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A Solvable 1+1 Dimensional U(1) gauge model

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A 1+1 dimensional $U(1)$ gauge model is proposed and the spectrum with the energy eigenstates represented in terms of fermion operators is exactly solved.

1. INTRODUCTION

Up to now, only a few gauge models are solvable except for the Schwinger model[1]. Although the spectrum of the Schwinger model is solvable with different methods, no one can find the energy eigenstates represented in fermion operators[2]. Our understanding about the structure of the gauge theory is still very limited.

Recently the authors have found some exact solutions for the 1+1 dimensional lattice gauge theory in which the energy eigenstates represented in fermion operators were given[3]. These results give us more information about the gauge theory. In this paper our work is generalized to continuum space. A 1+1 dimensional $U(1)$ gauge model is proposed and the energy eigenstates represented in fermion operators are exactly solved in the Hamiltonian formalism.

2. THE MODEL AND THE GROUND STATE

We adopt the Hamiltonian formalism and discuss the $U(1)$ gauge group only. For the gauge fields, we always choose the temporal gauge

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$$A_0 = 0. \tag{2.1}$$

Let the Hamiltonian which describes a non-relativistic 1+1 dimensional $U(1)$ gauge model be

$$H = \frac{1}{2}\int dx E(x)^2 - :F\int dx\bar{\psi}(x)(\partial_x + ieA(x))^2\psi(x):, \tag{2.2}$$

where e is a real coupling constant, F a real positive parameter, $A(x)$ the space component of the gauge field and $E(x)$ the electric field, and $A(x)$ and $E(x)$ satisfy the commutation relation

$$[A(x), E(x')] = -i\delta(x - x'), \tag{2.3}$$

$\psi(x)$ and $\psi^+(x)$ are the fermion fields with two components,

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta^+(x) \end{pmatrix}, \quad \psi^+(x) = (\xi^+(x)\,\eta(x)), \tag{2.4}$$

$\psi(x)$ and $\psi^+(x)$ satisfy

$$\{\xi^+(x), \xi(x')\} = \delta(x - x'), \quad \{\eta^+(x), \eta(x')\} = \delta(x - x'), \tag{2.5}$$

γ_0 is taken as

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.6}$$

and the normal ordering acts only on the fermion creation and destruction operators defined by (2.4). In fact, H can be rewritten as

$$H = \frac{1}{2}\int dx E(x)^2$$
$$- F\int dx[\xi^+(x)(\partial_x + ieA(x))^2\xi(x) + ((\partial_x + ieA(x))^2\eta^+(x))\eta(x)]. \tag{2.7}$$

It is easy to determine that H keeps invariant under the following transformations:

(1) Local $U(1)$ gauge transformation in spatial direction

$$A(x) \to A(x) - \partial_x\theta(x), \quad \psi(x) \to e^{ie\theta(x)}\psi(x). \tag{2.8}$$

(2) Space reflection

$$x \to -x, \quad A(x) \to -A(x), \quad \psi(x) \to \psi(x). \tag{2.9}$$

where we must note that $\psi(x)$ is not transformed as the usual case $\psi(x) \to \gamma_0\psi(x)$.

(3) Global γ_0 transformation

$$\psi(x) \rightarrow e^{i\alpha\gamma_0}\psi(x), \tag{2.10}$$

where α is a real constant.

Now we define the state $|0>$ as

$$E(x)|0\rangle = 0, \ \xi(x)|0\rangle = 0, \ \eta(x)|0\rangle = 0. \tag{2.11}$$

Obviously

$$H|0\rangle = 0, \tag{2.12}$$

$|0>$ is an eigenstate of H with zero energy. Neglecting the surface term at infinity, we can deduce

$$H = \frac{1}{2}\int dx E(x)^2 + F\int dx((\partial_x - ieA(x))\xi^+(x))(\partial_x + ieA(x))\xi(x)$$
$$+ F\int dx((\partial_x + ieA(x))\eta^+(x))(\partial_x - ieA(x))\eta(x). \tag{2.13}$$

Since

$$E(x)^+ = E(x),$$
$$((\partial_x + ieA(x))\xi(x))^+ = (\partial_x - ieA(x))\xi^+(x),$$
$$((\partial_x - ieA(x))\eta(x))^+ = (\partial_x + ieA(x))\eta^+(x), \tag{2.14}$$

H is positive definite. Therefore $|0>$ is an exact ground state of H.

3. THE SPECTRUM

Because of the symmetry of H, we need only consider the state with a pair of fermion and anti-fermion. In general, a gauge invariant and translational invariant state can be written as

$$|E\rangle = \int dx dx' f_E(x - x')\xi^+(x)e^{ie\int_x^{x'} A(t)dt}\eta^+(x')|0\rangle. \tag{3.1}$$

Demanding that $|E>$ satisfy the eigenvalue equation of H

$$H|E\rangle = E|E\rangle, \tag{3.2}$$

we can easily derive the equation about $f_E(x)$

$$\frac{1}{2}e^2|x|f_E(x) - 2F\partial_x^2 f_E(x) = Ef_E(x). \tag{3.3}$$

It can be seen from (3.3), that the fermion and anti-fermion interact with the linear potential. Using the condition that $f_E(x)$ must be finite at infinity, we can obtain

$$f_E(x) = \begin{cases} \dfrac{c^{(+)}}{\sqrt{\pi}} \displaystyle\int_0^\infty dp \cos\left(\frac{4F}{3e^2} p^3 + \left(x - \frac{2E}{e^2}\right)p\right) & (x > 0) \\ \dfrac{c^{(-)}}{\sqrt{\pi}} \displaystyle\int_0^\infty dp \cos\left(\frac{4F}{3e^2} p^3 + \left(-x - \frac{2E}{e^2}\right)p\right) & (x < 0), \end{cases} \tag{3.4}$$

where $c^{(\pm)}$ are constants. According to the definition of Airy function[4]

$$\phi(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty dp \cos\left(\frac{1}{3} p^3 + px\right), \tag{3.5}$$

$f_E(x)$ can be represented as

$$f_E(x) = \begin{cases} c^{(+)}\phi\left(\left(x - \frac{2E}{e^2}\right) \Big/ \left(\frac{4F}{e^2}\right)^{1/3}\right) & (x > 0) \\ c^{(-)}\phi\left(\left(-x - \frac{2E}{e^2}\right) \Big/ \left(\frac{4F}{e^2}\right)^{1/3}\right) & (x < 0). \end{cases} \tag{3.6}$$

Taking into account the continuity of $f_E(x)$ and $df_E(x)/dX$ at $x = 0$, for the state with even parity,

$$c^{(+)} = c^{(-)}, \tag{3.7}$$

the energy E is determined by the equation

$$\phi'\left(-\left(\frac{2E}{e^2}\right) \Big/ \left(\frac{4F}{e^2}\right)^{1/3}\right) = 0 \tag{3.8}$$

For the state with odd parity,

$$c^{(+)} = -c^{(-)}, \tag{3.9}$$

the energy E is determined by the equation

$$\phi\left(-\left(\frac{2E}{e^2}\right) \Big/ \left(\frac{4F}{e^2}\right)^{1/3}\right) = 0 \tag{3.10}$$

The first excited energy E_1 and the second excited energy E_2 given by (3.8) and (3.10) respectively are

$$E_1 = 0.51 e^2 \left(\frac{4F}{e^2}\right)^{1/3},$$
$$E_2 = 1.17 e^2 \left(\frac{4F}{e^2}\right)^{1/3}. \tag{3.11}$$

When $x \to +\infty$, the asymptotic behavior of Airy function is

$$\phi(x) \sim \frac{1}{2} x^{-1/4} e^{-\frac{2}{3} x^{3/2}}, \tag{3.12}$$

Therefore, from (3.6) we can see that the fermion and anti-fermion are confined in our model.

4. RESULTS AND DISCUSSIONS

(1) A 1+1 dimensional $U(1)$ gauge model is proposed in this paper. There exist fermions and anti-fermions even though the model is non-relativistic. A similar concept is widely used in solid state physics.

(2) The spectrum with the energy eigenstates represented in fermion operators is exactly solved. The results show that the fermions are confined by the linear potential.

(3) The model is still solvable even when the mass term of fermions: $m\int dx \psi^+(x)\, \gamma_0 \psi(x)$: is introduced. In fact, we just need to let $E \to E + 2m$ in the massive case.

(4) The significance of our results is that we can solve not only the spectrum, but also the energy eigenstates represented in fermion operators. This will be helpful for the further study of the gauge theory, including the lattice gauge theory, and even solid state physics.

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