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Excited nucleon electromagnetic form factors from broken spin-flavor symmetry

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Abstract A group theoretical derivation of a relation between the $\mathrm{N}\rightarrow\Delta$ charge quadrupole transition and neutron charge form factors is presented.

Key words electromagnetic form factors, group theory, spin-flavor symmetry, symmetry breaking

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1 Introduction

A milestone in the development of strong interaction theory was the proposition[1] that strong interactions not only conserve isospin and strangeness but are also approximately invariant under the higher $SU(3)$ flavor symmetry. The latter combines baryon isospin multiplets with different isospin I and strangeness S to larger degenerate multiplets of particles with the same spin J and parity P, e.g. to a baryon octet with spin 1/2 and a baryon decuplet with spin 3/2. Although broken, flavor symmetry leads to a number of remarkable predictions such as the Gell-Mann-Okubo relation for octet baryons and the equal spacing rule for decuplet baryons, both of which are well satisfied in nature.

A still higher symmetry is obtained when $SU(3)_F$ flavor and $SU(2)_J$ spin symmetries are embedded into the larger $SU(6)$ spin-flavor group, in which case new generators link the previously unconnected flavor and spin symmetries[2—4]. The assumption of an underlying spin-flavor symmetry of strong interactions has far greater predictive power than individual flavor and spin symmetries represented by the direct product group $SU(2)_J\times SU(3)_F$. For example, within $SU(6)$ it is not only possible to combine the spin 1/2 flavor octet (2× 8 states) and spin 3/2 flavor decuplet (4× 10 states) into a **56** dimensional spin-flavor supermultiplet, but also to connect observables of different spin tensor rank such as charge radii (rank 0 tensors) and quadrupole moments (rank 2 tensors) that remain unrelated by the direct product group.

Numerous successes, for example $\mu_p/\mu_n = -3/2$ for the ratio of proton and neutron magnetic moments[4], affirm that $SU(6)$ is a useful symmetry in baryon physics. We now understand that the underlying field theory of strong interactions, quantum chromodynamics (QCD), possesses a spin-flavor symmetry which is exact in the large N_c limit[5, 6], where N_c denotes the number of colors, and that for finite N_c spin-flavor symmetry breaking operators can be classified according to the powers of $1/N_\mathrm{c}$ associated with them. As a result, one obtains a rigorous energy scale independent perturbative expansion scheme for QCD processes[7].

Previously, we abstracted from the quark model with two-body exchange currents a relation between the inelastic $\mathrm{N}\rightarrow\Delta$ quadrupole and the elastic neutron charge form factors[8]

$$G_{C2}^{\mathrm{N}\rightarrow\Delta}(Q^2) = -\frac{3\sqrt{2}}{Q^2}G_C^n(Q^2), \tag{1}$$

which in the limit of zero photon momentum transfer reduces to a relation between the $\mathrm{N}\rightarrow\Delta$ transition quadrupole moment and the neutron charge radius[9]

$$Q_{\mathrm{N}\rightarrow\Delta} = \frac{1}{\sqrt{2}}r_n^2\,. \tag{2}$$

Here, $\mathrm{N}\rightarrow\Delta$ stands for both $\mathrm{p}\rightarrow\Delta^+$ and $\mathrm{n}\rightarrow\Delta^0$ transitions. Comparison of these relations with experiment shows good agreement from low to high momentum transfers[10].

It has been pointed out that the derivation of

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Eq. (1) relies only on the spin-flavor structure of the wave functions and operators involved, i.e., only on general algebraic properties of the quark model and not on specific assumptions, such as values for quark masses, coupling constants, etc. Therefore, it appears that it should be derivable within the framework of an abstract $SU(6)$ tensor analysis, similar to the derivation of the Beg-Lee-Pais relation $\mu_{\mathrm{p}\to\Delta^+} = 2\sqrt{2}\mu_{\mathrm{p}}/3$ between the $\mathrm{p}\to\Delta^+$ transition and proton magnetic moments. The purpose of this paper is to present a group theoretical derivation of Eq. (2). The generalization to finite momentum transfers will be presented elsewhere.

2 Spin-flavor symmetry analysis

We start from the observation that the N(939) and $\Delta(1232)$ are members of the same **56** dimensional $SU(6)$ ground state multiplet of spin-flavor symmetry. If the symmetry were exact, N and Δ baryons would have the same mass. Spin-dependent operators in the Hamiltonian H break $SU(6)$ symmetry and lift the degeneracy between N and Δ masses. Moreover, they connect the symmetry breaking in the flavor octet to the symmetry breaking in the flavor decuplet.

Similarly, in the $SU(6)$ symmetry limit the charge form factors $G^{\mathrm{n}}_{\mathrm{C}}(Q^2)$ and $G^{\mathrm{N}\to\Delta}_{\mathrm{C2}}(Q^2)$ are exactly zero. In the following we will see that spin-dependent terms in the charge operator ρ break $SU(6)$ symmetry and lead to nonzero neutron and $\mathrm{N}\to\Delta$ charge form factors, which are related as in Eq. (1) because the group algebra connects the breaking of the symmetry in $G^{\mathrm{n}}_{\mathrm{C}}(Q^2)$ to the symmetry breaking in $G^{\mathrm{N}\to\Delta}_{\mathrm{C2}}(Q^2)$.

A basic assumption in a group-theoretical analysis is that quantum mechanical operators and states have definite transformation properties, i.e., they transform according to certain irreducible representations (reps) of the underlying symmetry group. A general matrix element $\mathcal{M}$ of an operator $\mathit{\Omega}^R$ evaluated between baryon ground states reads

$$\mathcal{M} = \langle \mathbf{56}|\, \mathit{\Omega}^R \,|\mathbf{56}\rangle, \tag{3}$$

where R is the dimension of the irreducible rep associated with the considered operator.

An allowed symmetry breaking operator $\mathit{\Omega}^R$ acting on the baryon ground state multiplet must transform according to one of the irreducible reps R contained in the direct product[3]

$$\bar{\mathbf{56}} \times \mathbf{56} = \mathbf{1} + \mathbf{35} + \mathbf{405} + \mathbf{2695}. \tag{4}$$

Here, the **1** dimensional rep on the right-hand side corresponds to an $SU(6)$ symmetric operator, while the remaining reps characterize respectively, first, second, and third order $SU(6)$ symmetry breaking operators. Operators transforming according to other $SU(6)$ reps not contained in this product will lead to vanishing matrix elements when evaluated between states belonging to the **56**.

In terms of quark degrees of freedom, one can think of these operators as being constructed from quark-antiquark bilinears transforming according to the adjoint **35** dimensional rep of $SU(6)$, arising from the direct product of two fundamental reps $\mathbf{6}\times\bar{\mathbf{6}} = \mathbf{1}+\mathbf{35}$. Then, on the right-hand side of Eq. (4), the **1** is associated with a zero-quark operator (constant), and the **35**, **405**, and **2695**, are respectively connected with one-, two-, and three-quark operators[11].

First order $SU(6)$ symmetry breaking operators, i.e., one-quark operators, which are constructed from the 35 generators of the $SU(6)$ group[1)], do not lift the degeneracy between N and Δ masses and do not generate nonzero neutral baryon charge radii and nonvanishing baryon quadrupole moments.

In order to split the supermultiplet and to differentiate between spin 1/2 flavor octet and spin 3/2 flavor decuplet masses, the $SU(6)$ symmetry breaking part of H must be spin-dependent. On the other hand, the Hamiltonian transforms as an overall spin scalar. Therefore, we need at least second order $SU(6)$ symmetry breaking operators, i.e., in the simplest case, scalar products of two $SU(2)_J$ generators in order for H to satisfy the two conditions of being spin-dependent and a spin tensor of rank 0 at the same time.

Analogously, to explain the nonzeroness of the neutron charge and decuplet quadrupole form factors it is required that the charge operator ρ be spin-dependent. At the same time, for Coulomb multipoles of ρ we need spin tensor operators of even rank in order to satisfy the time reversal and parity invariances of the electromagnetic interaction. Thus, again at least second order $SU(6)$ symmetry breaking operators are required for these observables.

Second order $SU(6)$ symmetry breaking two-quark operators can be constructed from direct products of one-quark operators. A general two-quark spin-flavor operator transforms according to one of the irreducible reps found in the direct product $\mathbf{35}\times\mathbf{35}$[3]

$$\mathbf{35}\times\mathbf{35} = \mathbf{1}+\mathbf{35}+\mathbf{35}+\mathbf{189}+\mathbf{280}+\bar{\mathbf{280}}+\mathbf{405}. \tag{5}$$

1) For one particle, the 35 generators of $SU(6)$ are composed of three pure spin operators $\boldsymbol{\sigma}_i/2$ with $i=1,2,3$, eight pure flavor operators $\boldsymbol{\lambda}_k/2$ with $k=1,\cdots,8$, and 24 combined spin-flavor operators $(\boldsymbol{\sigma}_i/2)(\boldsymbol{\lambda}_k/2)$.

Two-quark operators transforming according to the **1** and **35** dimensional $SU(6)$ reps can be reduced to constants and one-quark operators in spin space, so that only the four higher dimensional reps on the right hand side of Eq. (5) remain. Of these only the **405** rep appears in the direct baryon ground state product $\bar{\mathbf{56}} \times \mathbf{56}$ according to Eq. (4). Therefore, within the **56** an allowed two-quark operator must necessarily transform according to the **405** dimensional rep of $SU(6)$.

To have a better understanding of the type of operators involved we perform a multipole expansion of the relevant two-quark charge density $\rho_{[2]}$ in spin-flavor space up to quadrupole terms

$$\rho_{[2]} = a\,\mathcal{S}_{[2]} + b\mathcal{T}_{[2]}, \qquad (6)$$

where the spin scalar $\mathcal{S}_{[2]}$ and spin tensor $\mathcal{T}_{[2]}$ operators are defined as

$$\begin{aligned} \mathcal{S}_{[2]} &= -B\sum_{i\neq j} e_i\,\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j, \\ \mathcal{T}_{[2]} &= -B\sum_{i\neq j} e_i\,(3\,\sigma_{i\,z}\sigma_{j\,z} - \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j). \end{aligned} \qquad (7)$$

Here, the constant B parametrizes the color and orbital matrix elements, and $e_i = (1+3\,\tau_{3\,i})/6$ is the quark charge. Furthermore, $\boldsymbol{\sigma}_i$ and $\boldsymbol{\tau}_i$ are the spin and isospin Pauli matrices of the i-th quark. To first order flavor (isospin) breaking, Eq. (6) represents the most general two-quark charge operator in spin-flavor space. These tensors have been used in calculating the **56** baryon ground state charge radii and quadrupole moments[12—15].

We will see shortly that as a consequence of the underlying $SU(6)$ spin-flavor symmetry, the spin scalar and spin tensor terms in Eq. (6) have fixed relative strengths $a/b = -2$. An evaluation of Eq. (6) between N and Δ spin-flavor wave functions leads then straightforwardly to the following results

$$\begin{aligned} r_n^2 &= 4\,B\,a, \\ Q_{\mathrm{N}\to\Delta} &= 2\sqrt{2}\,B\,a, \end{aligned} \qquad (8)$$

from which Eq. (2) is readily established.

Next, without reference to the quark model, we show that the spin tensors of rank 0 and 2 in Eq. (6) are different components of a general $SU(6)$ tensor of dimension **405** which are linked to each other by the group algebra. A decomposition of the tensor $\Omega_{\mathbf{405}}$ into subtensors with definite transformation properties with respect to the flavor and spin subgroups of $SU(6)$ reads

$$\begin{aligned} \mathbf{405} = {} & (\mathbf{1},\mathbf{1}) + (\mathbf{8},\mathbf{1}) + (\mathbf{27},\mathbf{1}) + \\ & 2\,(\mathbf{8},\mathbf{3}) + (\mathbf{10},\mathbf{3}) + (\bar{\mathbf{10}},\mathbf{3}) + (\mathbf{27},\mathbf{3}) + \\ & (\mathbf{1},\mathbf{5}) + (\mathbf{8},\mathbf{5}) + (\mathbf{27},\mathbf{5}), \end{aligned} \qquad (9)$$

where the first and second entry in the parentheses refers to the dimensions of the $SU(3)_F$ and $SU(2)_J$ representations respectively[16]. Thus, spin-flavor symmetry breaking proceeds along the chain $SU(6) \supset SU(3)_F \times SU(2)_J \supset SU(2)_I \times U(1)_Y \times SU(2)_J$, where in a first step $SU(6)$ symmetry is broken into $SU(3)_F \times SU(2)_J$, and in a second step $SU(3)_F$ symmetry is reduced to $SU(2)_I \times U(1)_Y$, i.e., an uncorrelated product of isospin and hypercharge symmetries.

For Coulomb multipoles, we are restricted to spin tensors of even rank, and the second line in Eq. (9) need not concern us here. Furthermore, we confine ourselves to flavor octet tensors appropriate for electromagnetic interaction operators. Eq. (9) shows that there is a unique spin scalar $\mathcal{S}$ transforming as $(\mathbf{8},\mathbf{1})$ and a unique spin tensor $\mathcal{T}$ transforming as $(\mathbf{8},\mathbf{5})$ and both are united in a common $SU(6)$ tensor with dimension **405**.

We expand the baryon charge density operator ρ into Coulomb multipoles[17] up to quadrupole terms

$$\rho = \sum_J i^J\,\hat{J}\,T_0^{CJ} = \rho^{C0} + \rho^{C2}, \qquad (10)$$

with $\hat{J} = \sqrt{2J+1}$ and where we have suppressed the momentum dependence of the multipole operators and an overall factor $\sqrt{4\pi}$. The Coulomb multipole operators T_0^{CJ} are irreducible tensors of rank J and correspond to the second order $SU(6)$ symmetry breaking tensors $\Omega^{\mathbf{405}}_{(\mu\,s)}$. We can now identify

$$\begin{aligned} \rho^{C0} &\sim \Omega^{\mathbf{405}}_{(\mathbf{8},\mathbf{1})}, \\ \rho^{C2} &\sim -\sqrt{5}\,\Omega^{\mathbf{405}}_{(\mathbf{8},\mathbf{5})}. \end{aligned} \qquad (11)$$

The two-quark operators $\mathcal{S}_{[2]}$ and $\mathcal{T}_{[2]}$ in Eq. (6) have just the same transformation properties and are recognized here as different components of a common **405** dimensional tensor operator $\Omega^{\mathbf{405}}$.

According to the generalized Wigner-Eckart theorem, the matrix elements of $\Omega^{\mathbf{405}}$ evaluated between the **56** multiplet can be factorized into a common reduced matrix element (indicated by a double bar), which is the same for the entire multiplet, and an $SU(6)$ Clebsch-Gordan (CG) coefficient

$$\begin{aligned} \mathcal{M} = {} & \langle \mathbf{56}_{\nu_f} |\, \Omega^{\mathbf{405}}_{\nu}\, | \mathbf{56}_{\nu_i} \rangle = \\ & \langle \mathbf{56} ||\, \Omega^{\mathbf{405}}\, || \mathbf{56} \rangle \begin{pmatrix} \mathbf{56} & \mathbf{405} & \mathbf{56} \\ \nu_i & \nu & \nu_f \end{pmatrix}. \end{aligned} \qquad (12)$$

The latter provide relations between the matrix elements of different components of the irreducible tensor operator $\Omega^{\mathbf{405}}_{\nu}$ and the individual states of the **56** dimensional baryon ground state supermultiplet, which are labelled by ν_i and ν_f. Because $SU(6)$ is a rank five group, the label ν comprises five quantum numbers to uniquely specify a state, three for $SU(3)$, e.g. total isospin T, isospin projection T_z, and hypercharge Y, and two for $SU(2)$, e.g. total angular momentum J and its projection J_z.

The $SU(6)$ CG coefficient can be split into a unitary scalar factor $f^{\mathbf{405}}_{(\mu,s)}$ and a product of $SU(3)_F$ and $SU(2)_J$ CG coefficients as

$$\begin{pmatrix}\mathbf{56} & \mathbf{405} & \mathbf{56} \\ \nu_f & \nu & \nu_i\end{pmatrix} = f^{\mathbf{405}}_{(\mu,s)}\begin{pmatrix}\boldsymbol{\mu}_f & \boldsymbol{\mu} & \boldsymbol{\mu}_i \\ \rho_f & \rho & \rho_i\end{pmatrix} (J_i\, J_{i,3} J\, J_3 | J_f\, J_{f,3}), \qquad (13)$$

where μ and $s=2J+1$ denote the dimensionalities of the $SU(3)$ and $SU(2)$ reps. The $SU(3)_F$ CG coefficient label ρ comprises the three quantum numbers $\rho=(Y,T,T_z)$. Note that the $SU(6)$ scalar factor $f^{\mathbf{405}}_{(\mu,s)}$, depends only on the dimensionalities of the $SU(6)$, $SU(3)_F$ and $SU(2)_J$ reps involved but not on the $SU(3)$ and $SU(2)$ labels ρ and J_z.

Now, consider the two $SU(6)$ matrix elements, which are of interest here

$$r_n^2 = \langle \mathbf{56}_n | \Omega^{\mathbf{405}}_{(\mathbf{8},\mathbf{1})} | \mathbf{56}_n \rangle = r\left(-\frac{2}{\sqrt{10}}\right)\left[\frac{1}{\sqrt{3}}\left(-\sqrt{\frac{1}{20}}\right)-\sqrt{\frac{3}{20}}\right] = r\frac{2\sqrt{6}}{15},$$

$$Q_{\mathrm{p}\to\Delta^+} = -\sqrt{5}\langle \mathbf{56}_{\Delta^+} | \Omega^{\mathbf{405}}_{(\mathbf{8},\mathbf{5})} | \mathbf{56}_p \rangle = (-\sqrt{5})r\left(\frac{1}{\sqrt{10}}\right)\left[\frac{2}{\sqrt{15}}\right]\left(-\frac{2}{\sqrt{10}}\right) = r\frac{2\sqrt{3}}{15}, \qquad (14)$$

where $r=\langle \mathbf{56} || \Omega^{\mathbf{405}} || \mathbf{56} \rangle$ is the $SU(6)$ reduced matrix element. The factor of -2 between the rank 0 (charge monopole) and rank 2 (charge quadrupole) tensors is reflected by the $SU(6)$ scalar factors[11, 18] $f^{\mathbf{405}}_{(8,1)} = -2/\sqrt{10}$ and $f^{\mathbf{405}}_{(8,5)} = 1/\sqrt{10}$. The $SU(3)_F$ flavor[19] and $SU(2)_J$ spin CG coefficients are explicitly shown. In the case of the neutron charge radius, the two terms in the brackets correspond to $SU(3)$ CG with sublabels $\rho=(0,0,0)$ and $\rho=(0,1,0)$. As usual, the isosinglet part of a flavor octet charge operator is multiplied by $1/\sqrt{3}$. From Eq. (14) and Eq. (15) we obtain Eq. (2). We could have arrived at this result much faster by noting that the spin-isospin Clebsch-Gordan coefficients have already been calculated in Eq. (8) so that the $SU(6)$ scalar factor would have sufficed to establish Eq. (2).

3 Summary

We have seen that for the present application to the neutron charge radius and the $\mathrm{N}\to\Delta$ quadrupole moment, where first order $SU(6)$ symmetry breaking does not contribute, the $J=0$ and $J=2$ multipole components of the charge density ρ transform as the second order $SU(6)$ symmetry breaking tensors $\Omega^{\mathbf{405}}_{(8,1)}$ and $\Omega^{\mathbf{405}}_{(8,5)}$ respectively. In contrast to separate $SU(2)_J$ and $SU(3)_F$ symmetries or their direct product $SU(2)_J\times SU(3)_F$, broken spin-flavor $SU(6)$ symmetry provides a definite relation between spin operators of different tensor rank that belong to the same $SU(6)$ tensor. As a result we obtain the relation between the neutron charge radius and the $\mathrm{N}\to\Delta$ quadrupole moment of Eq. (2) from a general $SU(6)$ symmetry analysis. We hope to address a generalization of this derivation including third order $SU(6)$ symmetry breaking in a future communication.

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