Topological approach to examine the singularity of the axial-vector current in an Abelian gauge field theory (QED)

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Abstract A topological way to distinguish divergences of the Abelian axial-vector current in quantum field theory is proposed. By using the properties of the Atiyah-Singer index theorem, the non-trivial Jacobian factor of the integration measure in the path-integral formulation of the theory is connected with the topological properties of the gauge field. The singularity of the fermion current related to the topological character can be correctly examined in a gauge background.

Key words axial-vector current, Ward-Takahashi identity, anomaly, Atiyah-Singer index theorem

PACS 11.40.Dw, 11.30.Ly, 11.15.Tk

1 Introduction

The analysis of the current conservation equation for the axial-vector current in quantum field theory shows that the origin of the occurrence of anomalies in the Ward-Takahashi (WT) type identity in gauge field theory such as QED is a twofold one[1]. First, from the path-integral viewpoint, the anomaly term associated with this identity can be understood in the path-integral formulation of quantum gauge theory as a consequence of the fact that the functional measure is not invariant with respect to the relevant local group transformations on the fields[2]. The mathematical explanation of such an anomaly is directly related to the Atiyah-Singer index theorem[3], which explains why the anomaly only arises in theories where fermions couple to gauge fields and indicates a connection between topology and the anomaly. In the case of Abelian gauge theory, the behavior of the anomaly function of the Abelian anomaly is linked with the index of the Dirac operator in a gauge background[4, 5].

Besides this, another type of anomaly may appear in the generalized Ward-Takahashi identity, which is related to the product of quantum operators such as fermion currents. Such an issue has already been presented in the expansion of the generating functional under the chiral gauge transformation of field variables, which is properly treated in the path-integral formulation of Abelian gauge field theory[5]. For instance, in the derivation of a conservation equation for the axial current, the differential equation of motion for the fermion fields can not be used to reduce the fermion-photon vertex in the QED in the most straightforward way[4]. A closer look at the manipulations reveals some subtleties, which come from the fact that the axial vector current is a composite operator built out of fermion fields. The product of the local operators is often singular, which can destroy symmetries of the classical equations of motion[6]. Thus the hope is to find a means to distinguish the divergences of the axial-vector current coupling with the gauge field. In this paper we concentrate on developing a topological method to examine the singularity of the Abelian axial-vector current by means of the path-integral and topological viewpoints. An explicit proof of the argument from the topological point of view will be presented.

Received 2 June 2008, Revised 2 November 2008

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2 The functional measure and the Abelian anomaly

For this purpose, we now display the path-integral derivation of the Jacobian of the integral measure in the background of Abelian gauge theory, which gives rise to an anomalous contribution of integral measure to the generalized Ward-Takahashi identities. (see Refs. [2, 5] for details). An infinitesimal local chiral transformation of fermion fields is given by

$$\begin{aligned} \psi'(x) &= \mathrm{e}^{\mathrm{i}\theta(x)\gamma_5}\psi(x), \\ \overline{\psi}'(x) &= \overline{\psi}(x)\mathrm{e}^{\mathrm{i}\theta(x)\gamma_5}, \end{aligned} \tag{1}$$

where the group parameter $\theta(x)$ is a real function. Then the fermion fields in a basis of eigenstates of the regulation operator H_ψ can be decomposed

$$\begin{aligned} \psi'(x) &= \sum_n c'_n\phi_n(x), \\ \bar{\psi}'(x) &= \sum_n \bar{c}'_n\phi_n(x), \\ \delta_{nm} &= \int \mathrm{d}^4x\phi_n^\dagger\phi_m. \end{aligned} \tag{2}$$

The expansion coefficients of ψ and ψ' are related by an infinitesimal linear transformation, computed as follows

$$c'_n(x) = \sum_m \int \mathrm{d}^4x\phi_n^\dagger(x)\mathrm{e}^{\mathrm{i}\theta(x)\gamma_5}\phi_m c_m, \tag{3}$$

where, in general, the sum extends over repeated indices.

Note that a specific application of Fujikawa's idea leads to the path-integral derivation of the axial anomaly. Under the above transformation (1), the functional measure $\mathrm{d}\mu$ over ψ, $\bar{\psi}$ has the transformation property

$$\mathrm{d}\mu = [D\overline{\psi}][D\psi] = \prod_n \mathrm{d}\bar{c}_n \prod_m \mathrm{d}c_m \tag{4}$$

and

$$\mathrm{d}\mu' = \prod_n \mathrm{d}\bar{c}'_n \prod_m \mathrm{d}c'_m = J_{\mathrm{A}}\mathrm{d}\mu \ , \tag{5}$$

where $\mathrm{d}\mu'$ is the functional measure over ψ', $\bar{\psi}'$, J_{A} is the Jacobian determinant of the corresponding transformation of the fermion variables (1).

Subsequently the variation of the Jacobian is evaluated

$$\begin{aligned} J_{\mathrm{A}}(x) &= \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\mathcal{A}(x)\theta(x)}\mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4(x)\bar{\mathcal{A}}(x)\theta(x)} = \\ & \mathrm{e}^{-\frac{\mathrm{i}}{8\pi^2}\int \mathrm{d}^4x\theta(x)\varepsilon_{\mu\nu\rho\sigma}\mathrm{Trace}(F_{\mu\nu}F_{\rho\sigma})} \ , \end{aligned} \tag{6}$$

where $\mathcal{A}(x)$ is the anomaly function.

To prepare for the discussion of properties of the axial-vector current, let's define the function μ which is the operator product of the anticommuting coefficient (for Grassman variables see (3))

$$\mu = \prod_n \bar{c}_n \prod_m c_m. \tag{7}$$

After the transformation (1), the expression of the function μ is changed explicitly into

$$\begin{aligned} \mu' &= \prod \bar{c}'_n \prod c'_m = \\ & \prod_n \sum_m \int \mathrm{d}^4x\phi_n^\dagger(x)\mathrm{e}^{\mathrm{i}\gamma_5\theta(x)}\phi_m c_m \times \\ & \prod_l \sum_s \int \mathrm{d}^4x\phi_l^\dagger(x)\mathrm{e}^{\mathrm{i}\gamma_5\theta(x)}\phi_s c_s = \widetilde{J}_{\mathrm{A}}\mu. \end{aligned} \tag{8}$$

Combining equations (5) and (8), by virtue of the properties of the Grassmann algebra, yields the following identity

$$\widetilde{J}_{\mathrm{A}} = J_{\mathrm{A}}^{-1} \ . \tag{9}$$

It shows that the Jacobian $\widetilde{J}_{\mathrm{A}}$ of the transformation of the operator product μ is linked with the inverse of the Jacobian of the corresponding transformation of the measure.

3 Topological properties of the Abelian anomaly and the axial current

From the topological viewpoint[7, 8], the non-perturbative effect of the Abelian anomaly associated with WT type identities is related to the topological character in the presence of the topologically non-trivial field configuration. The topological exposition of a quantum anomaly is addressed by the Atiyah-Singer index theorem in a gauge background. It indicates a connection between a functional of the gauge field and the numbers of zero modes of the Dirac operator $\gamma^\mu(\partial_\mu - \mathrm{i}gB_\mu(x))$. That is

$$\begin{aligned} & \int (\mathrm{d}^4x)_{\mathrm{E}}\mathcal{A}^{[\nu]}(x) = \nu_+ - \nu_- = \\ & -\frac{1}{32\pi^2}\int \mathrm{d}^4x\varepsilon_{\mu\nu\rho\sigma}\mathrm{Trace}(F_{\mu\nu}F_{\rho\sigma}). \end{aligned} \tag{10}$$

The fact reveals the important information that the integral of the anomaly function $\mathcal{A}^{[\nu]}(x)$ can not change smoothly under variations in the gauge field. The Abelian anomaly arisen here is a local quantity, because it is a consequence of short distance singularities. This argument is verified since the anomaly function in (10) can be written as a total derivative

$$-\frac{1}{32\pi^2}\int \mathrm{d}^4x\varepsilon_{\mu\nu\rho\sigma}\mathrm{Trace}(F_{\mu\nu}F_{\rho\sigma})=4\partial_\mu(\varepsilon_{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma). \quad (11)$$

The space integral of the anomaly function depends on only the behavior of the gauge field at the boundaries. This property already indicates a connection of topology and anomalies[8]. Thus between, we can further study the property of the axial-vector current by using the above argument, which is used to examine the singularities of the operator products of the current.

We now turn to the form of the fermion current. The axial-vector current is defined as[9]

$$J_{5\mu}(x)=\overline{\psi}(x)\Gamma\psi(x)=\sum_{nm}\overline{c}_n c_m\phi_n^\dagger\Gamma\phi_m, \quad (12)$$

where the letter Γ indicates Dirac matrix product $\mathrm{i}\gamma_5\gamma_\mu$. After the local chiral transformations, the fermion current coupling to the gauge field will appear as an interaction term in the expansion of the variation of the action, and is changed now into

$$J'_{5\mu}(x)=\overline{\psi}'(x)\Gamma\psi'(x)=\sum \overline{c}'_n c'_m\phi_n^\dagger\Gamma\phi_m. \quad (13)$$

To specify the relation between the fermion current and the corresponding integral measure, let's construct the square functions of the axial-vector current over ψ, $\bar{\psi}$:

$$J^2(x)=\sum_{nm}\overline{c}_n c_m\phi_n^\dagger\Gamma\phi_m\sum_{n'm'}\overline{c}_{n'}c_{m'}\phi_{n'}^\dagger\Gamma\phi_{m'}. \quad (14)$$

In the same way, the square function over ψ', $\bar{\psi}'$ can be defined as

$$J'^2(x)=\sum_{nm}\overline{c}'_n c'_m\phi_n^\dagger\Gamma\phi_m\sum_{n'm'}\overline{c}'_{n'}c'_{m'}\phi_{n'}^\dagger\Gamma\phi_{m'}\ . \quad (15)$$

Obviously due to the property of the Grassmann algebra, expression (15) implies that $J'^2(x)$ links with the operator product μ' (7), and is only the function of μ'. That is, the topological properties of the axial-vector current $J'(x)$ are related to the Jacobian of the corresponding integral measure.

For simplicity, we take a two-dimensional eigenspace of the regulation operator H_ψ, as an example, to illustrate the link between the Jacobian of the integral measure and the topological property of the fermion current.

In the path-integral formulation of gauge fields, the functional measure over ψ', $\bar{\psi}'$ can be read off from the chiral transformation (1)

$$\mathrm{d}\mu'=\mathrm{d}\overline{c}'_1\mathrm{d}\overline{c}'_2\mathrm{d}c'_1\mathrm{d}c'_2, \quad (16)$$

and the operator product μ' is given by

$$\mu'=\overline{c}'_1\overline{c}'_2c'_1c'_2, \quad (17)$$

with the Grassmann expansion coefficients of ψ and $\overline{\psi}$ given by

$$\begin{aligned} c'_1 &= c_1\int\phi_1^\dagger\mathrm{e}^\Gamma\phi_1+c_2\int\phi_1^\dagger\mathrm{e}^\Gamma\phi_2, \\ c'_2 &= c_1\int\phi_2^\dagger\mathrm{e}^\Gamma\phi_1+c_2\int\phi_2^\dagger\mathrm{e}^\Gamma\phi_2. \end{aligned} \quad (18)$$

Introducing the symbols

$$\int_{mn}=\int \mathrm{d}^4x\phi_m^\dagger\mathrm{e}^\Gamma\phi_n,\quad \int_{mn}^\dagger=\int \mathrm{d}^4x(\phi_m^\dagger\mathrm{e}^\Gamma\phi_n)^\dagger, \quad (19)$$

expansion (18) becomes

$$\begin{aligned} \mu' &= \overline{c}_1\overline{c}_1c_1c_1\int_{11}^\dagger\int_{11}^\dagger\int_{11}\int_{11}+\overline{c}_2\overline{c}_1c_1c_1\int_{11}^\dagger\int_{12}^\dagger\int_{11}\int_{11}+ \\ &\quad \cdots+\overline{c}_2\overline{c}_2c_2c_2\int_{22}^\dagger\int_{22}^\dagger\int_{22}\int_{22}\ . \end{aligned} \quad (20)$$

The corresponding expansion of the current can be implemented in the eigenspace

$$\begin{aligned} J'(x) &= (\overline{c}'_1\phi_1^\dagger+\overline{c}'_2\phi_2^\dagger)\Gamma(c'_1\phi_1+c'_2\phi_2)= \\ &\quad \overline{c}'_1c'_1\phi_1^\dagger\Gamma\phi_1+\overline{c}'_1c'_2\phi_1^\dagger\Gamma\phi_2+ \\ &\quad \overline{c}'_2c'_1\phi_2^\dagger\Gamma\phi_1+\overline{c}'_2c'_2\phi_2^\dagger\Gamma\phi_2. \end{aligned} \quad (21)$$

Then the square of the axial-vector current J'^2 is written out as

$$\begin{aligned} J'^2(x) &= (\overline{c}'_1c'_1\phi_1^\dagger\Gamma\phi_1+\overline{c}'_1c'_2\phi_1^\dagger\Gamma\phi_2+\overline{c}'_2c'_1\phi_2^\dagger\Gamma\phi_1+ \\ &\quad \overline{c}'_2c'_2\phi_2^\dagger\Gamma\phi_2)(\overline{c}'_1\phi_1^\dagger+\overline{c}'_2\phi_2^\dagger)\Gamma(c'_1\phi_1+c'_2\phi_2)= \\ &\quad \overline{c}'_1\overline{c}'_1c'_1c'_1\phi_1^\dagger\Gamma\phi_1\phi_1^\dagger\Gamma\phi_1+ \\ &\quad \overline{c}'_2\overline{c}'_1c'_1c'_1\phi_2^\dagger\Gamma\phi_1\phi_1^\dagger\Gamma\phi_1+\cdots+ \\ &\quad \overline{c}'_2\overline{c}'_2c'_2c'_2\phi_2^\dagger\Gamma\phi_2\phi_2^\dagger\Gamma\phi_2. \end{aligned} \quad (22)$$

Further, by taking the property of the Grassmann algebra into account, the calculus leads straightforwardly to the result

$$\begin{aligned} J'^2(x) &= 2\overline{c}'_1\overline{c}'_2c'_1c'_2\phi_1^\dagger\Gamma\phi_1\phi_2^\dagger\Gamma\phi_2+ \\ &\quad 2\overline{c}'_1\overline{c}'_2c'_1c'_2\phi_1^\dagger\Gamma\phi_2\phi_2^\dagger\Gamma\phi_1= \\ &\quad -2\mu'\phi_1^\dagger\Gamma\phi_1\phi_2^\dagger\Gamma\phi_2+2\mu'\phi_1^\dagger\Gamma\phi_2\phi_2^\dagger\Gamma\phi_1= \\ &\quad -2\tilde{J}_\mathrm{A}\mu\phi_1^\dagger\Gamma\phi_1\phi_2^\dagger\Gamma\phi_2+2\tilde{J}_\mathrm{A}\mu\phi_1^\dagger\Gamma\phi_2\phi_2^\dagger\Gamma\phi_1= \\ &\quad -2J_\mathrm{A}^{-1}\mu\phi_1^\dagger\Gamma\phi_1\phi_2^\dagger\Gamma\phi_2+2J_\mathrm{A}^{-1}\mu\phi_1^\dagger\Gamma\phi_2\phi_2^\dagger\Gamma\phi_1. \end{aligned} \quad (23)$$

Finally, we get the desired result. Clearly the topological property of products of the fermion currents $J^2(x)$ (i.e. $J(x)$) is related to the corresponding Jacobian factor. In other words, the singularity of the product of the fermion operators can be examined by the Jacobian factor of the integral measure under chiral transformation of the fermion variables. In the above case, the straightforward calculus shows that the axial-vector current $J^5_\mu = \bar{\psi}(x)\mathrm{i}\gamma^5\gamma_\mu\psi(x)$ coupling with the gauge field is singular, which is consistent with previous analysis made by a perturbational method[10].

4 Conclusion

We have presented a topological method to examine the singularity of axial-vector currents coupling to a gauge field by making use of the topological properties of anomalies in quantum gauge theories. Through analyzing the properties of the Atiyah-Singer index theorem and using the Grassman algebra rules, the connection between the Jacobian of the integral measure and the singularity of the fermion current has been illustrated here only for the Abelian gauge fields. For verifying the argument, we give an example that the Dirac spinor can be decomposed in a two-dimensional eigenspace of the regulation factor, and show that the axial-vector current is indeed singular in the simple situation of this approach. The proposal provides a useful procedure dealing with the reduction of the interaction vertex in the Dyson-schwinger equation by using the Dirac differential equation of motion[11]. In fact, the method is valid in general, and can be used in investigating other issues relevant to the singularity of the axial-vecor currents in quantum gauge theory[12].

The authors wish to thank Professor Wang Qing (Tsinghua University in China) for valuable discussions.

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