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Comments on Variational Method Used in Accelerating Structure

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Abstract The variational theory for general accelerating structure is studied in detail. The writer wishes to point out some errors in previous papers and explains some important points on the application of the variational expression.

Key words variational method, accelerating structure, electromagnetic field

1 Introduction

The variational method has high accuracy and needs small memory space. Therefore, many people used variational method to study accelerating structure in the past decades[1—4], especially for disk-loaded structure in linac. Although they gave good results, there were some errors in these papers about the variational theory. For example, Masao Nakamura[1] did not consider the non-metal condition at the two ends of the disk-loaded structure, and there is other mathematic errors in the procedure to prove the variational formula. Wang Boci[2] pointed out these problems and obtained the variational formula with new method. He first assumes that the electromagnetic field satisfies Maxwell equations, metal boundary condition on metal and periodic condition at the two ends of one cell, then the variational formula $\delta J = 0$ is proved by using the metal boundary condition $\boldsymbol{E} \times \boldsymbol{n} = 0$ and periodic condition. This is not what we want to prove. The correct way is to prove that the fields will satisfy Maxwell equations and boundary condition if $\delta J = 0$. And the metal boundary condition $\boldsymbol{E} \times \boldsymbol{n} = 0$ can't be used as a known condition because the trial function doesn't need satisfy this condition as in Ref.[3]. This is explained in the next section. Yao[4] also derived the variational formula. He first proved that the Maxwell equation will be satisfied by selecting one special function $\boldsymbol{\eta}$ (Eq.(5.2.14) in Ref.[4]) on metal boundary,

$$\boldsymbol{\eta} \times \boldsymbol{n} = \boldsymbol{\eta}^* \times \boldsymbol{n} = 0, \tag{1}$$

here $\boldsymbol{n}$ is the unit vector outward normal to surface of the accelerating structure. Then he concluded that the metal boundary condition will be satisfied for an arbitrary $\boldsymbol{\eta}$ by using Maxwell equation that is derived from Eq.(1). This is inconsistent with Eq.(1) that requires a special function $\boldsymbol{\eta}$.

The variational method for general accelerating structure is studied in this paper and some important points on the application of the variational formula will be explained.

2 Variational Theory for Electromagnetic Field in Structure

The Maxwell equations for field with time dependence of $e^{j\omega t}$ in vacuum space are

$$\nabla \times Z_0 \boldsymbol{H} = jk\boldsymbol{E}, \tag{2}$$

$$\nabla \times \boldsymbol{E} = -jkZ_0 \boldsymbol{H}, \tag{3}$$

with boundary conditions on metal as

$$\nabla \times \boldsymbol{H} \times \boldsymbol{n} = 0 \text{ or } \boldsymbol{n} \times \boldsymbol{E} = 0, \tag{4}$$

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where $k = \omega\sqrt{\varepsilon_0\mu_0}$ is the propagation constant and $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ the intrinsic impedance in free space. We can obtain the following equation for magnetic field from the above Maxwell equations,

$$\nabla\times(\nabla\times\boldsymbol{H}) - k^2\boldsymbol{H} = 0. \tag{5}$$

The field $Z_0\boldsymbol{H}$, which satisfies Eq. (5) and the boundary condition (4), can be found by making the following variational form minimum

$$J(Z_0\boldsymbol{H}) = \int\{(\nabla\times Z_0\boldsymbol{H})(\nabla\times Z_0\boldsymbol{H})^* - k^2(Z_0\boldsymbol{H})(Z_0\boldsymbol{H})^*\}\mathrm{d}V, \tag{6}$$

as shown below.

Let us define $\boldsymbol{H}_1$ as $\boldsymbol{H}$ which satisfies equation

$$\delta J(Z_0\boldsymbol{H}) = 0, \tag{7}$$

where δ means stationary. Consider $\boldsymbol{H}$, which is nearly equal to $\boldsymbol{H}_1$ but slight deviated from $\boldsymbol{H}_1$ by $\alpha\boldsymbol{\eta}$ where $\boldsymbol{\eta}$ is a arbitrary function,

$$\boldsymbol{H} = \boldsymbol{H}_1 + \alpha\boldsymbol{\eta}. \tag{8}$$

Substituting Eq.(8) into Eq.(6), we obtain

$$J(\boldsymbol{H}_1 + \alpha\boldsymbol{\eta}) = \int\{(\nabla\times(\boldsymbol{H}_1 + \alpha\boldsymbol{\eta}))(\nabla\times(\boldsymbol{H}_1 + \alpha\boldsymbol{\eta}))^* - k^2(\boldsymbol{H}_1 + \alpha\boldsymbol{\eta})(\boldsymbol{H}_1 + \alpha\boldsymbol{\eta})^*\}\mathrm{d}V. \tag{9}$$

Because $\boldsymbol{H}_1$ satisfies Eq.(7), we have

$$\frac{\partial}{\partial\alpha}J(\boldsymbol{H}_1 + \alpha\boldsymbol{\eta})\Big|_{\alpha=0} = 0. \tag{10}$$

Substituting Eq.(9) into Eq.(10), we obtain

$$\int\{(\nabla\times\boldsymbol{\eta})(\nabla\times\boldsymbol{H}_1)^* + (\nabla\times\boldsymbol{H}_1)(\nabla\times\boldsymbol{\eta})^* - k^2(\boldsymbol{\eta}\boldsymbol{H}_1^* + \boldsymbol{H}_1\boldsymbol{\eta}^*)\}\mathrm{d}V = 0. \tag{11}$$

From the vector identities

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\nabla\times\boldsymbol{A} - \boldsymbol{A}\cdot\nabla\times\boldsymbol{B}, \tag{12}$$

we obtain the following equations by using $(\nabla\times\boldsymbol{H}_1)(\nabla\times\boldsymbol{H}_1^*)$ instead of $\boldsymbol{A}$ and $\boldsymbol{\eta}(\boldsymbol{\eta}^*)$ instead of $\boldsymbol{B}$ in Eq.(12),

$$(\nabla\times\boldsymbol{\eta})\cdot(\nabla\times\boldsymbol{H}_1)^* = -\nabla\cdot[(\nabla\times\boldsymbol{H}_1)^*\times\boldsymbol{\eta}] + \boldsymbol{\eta}\cdot\nabla\times(\nabla\times\boldsymbol{H}_1)^*, \tag{13}$$

$$(\nabla\times\boldsymbol{\eta})^*\cdot(\nabla\times\boldsymbol{H}_1) = -\nabla\cdot[(\nabla\times\boldsymbol{H}_1)\times\boldsymbol{\eta}^*] + \boldsymbol{\eta}^*\cdot\nabla\times(\nabla\times\boldsymbol{H}_1). \tag{14}$$

Putting Eqs.(13,14) into Eq. (11), we obtain

$$\int[\nabla\times(\nabla\times\boldsymbol{H}_1) - k^2\boldsymbol{H}_1]\cdot\boldsymbol{\eta}^*\mathrm{d}V - \int\nabla\cdot[(\nabla\times\boldsymbol{H}_1)\times\boldsymbol{\eta}^*]\mathrm{d}V + \mathrm{CC.} = 0, \tag{15}$$

where CC. means complex conjugate of the rest part. The second term in the above equation can be rewritten as

$$\int\nabla\cdot[(\nabla\times\boldsymbol{H}_1)\times\boldsymbol{\eta}^*]\mathrm{d}V = \int(\nabla\times\boldsymbol{H}_1)\times\boldsymbol{\eta}^*\cdot\boldsymbol{n}\mathrm{d}S, \tag{16}$$

where $\boldsymbol{n}$ is the unit vector outward normal to the surface S. Therefore Eq. (15) can be expressed as

$$\int[\nabla\times(\nabla\times\boldsymbol{H}_1) - k^2\boldsymbol{H}_1]\cdot\boldsymbol{\eta}^*\mathrm{d}V - \int(\nabla\times\boldsymbol{H}_1)\times\boldsymbol{\eta}^*\cdot\boldsymbol{n}\mathrm{d}S + \mathrm{CC.} = 0. \tag{17}$$

Using the identities

$$\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C} = \boldsymbol{C}\cdot\boldsymbol{A}\times\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{C}\times\boldsymbol{A}, \tag{18}$$

the second term in Eq.(17) can further be transformed as

$$\int(\nabla\times\boldsymbol{H}_1)\times\boldsymbol{\eta}^*\cdot\boldsymbol{n}\mathrm{d}S = \int(\boldsymbol{n}\times\nabla\times\boldsymbol{H}_1)\cdot\boldsymbol{\eta}^*\mathrm{d}S = \int(\boldsymbol{\eta}^*\times\boldsymbol{n})\cdot(\nabla\times\boldsymbol{H}_1)\mathrm{d}S. \tag{19}$$

The above integration can be divided into two parts in general. One is on the metal boundary and another is on the non-metal boundary.

Suppose $\boldsymbol{H}_1$ has the same transverse component as $\boldsymbol{H}$ in Eq.(5) on the non-metal boundary, which means $\boldsymbol{H}_1$ satisfies the transverse boundary in Eq.(5),

$$\boldsymbol{H}_{1\perp} = \boldsymbol{H}_{\perp}. \tag{20}$$

Because $\boldsymbol{H} = \boldsymbol{H}_1 + \alpha\boldsymbol{\eta}$, therefore, $\boldsymbol{\eta}$ has zero transverse component, so the integration on non-metal boundary in Eq.(19) will become zero. Under such condition, Eq.(17) can be rewritten as

$$\int[\nabla\times(\nabla\times\boldsymbol{H}_1) - k^2\boldsymbol{H}_1]\cdot\boldsymbol{\eta}^*\mathrm{d}V - \int^{\mathrm{metal}}[\boldsymbol{n}\times(\nabla\times\boldsymbol{H}_1)]\cdot\boldsymbol{\eta}^*\mathrm{d}S + \mathrm{CC.} = 0. \tag{21}$$

Because $\boldsymbol{\eta}$ is an arbitrary function, we can argue that for Eq.(21) to be true for any $\boldsymbol{\eta}$ the magnetic field $\boldsymbol{H}_1$ must satisfy

$$\nabla\times(\nabla\times\boldsymbol{H}_1) - k^2\boldsymbol{H}_1 = 0, \tag{22}$$

$$\nabla\times\boldsymbol{H}_1\times\boldsymbol{n} = 0, \text{ on metal boundary}. \tag{23}$$

Therefore we can conclude that if $\boldsymbol{H}_1$ satisfies Eqs.(7) and (20) on the non-metal boundary, then $\boldsymbol{H}_1$ satisfies Eq.(5) and metal boundary condition (4). There is not any limitation on trial functions if all the boundaries are composed by perfect metal material. If some of the boundary is non-metal boundary, then the trial functions should satisfies Eq.(20) on the non-metal boundary. It means that the transverse magnetic field component given by trial functions should be equal to the value of the true fields there. In the periodic disk-loaded structure case, the trial functions for magnetic field should fulfill the Floquet condition at the two end sides of one cell,

$$\boldsymbol{H}(r,\theta,z+D) = \boldsymbol{H}(r,\theta,z)\mathrm{e}^{-\mathrm{j}\phi_0}, \tag{24}$$

where D is the periodic structure length and ϕ_0 the phase shift over the period.

We can obtain similar variational expression for electronic field as

$$J(\boldsymbol{E}) = \int\{(\nabla\times\boldsymbol{E})(\nabla\times\boldsymbol{E})^* - k^2\boldsymbol{E}\boldsymbol{E}^*\}\mathrm{d}V. \tag{25}$$

But Eq.(25) requires that the trial functions should satisfy the metal boundary condition (4) on metal boundary. This will bring some difficulty when we choose the trial functions. Therefore we usually use Eq.(6) as our variational expression.

Using Eq.(2), the variational form Eq.(6) can be rewritten as

$$J(Z_0\boldsymbol{H}) = k^2\int[\boldsymbol{E}\cdot\boldsymbol{E}^* - (Z_0\boldsymbol{H})(Z_0\boldsymbol{H})^*]\mathrm{d}V. \tag{26}$$

Using the relationship

$$\nabla\cdot[Z_0\boldsymbol{H}^*\times(\nabla\times Z_0\boldsymbol{H})] = (\nabla\times Z_0\boldsymbol{H})\cdot(\nabla\times Z_0\boldsymbol{H})^* - k^2(Z_0\boldsymbol{H})\cdot(Z_0\boldsymbol{H})^*, \tag{27}$$

Eq.(6) can also be written as

$$\begin{aligned} J(Z_0\boldsymbol{H}) &= \int\nabla\cdot[(Z_0\boldsymbol{H})^*\times(\nabla\times Z_0\boldsymbol{H})]\mathrm{d}V = \\ &\int(Z_0\boldsymbol{H})^*\times(\nabla\times Z_0\boldsymbol{H})\cdot\boldsymbol{n}\mathrm{d}S = \\ &\mathrm{j}k\int(Z_0\boldsymbol{H})^*\times\boldsymbol{E}\cdot\boldsymbol{n}\mathrm{d}S. \end{aligned} \tag{28}$$

If the energy is conserved, then the integral on the left-hand side of Eq.(28) is zero when the integral is taken over the whole structure. It can be proven that the integral is also zero for a single cell. The integral on the metal surface is zero for a real field. On the two end surfaces of the cell, the Floquet condition gives

$$(\boldsymbol{H}^*\times\boldsymbol{E})|_{z=-D/2} = (\boldsymbol{H}^*\times\boldsymbol{E})|_{z=D/2}. \tag{29}$$

Since the normal $\boldsymbol{n}$ at the left and right end surface is $-\hat{z}$ and $\hat{z}$, respectively, the integration on the two end surfaces cancel each other to make J zero. It also can be proven that the integration on the right of Eq.(28) is zero for a real field when the boundary is perfect conductor, which means there are equal mean stored electric energy and magnetic energy in the structure for an electromagnetic resonator. Therefore, the variational problem Eq.(28) is equivalent to seeking for fields with zero minimum value of J, and the resulting field will has equal amounts of stored electric and magnetic energy.

From Eqs.(6,7), we can obtain new variational expression

$$k^2 = \min \frac{\int (\nabla \times Z_0 \boldsymbol{H})(\nabla \times Z_0 \boldsymbol{H})^* \mathrm{d}V}{\int (Z_0 \boldsymbol{H})(Z_0 \boldsymbol{H})^* \mathrm{d}V}. \tag{30}$$

This equation gives the resonant frequency of an electromagnetic resonator that includes only vacuum in it. It means resonant frequency is a minimum value of the above expression. Ref.[3] used Eq.(28) as the variational expression because it is simpler than Eq.(6). Refs.[1,2] used Eq.(30) as the variational expression and Ref.[4] used $\frac{\mathrm{j}}{4\pi k}\int (Z_0 \boldsymbol{H})^* \times \boldsymbol{E} \cdot \boldsymbol{n} \mathrm{d}S$.

3 Application of Variational Method

Some important aspects on the application of the variational method will be pointed out in this section. We use tapered disk-loaded waveguide as an example. It is often used in linac for acceleration and studied in Refs.[1—4] by using variational method. The application is same for other kinds of structures.

The general variational expression for magnetic field in Eqs.(28) and (30) can be used as variational expression for disc-loaded structure and other kind of cavity directly as in Refs.[1—4]. Here, the writer points some important issue about the boundary.

There are two kinds of non-metal boundary condition in disk-loaded structure. One is the two end sides of one cell where the periodic condition of magnetic field should be satisfied for the periodic structure as Eq.(24). The total structure is usually divided into a few subregions. Therefore, another non-metal boundary is the interface between the different regions. Different trial functions are used in the different regions as in the Refs.[1—4]. The fields should be continuous across the interface between the subregions. We can obtain the fields matching conditions by equating the two tangential components of the magnetic fields at interface. However, we shouldn't match the electric field at the interface because Eqs.(28) and (30) are derived from the Eq.(6) which requires that the magnetic field should satisfy the boundary condition instead of electric field. This is different from the mode matching method where we can match both the electric field and magnetic field.

The transverse magnetic field boundary should be satisfied at each end of cell for a single cell cavity or for a periodic structure. We should set magnetic field boundary instead of electron field boundary here for the same reason as explained in the above paragraph.

If we use variational expression of electric field as in Eq.(25), then the electric field boundary and continuity across the interface of different region should be satisfied.

4 Conclusion

The variational expression for electromagnetic field in a structure is studied in detail. It can be used in any structure. The main points for boundary condition on the usage of these variational expressions are pointed out.

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变分原理在加速结构中应用的论证

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摘要 详细地论述了加速结构中的变分原理,作者希望能指正已发表的有关文章的不足之处,并就变分方法在加速结构中的应用提出了几点注意事项.

关键词 变分法 加速结构 电磁场

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