Landau problem in noncommutative quantum mechanics*

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Abstract The Landau problem in non-commutative quantum mechanics (NCQM) is studied. First by solving the Schrödinger equations on noncommutative (NC) space we obtain the Landau energy levels and the energy correction that is caused by space-space noncommutativity. Then we discuss the noncommutative phase space case, namely, space-space and momentum-momentum non-commutative case, and we get the explicit expression of the Hamiltonian as well as the corresponding eigenfunctions and eigenvalues.

Key words quantum mechanic, noncommutative

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1 Introduction

Recently, there has been much interest in the study of physics on noncommutative (NC) space[1—7], not only because the NC space is necessary when one studies the low energy effective theory of D-brane with B field background, but also because in the very tiny string scale or at very high energy situation, the effects of non commutativity of both space-space and momentum-momentum may appear. There are many papers devoted to the study of various aspects of quantum mechanics on noncommutative space with usual (commutative) time coordinate.

In the noncommutative (NC) space the coordinate and momentum operators satisfy the following commutation relations

$$[\hat{x}_i,\hat{x}_j]=\mathrm{i}\Theta_{ij},\quad [\hat{p}_i,\hat{p}_j]=0,\quad [\hat{x}_i,\hat{p}_j]=\mathrm{i}\hbar\delta_{ij}, \tag{1}$$

where $\hat{x}_i$ and $\hat{p}_i$ are the coordinate and momentum operators on a NC space. Refs. [8, 9] showed that $\hat{p}_i=p_i$, and $\hat{x}_i$ have the representation form

$$\hat{x}_i=x_i-\frac{1}{2\hbar}\Theta_{ij}p_j,\quad i,j=1,2,...,n. \tag{2}$$

The case of both space-space and momentum-momentum noncommuting[8, 9] is different from the case of only space-space noncommuting. Thus in the noncommutative (NC) phase space the momentum operator in Eq. (1) satisfies the following commutation relations

$$[\hat{p}_i,\hat{p}_j]=\mathrm{i}\bar{\Theta}_{ij},\quad i,j=1,2,...,n. \tag{3}$$

Here $\{\Theta_{ij}\}$ and $\{\bar{\Theta}_{ij}\}$ are totally antisymmetric matrices which represent the noncommutative property of the coordinate and momentum on noncommutative space and phase space, respectively, and play analogous role to $\hbar$ in the usual quantum mechanics. On NC phase space the representations of $\hat{x}$ and $\hat{p}$ in terms of x and p are given in Ref. [9] as follows

$$\begin{aligned}\hat{x}_i&=\alpha x_i-\frac{1}{2\hbar\alpha}\Theta_{ij}p_j,\\ \hat{p}_i&=\alpha p_i+\frac{1}{2\hbar\alpha}\bar{\Theta}_{ij}x_j,\quad i,j=1,2,...,n.\end{aligned} \tag{4}$$

The α here is a scaling constant related to the noncommutativity of phase space. When $\bar{\Theta}=0$, it leads $\alpha=1$[9], the NC phase space returns to the NC space, which is extensively studied in the literature, where the space-space is non-commuting, while momentum-momentum is commuting.

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Given the NC space or NC phase space, one should study its physical consequences. It appears that the most natural places to search the noncommutativity effects are simple quantum mechanics (QM) system. So far many interesting topics in NCQM such as hydrogen atom spectrum in an external magnetic field[10, 11], Aharonov-Bohm(AB) effect[12] in the presence of the magnetic field, the Aharonov-Casher effects[13], and Landau problem[14], as well as the Van de Waals interactions and photoelectric effect in noncommutative quantum mechanics[15] have been studied extensively. The purpose of this paper is to study the Landau problems on NC space and NC phase space, respectively, where both space-space and momentum-momentum noncommutativity could give additional contribution.

This paper is organized as follows: In Section 2, we study the Landau problem on NC space. By solving the Schrödinger equation in the presence of magnetic field we obtain all the energy levels. In Section 3, we investigate the Landau problem on NC phase space. By solving the Schrödinger equation in the presence of magnetic field, the additional terms related to the momentum-momentum noncommutativity is obtained explicitly. Conclusions are given in Section 4.

2 The Landau problem on NC space

In this section we consider the two dimensional Landau problem in the symmetric gauge on noncommutative space. Let us consider a charged particle, with electric charge q and mass μ, moving in two dimensions (say x-y plane), and under uniform magnetic field B perpendicular to the plane (say z direction). The magnetic vector potential has the form,

$$A_x=-\frac{1}{2}By, \quad A_y=\frac{1}{2}Bx, \quad A_z=0 \tag{5}$$

and then the Hamiltonian of the system has the following form,

$$\begin{aligned}H=&\frac{1}{2\mu}\left[\left(p_x^2+\frac{qB}{2c}y\right)^2+\left(p_y^2-\frac{qB}{2c}x\right)^2+p_z^2\right]=\\&\frac{1}{2\mu}(p_x^2+p_y^2)+\frac{1}{2}\mu\omega_{\rm L}^2(x^2+y^2)-\omega_{\rm L}l_z+\frac{1}{2\mu}p_z^2\,,\end{aligned} \tag{6}$$

where $\omega_{\rm L}=\dfrac{qB}{2\mu c}$, l_z is the z component of the orbital angular momentum and defined as $l_z=xp_y-yp_x$. The static Schrödinger equation on NC space is usually written as

$$H(x,p)*\psi=E\psi\,, \tag{7}$$

where the Moyal-Weyl (or star) product between two functions is defined by,

$$\begin{aligned}(f*g)(x)\;=\;&{\rm e}^{\frac{\rm i}{2}\Theta_{ij}\partial_{x_i}\partial_{x_j}}f(x_i)g(x_j)=\\&f(x)g(x)+\frac{\rm i}{2}\Theta_{ij}\,\partial_i f\,\partial_j g\big|_{x_i=x_j}\,,\end{aligned} \tag{8}$$

here $f(x)$ and $g(x)$ are two arbitrary functions. On NC space the star product can be replaced by a Bopp's shift[5], i.e. the star product can be changed into the ordinary product by replacing $H(x,p)$ with $H(\hat{x},p)$. Thus the Schrödinger Eq. (7) can be written as,

$$H(\hat{x}_i,p_i)\psi=H\left(x_i-\frac{1}{2\hbar}\Theta_{ij}p_j,p_i\right)\psi=E\psi\,. \tag{9}$$

In our case, the $H(\hat{x},p)$ has the following form

$$\begin{aligned}H(\hat{x},p)\;=\;&\frac{1}{2\mu}\left[\left(p_x^2+\frac{qB}{2c}\hat{y}\right)^2+\left(p_y^2-\frac{qB}{2c}\hat{x}\right)^2+p_z^2\right]=\\&\frac{1}{2\mu}\left\{\left[\left(1+\frac{qB}{4\hbar c}\theta\right)p_x+\frac{qB}{2c}y\right]^2+\right.\\&\left.\left[\left(1+\frac{qB}{4\hbar c}\theta\right)p_y-\frac{qB}{2c}x\right]^2+p_z^2\right\}=\\&\frac{1}{2\mu'}(p_x^2+p_y^2)+\frac{1}{2}\tilde{\mu}\tilde{\omega}_{\rm L}^2(x^2+y^2)-\\&\tilde{\omega}_{\rm L}l_z+\frac{1}{2\mu}p_z^2=H_{xy}+H_{l_z}+H_{\parallel}\,,\end{aligned} \tag{10}$$

where

$$H_{xy}=\frac{1}{2\tilde{\mu}}(p_x^2+p_y^2)+\frac{1}{2}\tilde{\mu}\tilde{\omega}_{\rm L}^2(x^2+y^2),$$

$$H_{l_z}=-\tilde{\omega}_{\rm L}l_z,\qquad H_{\parallel}=\frac{1}{2\mu}p_z^2\,,$$

$$\tilde{\mu}=\frac{\mu}{\left(1+\dfrac{qB}{4\hbar c}\theta\right)^2}\,,\quad \tilde{\omega}_{\rm L}=\frac{qB}{2\tilde{\mu}c\left(1+\dfrac{qB}{4\hbar c}\theta\right)}\,, \tag{11}$$

H_{xy} is the hamiltonian for two dimensional harmonic oscillator with mass $\tilde{\mu}$ and angular frequency $\tilde{\omega}_{\rm L}$. We now look for a basis of eigenvectors common to H_{xy} (eigenvalues E_{xy}), H_{l_z}(eigenvalues E_{l_z}), and $H_{\parallel}$ (eigenvalues $E_{\parallel}$). It is easy to show that the H_{xy}, H_{l_z}, and $H_{\parallel}$ commute with each other. Therefore the eigenvectors of $\{H_{xy},H_{l_z},H_{\parallel}\}$ will automatically be eigenvectors of H with eigenvalues

$$E=E_{xy}+E_{l_z}+E_{\parallel}\,. \tag{12}$$

The eigenvectors $\psi_k(z)\sim{\rm e}^{{\rm i}kz}$ of the momentum operator p_z are also eigenvectors of $H_{\parallel}$. Thus the eigenvalues of $H_{\parallel}$ are of the form

$$E_{\parallel}=\frac{\hbar^2k^2}{2\mu},\qquad -\infty<k<+\infty\,. \tag{13}$$

We see that the spectrum of $H_{\parallel}$ is continuous, the energy $E_{\parallel}$ can take any positive value or zero. This result implies that $H_\parallel$ describes the kinetic energy of a free particle moving along the oz (along the direction of magnetic field). The eigenfunctions $\psi_m(\varphi)\sim \mathrm{e}^{\mathrm{i}m\varphi}$, $m=0,\pm1,\pm2,\dots$ of l_z are also wave functions of H_{l_z}. Therefore the eigenvalues of H_{l_z} are

$$E_{l_z}=-m\hbar\tilde{\omega}_{\mathrm{L}}\,. \tag{14}$$

Thus, now we shall concentrate on solving the eigenvalue equation of H_{xy} of two-dimensional harmonic oscillator; note that the wave functions which we consider now depend on x and y, and not on z. The solution to Eq. (9) can be written as a product of the solution for a static harmonic oscillator with the phase factors responsible for the momentum and orbital angular momentum,

$$\begin{aligned}&\psi_{n_\rho mk}(\rho,\varphi,z)=R(\rho)\mathrm{e}^{\mathrm{i}m\varphi}\mathrm{e}^{\mathrm{i}kz},\\ &m=0,\pm1,\pm2,\dots,\qquad -\infty<k<+\infty\,.\end{aligned} \tag{15}$$

Inserting Eq. (15) into Eq. (9), and using cylindrical coordinate system, we can obtain the following radial equation for the two dimensional homogenous harmonic oscillator

$$\left[-\frac{\hbar^2}{2\tilde{\mu}}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\right)+\frac{1}{2}\mu\tilde{\omega}_{\mathrm{L}}^2\rho^2\right]R(\rho)=E_{xy}R(\rho)\,. \tag{16}$$

Solving Eq. (16), the eigenvalues of the Hamiltonian H_{xy} are

$$E_{xy}=(N+1)\hbar\tilde{\omega}_{\mathrm{L}}\,, \tag{17}$$

with $N=(2n_\rho+|m|), n_\rho=0,1,2,\cdots$; and the corresponding eigenfunctions are

$$R(\rho)=\rho^{|m|}F(-n_\rho,|m|+1,\zeta^2\rho^2)\mathrm{e}^{-\zeta^2\rho^2/2},\quad \zeta^2=\frac{\tilde{\mu}\tilde{\omega}_{\mathrm{L}}}{\hbar}\,. \tag{18}$$

Therefore the energy eigenfunctions are

$$\psi_{n_\rho mk}(\rho\varphi,z)=\rho^{|m|}F(-n_\rho,|m|+1,\zeta^2\rho^2)\mathrm{e}^{\mathrm{i}m\varphi+\mathrm{i}kz}\,. \tag{19}$$

The eigenvalues of the total Hamiltonian H are of the form

$$E=(N+1)\hbar\tilde{\omega}_{\mathrm{L}}-m\hbar\tilde{\omega}_{\mathrm{L}}+\frac{\hbar^2k^2}{2\mu}\,. \tag{20}$$

The corresponding levels are called Landau levels. Obviously, when $\theta=0$, then $\tilde{\mu}\to\mu$, $\tilde{\omega}_{\mathrm{L}}\to\omega_{\mathrm{L}}$, our results return to the space-space commuting case.

3 The Landau problem on NC phase space

The Bose-Einstein statistics in NCQM requires both space-space and momentum-momentum non-commutativity. Thus we should also consider the momentum-momentum non-commutativity when we deal with physical problems. The star product in Eq. (8) on NC phase space now is defined as

$$\begin{aligned}(f*g)(x,p)=&\ \mathrm{e}^{\frac{\mathrm{i}}{2\alpha^2}\Theta_{ij}\partial_i^x\partial_j^x+\frac{\mathrm{i}}{2\alpha^2}\bar{\Theta}_{ij}\partial_i^p\partial_j^p}f(x,p)g(x,p)=\\ &f(x,p)g(x,p)+\frac{\mathrm{i}}{2\alpha^2}\Theta_{ij}\partial_i^x f\partial_j^x g\big|_{x_i=x_j}+\\ &\frac{\mathrm{i}}{2\alpha^2}\bar{\Theta}_{ij}\partial_i^p f\partial_j^p g\big|_{p_i=p_j}\,,\end{aligned} \tag{21}$$

which can be replaced by a generalized Bopp's shift $x_i\to\hat{x}_i$, $p_i\to\hat{p}_i$ with $\hat{x}_i$ and $\hat{p}_i$ defined in Eq. (4). Thus on noncommutative phase space the Schrödinger Eq. (9) can be written as,

$$\begin{aligned}&H(\hat{x}_i,\hat{p}_i)\psi=\\ &H\left(\alpha x_i-\frac{1}{2\hbar\alpha}\Theta_{ij}p_j,\alpha p_i+\frac{1}{2\hbar\alpha}\bar{\Theta}_{ij}x_j\right)\psi=E\psi\,.\end{aligned} \tag{22}$$

In two dimensions we have,

$$\begin{aligned}\hat{x}&=\alpha x-\frac{\theta}{2\hbar\alpha}p_y,\quad \hat{y}=\alpha y+\frac{\theta}{2\hbar\alpha}p_x,\\ \hat{p}_x&=\alpha p_x+\frac{\bar{\theta}}{2\hbar\alpha}y,\quad \hat{p}_y=\alpha p_y-\frac{\bar{\theta}}{2\hbar\alpha}x,\end{aligned} \tag{23}$$

The three parameters $\theta,\bar{\theta}$ and α represent the non-commutativity of the phase space, it is related by

$$\bar{\theta}=4\hbar^2\alpha^2(1-\alpha^2)/\theta\,, \tag{24}$$

so only two of them are free in the theory and they may depend on the space and energy scales. The Hamiltonian for the two dimensional Landau problem on noncommutative phase space in the symmetric gauge is

$$\begin{aligned}H(\hat{x},\hat{p})=&\frac{1}{2\mu}\left[\left(\hat{p}_x^2+\frac{qB}{2c}\hat{y}\right)^2+\left(\hat{p}_y^2-\frac{qB}{2c}\hat{x}\right)^2+\hat{p}_z^2\right]=\\ &\frac{1}{2\mu}\left\{\left[\left(\alpha+\frac{qB}{4\hbar\alpha c}\theta\right)p_x+\left(\frac{qB}{2c}\alpha+\frac{\bar{\theta}}{2\hbar\alpha}\right)y\right]^2+\right.\\ &\left.\left[\left(\alpha+\frac{qB}{4\hbar\alpha c}\theta\right)p_y-\left(\frac{qB}{2c}\alpha+\frac{\bar{\theta}}{2\hbar\alpha}\right)x\right]^2+p_z^2\right\}=\\ &\frac{1}{2\tilde{\mu}'}(p_x^2+p_y^2)+\frac{1}{2}\tilde{\mu}'\tilde{\omega}_{\mathrm{L}}'^2(x^2+y^2)-\tilde{\omega}_{\mathrm{L}}'l_z+\frac{1}{2\mu}p_z^2=\\ &H_{xy}'-\tilde{\omega}_{\mathrm{L}}'l_z+\frac{1}{2\mu}p_z^2\,,\end{aligned} \tag{25}$$

where

$$\tilde{\mu}'=\frac{\mu}{\left(\alpha+\dfrac{qB}{4\hbar\alpha c}\theta\right)^2}\,,\quad \tilde{\omega}_{\mathrm{L}}'=\frac{\dfrac{qB}{c}\alpha+\dfrac{\bar{\theta}}{\hbar\alpha}}{2\tilde{\mu}'\left(\alpha+\dfrac{qB}{4\hbar\alpha c}\theta\right)}\,, \tag{26}$$

H_{xy}' is the Hamiltonian for two dimensional harmonic oscillator with mass $\tilde{\mu}'$ and angular frequency $\tilde{\omega}_{\mathrm{L}}'$. In an analogous way as in NC space, the solution to Eq. (22) can be written as a product of the solution for a static harmonic oscillator with the phase factors responsible for the momentum and orbital angular momentum.

$$\psi_{n_\rho mk}(\rho,\varphi,z)=R(\rho)\mathrm{e}^{\mathrm{i}m\varphi}\mathrm{e}^{\mathrm{i}kz},$$
$$m=0,\pm1,\pm2,\ldots,\qquad -\infty<k<+\infty\,. \quad (27)$$

The eigenvalues of l_z and p_z are $m\hbar$ and $\hbar k$, respectively. Choosing cylindrical coordinate system, and inserting Eq. (27) into Eq. (22), we can obtain the following radial equation for the two dimensional homogenous harmonic oscillator

$$\left[-\frac{\hbar^2}{2\tilde{\mu}'}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\right)+\frac{1}{2}\mu\tilde{\omega}'^2_{\mathrm{L}}\rho^2\right]R(\rho)=E'_{xy}R(\rho)\,. \quad (28)$$

This eigenvalue equation of H'_{xy} leads to the wave functions

$$R(\rho)=\rho^{|m|}F(-n_\rho,|m|+1,\zeta'^2\rho^2)\mathrm{e}^{-\zeta'^2\rho^2/2},\quad \zeta'^2=\frac{\tilde{\mu}'\tilde{\omega}'_{\mathrm{L}}}{\hbar}\,. \quad (29)$$

with eigenvalue

$$E'_{xy}=(N+1)\hbar\tilde{\omega}'_{\mathrm{L}}\,, \quad (30)$$

where $N=(2n_\rho+|m|), n_\rho=0,1,2,\cdots$. Therefore the total eigenfunctions of the Hamiltonian H are of the form

$$\psi_{n_\rho mk}(\rho\varphi,z)=\rho^{|m|}F(-n_\rho,|m|+1,\zeta'^2\rho^2)\mathrm{e}^{\mathrm{i}m\varphi+\mathrm{i}kz}\,, \quad (31)$$

where the term $\mathrm{e}^{\mathrm{i}kz}$ describes a free particle moving along the magnetic field, and the particle energy is continuous. In the x-y plane, particle is confined in a harmonic potential, energy is discontinuous. The eigenvalues of the total Hamiltonian H are of the form

$$E=(N+1)\hbar\tilde{\omega}'_{\mathrm{L}}-m\hbar\tilde{\omega}'_{\mathrm{L}}+\frac{\hbar^2k^2}{2\mu}\,. \quad (32)$$

The corresponding levels are called Landau levels on NC phase space. Obviously, when $\theta\neq0$ and $\alpha=1$, it leads to $\bar{\theta}=0$ (refer to Eq. (24)), such that $\tilde{\mu}'\to\tilde{\mu}$, $\tilde{\omega}'_{\mathrm{L}}\to\tilde{\omega}_{\mathrm{L}}$, which is the space-space noncommuting case. When both $\theta=0$ and $\bar{\theta}=0$ then $\tilde{\mu}'\to\mu$, $\tilde{\omega}'_{\mathrm{L}}\to\omega_{\mathrm{L}}$, our results return to the case of usual quantum mechanics.

4 Conclusion

In this letter we study the Landau problem in NCQM. The consideration of the NC space and NC phase space produces additional terms. In order to obtain the NC space correction to the usual Landau energy levels, in Section 2, first, we give the Schrödinger equation in the presence of a uniform magnetic field; and then by solving the equation we derive all the energy levels. In order to obtain the NC phase space correction to the usual Landau problems, in Section 3, we solve the Schrödinger equation in the presence of a uniform magnetic field and obtain new terms which comes from the momentum-momentum noncommutativity.

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