Vector meson masses in two-dimensional $SU(N_C)$ lattice gauge theory with massive quarks*

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Abstract Using an improved lattice Hamiltonian with massive Wilson quarks a variational method is applied to study the dependence of the vector meson mass M_V on the quark mass m and the Wilson parameter r in two-dimensional $SU(N_C)$ lattice gauge theory. The numerical results show that for $N_C = 2, 3, 4, 5, 6, 7, \cdots$, in the scaling window $1 \leqslant 1/g^2 \leqslant 2$, M_V/g is approximately linear in m, but M_V/g obviously does not depend on r (this differs from the quark condensate). Particularly for $m \to 0$ our numerical results agree very well with Bhattacharya's analytical strong coupling result in the continuum, and the value of $\left(\partial M_V/\partial m\right)\big|_{m=0}$ in two-dimensional $SU(N_C)$ lattice gauge theory is very close to that in Schwinger model.

Key words $SU(N_C)$ lattice gauge theory, Wilson quark, improved Hamiltonian, vector meson mass

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1 Introduction

Lattice gauge field theory (LGT), which is based on the first principle, is the most reliable and powerful non-perturbative approach to QCD (quantum chromodynamics). LGT has two equivalent formulations, namely, the Lagrangian formulation and the Hamiltonian formulation. The advantage of the Hamiltonian formulation is that one can use it to compute not only the mass spectrum but also the wave function.

The systematical errors of LGT are mainly due to the finite value of the lattice spacing a. One possible way to tackle this problem is to improve the lattice action (or Hamiltonian), such as to reduce the finite a errors to higher orders in a. In recent years, we (Guo, Chen, Luo and Jiang) have been studying the problem of improvement of lattice Hamiltonian by adding local, neaest-neighbor or next-neaest-neighbor interaction terms to the lattice Hamiltonian: (i) For the Wilson quark sector, we constructed an improved Hamiltonian by reducing the errors from $O(a)$ to $O(a^2)$. This improvement has been tested successfully in the two-dimensional QCD by Jiang and Luo et al[1]. (ii) For the gluonic sector, Luo, Guo, Kröger, and Schütte constructed a simple improved Hamiltonian[2] to reduce the errors from $O(a^2)$ to $O(a^4)$. This has been tested successfully in the three-dimensional LGT by Jiang et al[3—5]. Our results[1, 3—5] have shown that the improved Hamiltonian indeed leads to much better results than the unimproved one.

Similar to the four-dimensional QCD, the two-dimensional QCD (corresponding to the two-dimensional gauge symmetry group $SU(N_C)$) shows also gluonic self-interactions and a rich mass spectrum because of the non-Abelian gauge interactions between gluons and quarks. Of course, the two-dimensional QCD is much simpler than the four-dimensional QCD. It has been extensively studied[1, 6—10] for the sake of testing various methods and mimicking the properties of four-dimensional QCD. Early in 1974, 't Hooft[6] did pioneering work on the two-dimensional QCD using the $1/N_C$ expansion. In 1982, Bhattacharya[8] investigated the two-dimensional QCD in the chiral limit $m \to 0$ (m is the quark mass) and in the strong coupling phase and obtained the vector meson mass

$$M_V = e\sqrt{(N_C+1)/(2\pi)}, \tag{1}$$

where e is the coupling constant (in Ref. [8], use M

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instead of M_V, use g instead of e, use n instead of N_C). The first lattice study of the two-dimensional QCD was done by Hamer[11] using Kogut-Susskind quarks. He computed for $SU(2)$ the mass spectrum in the 't Hooft phase.

In Ref. [1], we have studied the scaling behavior of the vector meson mass M_V in two-dimensional $SU(N_C)$ LGT by using the improved lattice Hamiltonian with massless Wilson quarks (m=0). In Ref. [12], we studied quark condensate $\langle\bar{\psi}\psi\rangle$ in two-dimensional $SU(N_C)$ lattice gauge theory with massive Wilson quarks ($m \neq 0$). In this paper we shall study the dependence of the vector meson mass M_V on the quark mass m and the Wilson paramete r in two-dimensional $SU(N_C)$ LGT using the improved lattice Hamiltonian and the variational method.

2 Improved lattice Hamiltonian with massive Wilson quark and the variational method

The improved lattice Hamiltonian with massive Wilson quark is[1]

$$\begin{aligned} &H = H_g + H_m + H_k + H_r\,,\\ &H_g = \frac{g^2}{2a}\sum_{x,j} E_j^\alpha(x)E_j^\alpha(x)\,, \quad H_m = m\sum_x \bar{\psi}(x)\psi(x)\,,\\ &H_k = \frac{1}{2a}\sum_{x,k}\left[\frac{4}{3}\bar{\psi}(x)\gamma_k U(x,k)\psi(x+k) - \frac{1}{6}\bar{\psi}(x)\gamma_k U(x,2k)\psi(x+2k)\right],\\ &H_r = \frac{r}{2a}\sum_{x,k}\left[\bar{\psi}(x)\psi(x) - \frac{4}{3}\bar{\psi}(x)U(x,k)\psi(x+k) + \frac{1}{3}\bar{\psi}(x)U(x,2k)\psi(x+2k)\right], \end{aligned} \tag{2}$$

where g is the dimensionless coupling constant which is related to the lattice spacing a and the invariant charge e by g=ea, r $(0 < r \leqslant 1)$ is the Wilson parameter, m is the free quark mass, $E_j^\alpha(x)$ is the color-electric field, $U(x,2k) = U(x,k)U(x+k,k)$, $U(x,k)$ are the gauge link variables, $k = \pm 1$, $j = 1$, $\gamma_{-k} = -\gamma_k$, γ_k is a Pauli matrix:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{3}$$

The two-component spinor $\psi(x)$ may be written as

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta^+(x) \end{pmatrix}, \tag{4}$$

and the bare vacuum $|0\rangle$ is defined by

$$\xi(x)\,|0\rangle = \eta(x)\,|0\rangle = E_j^\alpha(x)\,|0\rangle = 0\,. \tag{5}$$

The gauge field physical vacuum $|\Omega\rangle$ is defined by[1]

$$|\Omega\rangle = \exp(\mathrm{i}S)\,|0\rangle\,, \tag{6}$$

where

$$\begin{aligned} &S = \sum_{n=1}^{N_{\mathrm{trun}}} \theta_n S_n\,,\\ &S_1 = \mathrm{i}\sum_{x,k}\psi^+(x)\gamma_k U(x,k)\psi(x+k),\\ &S_2 = \mathrm{i}\sum_{x,k}\psi^+(x)\gamma_k U(x,2k)\psi(x+2k),\\ &S_3 = \mathrm{i}\sum_{x,k}\psi^+(x)\gamma_k U(x,3k)\psi(x+3k),\\ &\cdots \end{aligned} \tag{7}$$

here θ_1, θ_2, θ_3 $\cdots$ are the variational parameters. In this paper, we choose N_{trun}=3.

The vacuum energy is defined by

$$E_\Omega = \frac{\langle\Omega|H|\Omega\rangle}{\langle\Omega|\Omega\rangle}\,. \tag{8}$$

From the conditions of the minimization of the vacuum energy

$$\frac{\partial E_\Omega}{\partial\theta_1} = 0\,, \quad \frac{\partial E_\Omega}{\partial\theta_2} = 0\,, \quad \frac{\partial E_\Omega}{\partial\theta_3} = 0\,, \tag{9}$$

we obtain the nonlinear equations of the variational parameters θ_1, θ_2, θ_3.

The wave function of the flavor-singlet vector meson is constructed as a superposition of some operators (in this paper we choose three operators):

$$|V\rangle = \left[\lambda_0\sum_{x,k}\bar{\psi}(x)\gamma_1\psi(x) + \lambda_1\sum_{x,k}\bar{\psi}(x)\gamma_1 U(x,k)\psi(x+k) + \lambda_2\sum_{x,k}\bar{\psi}(x)\gamma_1 U(x,2k)\psi(x+2k)\right]|\Omega\rangle, \tag{10}$$

here λ_0, λ_1, λ_2 are the variational parameters.

The vector meson mass is defined by

$$M_V = \frac{\langle V|H|V\rangle}{\langle V|V\rangle} - \frac{\langle\Omega|H|\Omega\rangle}{\langle\Omega|\Omega\rangle}\,. \tag{11}$$

From the conditions of the minimization of the vector meson mass

$$\frac{\partial M_V}{\partial\lambda_0} = 0, \quad \frac{\partial M_V}{\partial\lambda_1} = 0, \quad \frac{\partial M_V}{\partial\lambda_2} = 0, \tag{12}$$

we get the linear equations of the variational parameters λ_0, λ_1 and λ_2.

From Eq. (9) and Eq. (12), we obtain M_V as a function of N_C, m, r and $1/g^2$: $M_V = f(N_C, r, m, 1/g^2)$.

3 Numerical results of the vector meson mass

Given the values of r, $1/g^2$ and N_C, we obtain the dependence of the vector meson mass M_V on the quark mass m. Our numerical results show that for the improved lattice Hamiltonian with massive Wilson quarks, in the scaling window $1 \leqslant 1/g^2 \leqslant 2$, the vector meson mass M_V/g is approximatively linear in m, but M_V/g obviously does not dependent on r.

Figures 1 and 2 show $M_V\, a/g$ as a function of ma for $N_C = 2, 3, 4, 5, 6, 7$; $1/g^2 = 2$; $r = 0.5$ and $r = 1$, respectively.

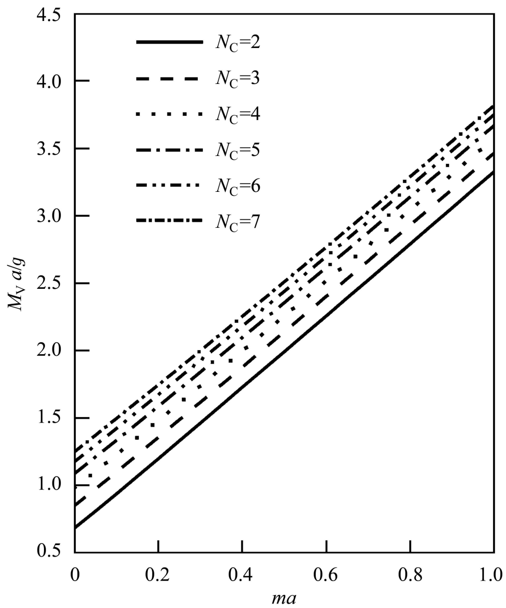

Fig. 1. $M_V\, a/g$ versus ma for $1/g^2 = 2$ and $r = 0.5$.

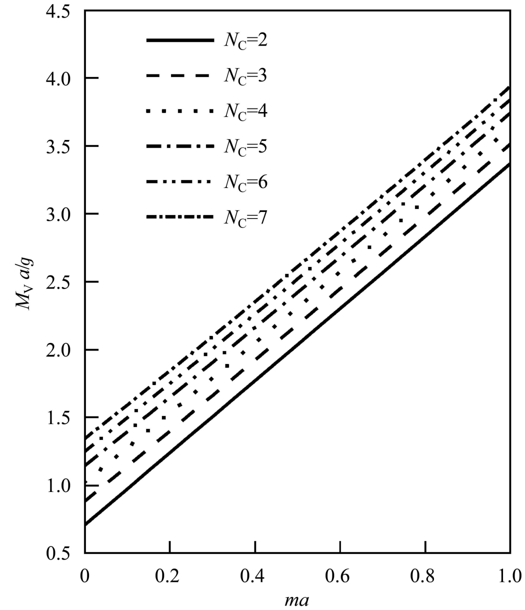

Fig. 2. $M_V\, a/g$ versus ma for $1/g^2 = 2$ and $r = 1$.

An interesting quantity is the value of $(\partial M_V/\partial m)\big|_{m=0}$. In the massive Schwinger model (which corresponds to the two-dimensional $U(1)$ gauge group with massive fermions) it can be calculated exactly to $(\partial M_V/\partial m)\big|_{m=0}$=1.78[13]. However, in the two-dimensional $SU(N_C)$ gauge group, the value of $(\partial M_V/\partial m)\big|_{m=0}$ cannot be computed exactly. In this paper, we present (according to our knowledge for the first time) a computation of $(\partial M_V/\partial m)\big|_{m=0}$ in the two-dimensional $SU(N_C)$ gauge group on the lattice. Table 1 gives the numerical results for $1/g^2 = 2$, $r = 0.5$ and $r = 1$.

Table 1. The rate of change of M_V with m at m=0 when $1/g^2 = 2$.

N_C		2	3	4	5	6	7	8	9	10	11	12
$\left(\partial M_V/\partial m\right)\big\|_{m=0}$	(r=0.5)	1.76	1.75	1.74	1.73	1.73	1.74	1.74	1.75	1.76	1.76	1.77
	(r=1)	1.86	1.82	1.79	1.77	1.75	1.75	1.74	1.74	1.75	1.75	1.76

As one can see, the values of $(\partial M_V/\partial m)\big|_{m=0}$ in the two-dimensional $SU(N_C)$ gauge group vary from 1.73 to 1.86, which is very close to the Schwinger model.

To compare our results with the analytical result from Ref. [8], we give also $M_V\ a/(g\sqrt{N_C+1})$ as a function of ma for $N_C = 2, 3, 4, 5, 6, 7$; $1/g^2 = 2$; $r = 0.5$ and $r = 1$, respectively. The results are shown in Figs. 3 and 4.

It can be seen that for $m \to 0$, $M_V\ a/(g\sqrt{N_C+1})$ (i.e. $M_V/(e\sqrt{N_C+1})$) approaches almost the same value for all $N_C = 2, 3, 4, 5, 6, 7, \cdots$, which is very close to Bhattacharya's analytical strong coupling result in the chiral limit[8]: $M_V/(e\sqrt{N_C+1}) = 1/\sqrt{2\pi} \approx 0.4$.

4 Conclusions and discussions

Using an improved lattice Hamiltonian and the variational method, we have studied in this work the dependence of the vector meson mass M_V on the quark mass m and the Wilson paramete r in the framework of two-dimensional $SU(N_C)$ lattice gauge theory with $N_C = 2, 3, 4, 5, 6, 7, \cdots$. The main results obtained are summarized in the following.

(a) In the scaling window $1 \leqslant 1/g^2 \leqslant 2$, the vector meson mass M_V/g is approximately linear in m, but M_V/g obviously does not dependent on r, which differs from the quark condensate[12] (if m is large, the quark condensate is obviously dependent on r).

(b) For $m \to 0$ our numerical results agree very well with Bhattacharya's analytical strong coupling result[8] in the chiral limit. This implies that our numerical results are reliable in the case of $m \neq 0$.

(c) In the two-dimensional $SU(N_C)$ gauge group $(\partial M_V/\partial m)\big|_{m=0}$ is very close to the Schwinger model. Taking into account the numerical errors, one may even be tempted to believe that the value of $(\partial M_V/\partial m)\big|_{m=0}$ in two-dimensional $SU(N_C)$ lattice gauge theory is equal to that in the Schwinger model. In the two-dimensional $SU(N_C)$ gauge group $(\partial M_V/\partial m)\big|_{m=0}$ cannot be obtained from the analysis of the strong coupling phase in the chiral limit[8]. Our results demonstrate once more the advantage of lattice gauge field theory.

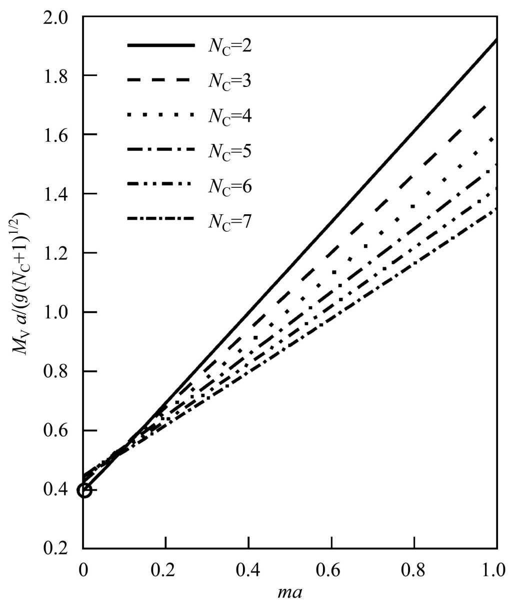

Fig. 3. $M_V\,a/(g\sqrt{N_C+1})$ versus ma for $1/g^2=2$ and $r=0.5$. The open circle gives the analytical result from Ref. [8].

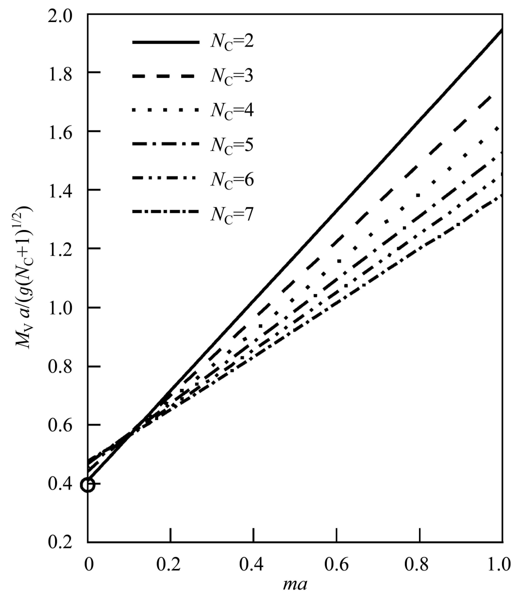

Fig. 4. $M_V\,a/(g\sqrt{N_C+1})$ versus ma for $1/g^2=2$ and $r=1$. The open circle gives the analytical result from Ref. [8].

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