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Unified Model Accommodating Three Generations of Fermions

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An $SO(17)$ unified model defined in ten dimensions is presented. The model is dimensionally reduced over the coset space $SU(3)/U(1)\times U(1)\times Z_2$, giving in four dimensions the standard model with three generations of light fermions.

Key words: coset space, dimensional reduce, standard model, grand unified theory.

1. INTRODUCTION

Although there is not a experimental result conflict with the standard model so far, the standard model still has some defect. Some of them are the origin of Higgs potential and the existence of chiral fermions. That is what we are interested in. Recently some authors presented an effective scheme to study these questions. That is the coset-space-dimension-reduction (CSDR) scheme. It has the advantage of fewer parameters and stronger prediction power. At present, some authors have presented $SO(13)$ [2] and $SO(16)$ [3] model. In this paper we present an $SO(17)$ unified model with three generations of light fermions.

This paper is organized as follows. In the next section, we briefly review the main content of the CSDR scheme. In Sect. 3 we introduce the $SO(17)$ unified model in detail. Finally, we discuss the result of this model.

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2. CSDR SCHEME

Consider in D-dimensions a manifold $M^D = M^4 + S/R$, where M^4 is a four-dimensional Minkowski space, S/R is a N-dimensional compact coset space. $D = 4 + N$. The action of the Yang-Mills Dirac theory which defined on M^D is

$$A = \int d^4x d^4y \sqrt{-g}\left(-\frac{1}{4}\mathrm{Tr}(F_{MN}F_{KA})g^{MK}g^{NA} + \frac{1}{2}\mathrm{i}\bar{\psi}\Gamma^M D_M\psi\right), \tag{1}$$

where x is a coordinate on M^4, y is a coordinate on coset space S/R. We assume that fields A_μ, ψ have an S-invariance, that means, any transformation under symmetries of S/R can be compensated by gauge transformations. Therefore, the dependence of the A_μ and ψ on the coordinates of the coset space is definitely determined. Integrating out the coordinate of the coset space, we can obtain the following lagrangian in four dimensions:

$$\begin{aligned} A = {} & C\int d^4x\left[-\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}\sum_a \mathrm{Tr}(D_\mu\phi_a D^\mu\phi_a - V(\phi)\right] \\ & + C\int d^4x\,\frac{1}{2}\mathrm{i}(\bar{\psi}\Gamma^\mu D_\mu\psi + \bar{\psi}\Gamma^a D_a\psi), \end{aligned} \tag{2}$$

where C is the volume of the coset space, D_μ and D_a are the appropriate covariant derivatives, the potential $V(\phi)$ is given by the formula

$$V(\phi) = \frac{1}{4}g^{ac}g^{bd}\mathrm{Tr}\{(f^d_{ab}\phi_d - [\phi_a, \phi_b])(f^e_{cd}\phi_e - [\phi_c, \phi_d])\}, \tag{3}$$

where ϕ_a are the scalar fields which are determined by the Lie algebra of the gauge group G, a, b, c and d, over S, f^d_{ab} the structure constants of S.

Imposing S invariance on all the fields, the number of space-time dimensions are consistently reduced so producing an effective theory in four dimensions. This scheme is called the coset space dimension reduction (CSDR) scheme. To construct an unified theory by CSDR, we need to choose a gauge group first and then to find the surviving scalar fields and surviving fermionic fields.

S-invariance requires that the gauge fields content on M^4 before dimensional reduction satisfies

$$A_\mu(x, y) = g(s)A_\mu(x, s^{-1}y)g^{-1}(s), \tag{4}$$

where $g(s)$ is the gauge transformation on S/R. After dimensional reduction, A_μ become four dimensional gauge field, and they are independent of y. We may see from (4), they commute with the image R_G of R in G. Therefore, the four dimensional gauge group H is the centralizer of R_G in G.

How can we find the surviving scalar fields? To satisfy the S-invariance, the ϕ in $V(\phi)$ must satisfy the constraints condition;

$$f^b_{ai}\phi_b - [\phi_a, \phi_i] = 0, \tag{5}$$

Where the index i runs only in the R-subalgebra of S. We can see from (5), some components of ϕ_a are zero, some are constants and the rest can be identified with components of the genuine Higgs fields. When the R can be embedded in G, we can find the genuine Higgs fields by group theory. Then we decompose the adjoint representations of S into the irreducible representation of R.

$$\mathrm{ad}S = \mathrm{ad}R + v, \tag{6}$$

Where $v = \Sigma S_k$, S_k is the irreducible representation of R. Again we give the decomposition of the adjoint representation of G into the irreducible representation of the $R_G \times H$:

$$\mathrm{ad}G = \sum(r_k, h_k), \tag{7}$$

where r_k and h_k are the irreducible representation of the R_G and H respectively. Then, there is h_k scalar multiplet for every pair (r_k, S_k) where r_k and S_k are identical representation.

To find the surviving fermions, we first embedded R in the Lorentz group $SO(N)$ of S/R, so that we can identify the decomposition rule of the vector representation N of $SO(N)$ with v (to see Eq.(6)). Then we take the spinor σ of $SO(N)$, and decompose it into the irreducible representation of R:

$$\sigma = \sum \sigma_j, \tag{8}$$

Again we decompose the representation F of G in which the fermions are assigned into the irreducible representation of $R \times H$

$$F = \sum(r_k, h_k), \tag{9}$$

therefore, it turns out that for every pair of (r_k, h_k), where r_k and σ_j are identical representation, there is an h_k multiplet of fermions (to see [5,6]).

3.MODEL

Considering the significance of ten dimensions in superstring, we present an $SO(17)$ gauge theory in ten dimensions, and we take $S/R = SU(3)/U(1) \times U(1)$ on coset space. According to previous section, the four-dimensional gauge group H is the centralizer of R in G. Therefore, we have to specify the embedding of $R = U(1) \times U(1)$ into $SO(17)$. Then we first give the decomposition of the gauge group $SO(17)$:

$$SO(17) \supset SO(10) \times SO(7)$$

Further we give the decomposition of $SO(10)$ and $SO(7)$:

$$SO(10) \supset SU(5) \times \tilde{U}(1)_{\mathrm{II}}$$
$$SO(7) \supset SU(4) \supset SU(3) \times \tilde{U}(1)_{\mathrm{I}}$$

On the whole we have

$$SO(17) \supset SU(5) \times SU(3) \times \tilde{U}(1)_{\mathrm{I}} \times \tilde{U}(1)_{\mathrm{II}}. \tag{10}$$

we identify the $\tilde{U}(1)_{\mathrm{I}} \times \tilde{U}(1)_{\mathrm{II}}$ as the image of R in G. Therefore the four-dimensions gauge group is $H = C_{SO(17)} = SO(5) \times SU(3)$. The $SU(5)$ may be considered as the gauge group of the ordinary Grand Unified Theory (GUT), which $SU(3)$ can play the role of horizontal symmetry which leads to three generation fermions.

In order to find the surviving scalar fields, we first give out the branching rule for the adjoint representation of $SO(17)$ under the $SU(5) \times SU(3) \times \tilde{U}(1)_{\mathrm{I}} \times \tilde{U}(1)_{\mathrm{II}}$:

$$\begin{aligned} 136 = {} & (1,10)(0,4) + (1,24)(0,0) + (1,1)(0,0) + (1,5)(0,2) \\ & + (3,5)(2,2) + (\bar{3},5)(-2,2) + (3,\bar{5})(2,-2) + (\bar{3},\bar{5})(-2,-2) \\ & + (1,\overline{10})(0,-4) + (1,\bar{5})(0,-2) + (8,1)(0,0) + (1,1)(0,0) \\ & + (3,1)(2,0) + (\bar{3},1)(-2,0) + (3,1)(-4,0) + (\bar{3},1)(4,0), \end{aligned} \tag{11}$$

The corresponding decomposition of the adjoint representation of $SU(3)$ under $R=\tilde{U}(1)_{\mathrm{I}}\times\tilde{U}(1)_{\mathrm{II}}$:

$$\begin{aligned}8 = &(0,0)+(0,0)+(1,3)+(-2,-2)\\ &+(-1,1)+(-1,-3)+(2,2)+(1,-1),\end{aligned} \tag{12}$$

Therefore according to the rules given in the previous section the surviving scalar fields in the four-dimensions theory transform under $H = SU(5)\times U(3)$ as a complex scalar fields $\beta = (3,5)$.

Now, let's find out the surviving fermionic fields. According to the previous section, we first set the fermions in the spinor representation F of $SO(17)$ in ten-dimensions. And also, in order to obtain the chiral fermions in the four-dimensions theory, we have to apply the Weyl condition and Majorana condition. We first give the decomposition of F into the irreducible representation of $SU(5)\times SU(3)\times\tilde{U}(1)_{\mathrm{I}}\times\tilde{U}(1)_{\mathrm{II}}$ and the branching rule:

$$\begin{aligned}256 = &(1,1)(3,-5)+(1,\bar{5})(3,3)+(1,10)(3,-1)+(3,1)(-1,-5)\\ &+(3,\bar{5})(-1,3)+(3,10)(-1,-1)+(1,1)(-3,-5)+(1,\bar{5})(-3,3)\\ &+(1,10)(-3,-1)+(\bar{3},1)(1,-5)+(\bar{3},\bar{5})(1,3)+(\bar{3},10)(1,-1)\\ &+(1,1)(3,5)+(1,5)(3,-3)+(1,\overline{10})(3,1)+(3,1)(-1,5)\\ &+(3,5)(-1,-3)+(3,\overline{10})(-1,1)+(1,1)(-3,5)+(1,5)(-3,-3)\\ &+(1,\overline{10})(-3,1)+(\bar{3},1)(1,5)+(\bar{3},5)(1,-3)+(\bar{3},\overline{10})(1,1),\end{aligned} \tag{13}$$

The corresponding branching rule of the $SO(6)$ spinor representation under $R = U(1)\times U(1)$:

$$4 = (0,0)+(1,3)+(2,-2)+(1,-1), \tag{14}$$

Comparing (13) with (14), we have the surviving fermions in four-dimensions, which transform as $(3,\bar{5})$ and $(\bar{3},10)$ under H.

So far we obtain an ordinary $SU(5)$ GUT model with three generation fermions and three complex scalar fields. How is the $SU(5)$ gauge group broken down to standard model? Because we do not a have Higgs multiplet which makes $SU(5)$ broken down to standard model here. However we can adopt the Wilson flux mechanism. In order to make wilson flux have a $SU(3)_c\times SU(2)_w\times U(1)_Y$ symmetry, U must be an element of the group generated by the hypercharge in $SO(5)$. The hypercharge in $SU(5)$ is

$$Y = \frac{2}{3}\begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix} \tag{15}$$

therefore it must take the form

$$U = \begin{pmatrix} \alpha & & & & \\ & \alpha & & & \\ & & \alpha & & \\ & & & \beta & \\ & & & & \beta \end{pmatrix} \tag{16}$$

Table 1

The commutation relations between the $SO(17)$ generators concerned with the Higgs potential.

$[Q_{ai}, Q_0] = -\frac{1}{\sqrt{10}} Q_{ai}$,	$[Q_{ai}, Q^{bj}] = -\delta^j_i \left(\frac{1}{2} \Lambda^a\right)^b_a Q_a$
$[Q_{ai}, Q'_0] = -\frac{1}{\sqrt{6}} Q_{ai}$,	$-\delta^b_a \left(\frac{1}{2} \lambda^\rho\right)^i_j Q_\rho + \frac{1}{\sqrt{10}} \delta^j_i \delta^b_a Q_0 + \frac{1}{\sqrt{6}} \delta^b_a \delta^j_i Q'_0$
$[Q^a_i, Q_0] = \frac{1}{\sqrt{10}} Q^a_i$,	$[Q^j_a, Q^b_i] = -\delta^j_i \left(\frac{1}{2} \Lambda^a\right)^b_a Q_a$
$[Q^a_i, Q'_0] = -\frac{1}{\sqrt{6}} Q^a_i$,	$+\delta^b_a \left(\frac{1}{2} \lambda^\rho\right)^i_j Q_\rho + \frac{1}{\sqrt{10}} \delta^i_j \delta^b_a Q_0 - \frac{1}{\sqrt{6}} \delta^b_a \delta^i_j Q'_0$

where $\alpha^3\beta^2 = 1$, $\alpha^2 = \beta^2 = 1$. If we replace the $SU(3)/U(1)\times U(1)$ by the non-simply connected coset space $SU(3)/U(1)\times U(1)\times \mathbf{Z}_2$, U will has the form of (16). Hence $SU(5)$ can be broken down to the standard model $SU(3)_c\times SU(2)_w\times U(1)_y$ by wilson flux.

Finality, we will study the mechanism of breaking the standard model $SU(3)_c\times SU(2)_w\times U(1)_y$ down to $SU(3)_c\times U(1)_{em}$. In that case we shall discuss the Higgs potential in detail. Formally, the Higgs potential takes the form (3). To express the formal Higgs potential of Eq.(3) in terms of the genuine Higgs field, we need the commutation relations between $SO(17)$ generators and $SU(3)$ generators. We first introduce the $SO(17)$ generators corresponding to the decomposition (11):

$$\{Q^\lambda_{SO(17)}\} = \{Q_0, Q_a, Q^a, Q_{ab}, Q^{ab}, Q^\alpha, Q_{ia}, Q^a_i, Q^i_a, Q^{ia}, Q'_0, Q_i, Q^i, \tilde{Q}_i, \tilde{Q}^i, Q^\rho\},$$

where a, b = 1, 2, ...5, α = 1, 2, ..., 24, and ρ = 1, 2, ..., 8. Their commutation relations concerned with the Higgs potential are listed in Table 1.

Where Λ^α are twenty four 5×5 matrices of $SU(5)$, λ^ρ are eight 3×3 Gell-mann matrices, the normalization conditions are $\mathrm{Tr}(\Lambda^\alpha\Lambda^\beta) = 2\delta^{\alpha\beta}$ and $\mathrm{Tr}(\lambda^\rho\lambda^\sigma) = 2\delta^{\rho\sigma}$.

Similarly, we introduce the $SU(3)$ generators corresponding to the decomposition of Eq. (12)

$$\{T^\beta_{SU(3)}\} = \{T_a, T'_a, T_1, T_2, T_3, T^1, T^2, T^3\},$$

their non-zero commutation relations are listed in the Table 2.

In order to express the formal Higgs potential of (3) in terms of the genuine Higgs fields, we denote the scalar fields as:

$$\phi_s = \{\phi_0, \phi'_0, \phi_1, \phi_2, \phi_3, \phi^1, \phi^2, \phi^3\}, \tag{17}$$

Therefore the solutions of the constraint conditions (5) are

$$\begin{aligned} \phi_0 &= \frac{\sqrt{10}}{2}\left(1 - \frac{5}{\sqrt{3}}\right) Q_0, \\ \phi'_0 &= -\frac{\sqrt{6}}{2}(1+\sqrt{3})Q'_0, \\ \phi_1 &= \beta_{ia}Q^{ia}, \quad \phi^1 = \beta^{ia}Q_{ia}, \end{aligned} \tag{18}$$

and all others vanish.

Table 2

The non-zero commutation relations between the $SU(3)$ generators corresponding to the decomposition Eq.(12).

$[T_1, T_0] = -\frac{1}{2}\left(\frac{5}{3}\sqrt{3} - 1\right)T_1,$	$[T_1, T^1] = \frac{3}{4}(\sqrt{3} - 1)T_0 + \frac{1}{4}(3\sqrt{3} + 5)T'_0,$
$[T_1, T'_0] = -\frac{1}{2}(\sqrt{3} + 1)T_1,$	$[T_2, T^2] = -\frac{3}{4}(\sqrt{3} + 1)\,T_0 - \frac{1}{4}(3\sqrt{3} - 5)T'_0,$
$[T_2, T_0] = \frac{1}{2}\left(\frac{5}{3}\sqrt{3} + 1\right)T_2,$	$[T_3, T^3] = -\frac{1}{2}(3T_0 - 5T'_0),$
$[T_2, T'_0] = -\frac{1}{2}(-\sqrt{3} + 1)T_2,$	$[T_1, T_2] = \sqrt{2}T_3,$
$[T_3, T_0] = T_3,$	$[T_2, T^3] = \sqrt{2}T^1,$
$[T_3, T'_0] = -T_3,$	$[T^1, T_3] = \sqrt{2}T_2,$

In general, for the traces of products of generators appearing in Eq. (3), we can take

$$\mathrm{Tr}\{Q_0^2\} = \mathrm{Tr}\{Q_0'^2\} = \mathrm{Tr}\{Q_{ia}Q^{ia}\} = 2.$$

In order to determine the matrix $h = \| g^{ab} \|$ in the Higgs potential, we assume that h is $\tilde{U}(1)_{\mathrm{I}} \times \tilde{U}(1)_{\mathrm{II}}$ invariant, then we have

$$g_{aa} = g^{aa} = \frac{1}{R_a^2}, \quad g_{ab} = g^{ab} = 0, \quad a \neq b, a, b = 1, 2, 3,$$

The radius parameters $R_{(1)}$, $R_{(2)}$ and $R_{(3)}$ correspond to the multiplet ϕ_1, ϕ_2, ϕ_3 in Eq. (17) respectively. Because the metric is real, the radius parameter $R_{(a)}$ corresponding to ϕ^a is equal to $R_{(a)}$.

Combining the above result, we obtain:

$$V_{4-\mathrm{dim}}(\phi) = \frac{117 - 24\sqrt{3}}{g^2 R_{(1)}^4} + \frac{27 + 6\sqrt{3}}{g^2 R_{(2)}^4} + \frac{90}{g^2 R_{(3)}^4} + \frac{4}{g^2}\left(\frac{1}{R_{(2)}^2 R_{(3)}^2} - \frac{2}{R_{(1)}^4}\right)(\beta_{ia}\beta^{ia}) + \frac{1}{g^2 R_{(1)}^4}(\beta_{ia}\beta_{ia})^2, \tag{19}$$

where g is a coupling constant in ten-dimensions. We see from (19) that the electroweak symmetry breaking will occur in a wide range of radii. The electroweak symmetry breaking scale requires that $R_{(a)}$ must be Fermi scale ($\sim 10^{-15}$cm). But, in dimensional reduction theory, this compactification scale will be of the order of the Planck scale ($\sim 10^{-33}$cm). To be consistent with the standard model, we introduce the relation $R_{(1)}^2 = \sqrt{2}R_{(2)}R_{(3)}$ between the radii, so that classical vacuum expectation value is equal to zero. And spontaneous symmetry breaking of the electroweak sector of the theory can be achieved by high order radiative corrections.

4. CONCLUSION

We have proposed an $SO(17)$ unified model in ten-dimensional space-time. We obtained the four-dimensional space-time $SU(5) \times SU(3)$ unified theory with three generation fermions by the coset space dimensional reduction. As a four-dimensions grand unified gauge group, the $SU(5)$ is broken by the wilson flux mechanism to $SU(3)_c \times SU(2)_w \times U(1)_y$ standard model. Finally the standard model is broken to $SU(3) \times U(1)_{em}$ by the Higgs fields of geometrical origin (via the coset space dimensional reduction

procedure). To obtain the vacuum expectation value with the order of Fermi scale, we assume a relation $R_{(1)}^2=\sqrt{2}R_{(2)}R_{(3)}$. There is still a question in this model, that is, how can we make the proton lifetime consistent with experimental results?

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