

$$\left[Ric_{jl}-\frac{1}{2}g_{jl}S\right]+\left\{\frac{1}{2}B_{k\ \ jl}^{\ k}+B_{j\ \ lk}^{\ k}\right\}=8\pi GT_{jl}. \tag{83}$$

The gravitational field equation on Berwald space is obviously different from Einstein's field equation. The geometric part contains a nonsymmetric term. Thus, in general, the energy–momentum tensor T_{jl} is not symmetric. This means that the local Lorentz invariance is violated in general.

Here, we present the relation between Berwald space and Randers space. Kikuchi [25] proved that in a Randers space of Berwald type, one has

$$\tilde{b}_{i|j}\equiv\tilde{b}_{i,j}-\tilde{b}_k\tilde{\gamma}^k_{ij}=0, \tag{84}$$

where $\tilde{\gamma}^k_{ij}$ is the Christoffel symbols of the Riemannian metric $\tilde{a}\equiv\tilde{a}_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j$. In Randers space, one can derive straightforwardly the expression of the geodesic spray coefficients as

$$\begin{aligned} G^i \equiv & \gamma^i_{jk}y^j y^k=(\tilde{\gamma}^i_{jk}+l^i\tilde{b}_{j|k})y^j y^k+ \\ & (\tilde{a}^{ij}-l^i\tilde{b}^j)(\tilde{b}_{j|k}-\tilde{b}_{k|j})\alpha y^k, \end{aligned} \tag{85}$$

and the Chern connection as

$$\Gamma^i_{jk}=(N^i_j)_{y^k}+\frac{1}{2}g^{it}y_s(N^s_t)_{y^j y^k}. \tag{86}$$

It is not difficult to check that the geodesic spray coefficients satisfy that

$$\frac{1}{2}\frac{\partial G^i}{\partial y^j}=N^i_j. \tag{87}$$

Thus in Randers spaces of Berwald type, the geodesic spray coefficients reduce to

$$G^i = \tilde{\gamma}^i_{jk} y^j y^k. \tag{88}$$

The Chern connection reduces to

$$\Gamma^i_{jk} = \tilde{\gamma}^i_{jk}. \tag{89}$$

Then, the hh–curvature takes the form

$$R_j{}^i{}_{kl} = \frac{\partial \tilde{\gamma}^i_{jl}}{\partial x^k} - \frac{\partial \tilde{\gamma}^i_{jk}}{\partial x^l} + \tilde{\gamma}^i_{hk}\tilde{\gamma}^h_{jl} - \tilde{\gamma}^i_{hl}\tilde{\gamma}^h_{jk}. \tag{90}$$

Thus, it is very convenient to investigate the field equation in Randers space of Berwald type. This will be studied in our future work.

4 Conclusion and remarks

In this paper, we have set up a gravitation theory in a torsion freeness Berwald-Finsler space. The geometric part of the gravitational field equation is in general nonsymmetric. This fact indicates that the local Lorentz invariance is violated in the Finsler manifold. This is in good agreement with the discussions on special relativity in Finsler space [12, 13, 20].

However, problems still remain. How to construct a gravitation in general Finsler space is still a open question. It is well-known that in Riemannian space the sign of section curvature $K(x)$ determines the type of geometry near x (hyperbolic, flat or spherical). In the Finslerian landscape, the sign of $K(x,y)$ depends on the direction y of our line of sight. This makes it possible to encounter all three types of geometry during a survey. In such a cosmology model, one may wish to find a natural explanation for why the space of the universe is asymptotically flat.

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