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Derivation of Chiral Gauge Anomalies in Canonical Formalism

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Through a covariant regularization of the chiral fermion current, we derive the covariant fermion current divergence anomaly as well as the covariant Gauss law commutator anomaly of the chiral gauge theory in the canonical formalism.

Anomaly is such that the classical symmetry of a theory is not preserved after quantization. The chiral gauge anomaly is referred to the nonzero divergence of the chiral fermion current. L. Faddeev recently found that [1] taking the gauge theory as the projective representation of the gauge group, the fermion current divergence anomaly corresponds to the 1-cocycle of the gauge group, He pointed out that the 2-cocycle of the gauge group also related to the gauge anomaly, in the infinitesimal form, the anomaly is the nontrivial center extension of the Gauss law operator algebra of the gauge theory. The authors of Refs. [2,3] confirmed Faddeev's idea by calculating the Schwinger term of the gauge algebra through the perturbative BJL method. The result they got satisfies the 2-cocycle condition, and is called the consistent anomaly. The definition is in agreement with the consistent current divergence anomaly.

At first look, it seems that the most direct derivation of Gauss law anomaly is calculating the equal time commutators. Until now, however, no one has arrived at Faddeev's result with a fixed time approach [4]. Besides, the covariant form of anomalies is undoubtedly important for the anomaly study. In fact, these two problems are both related to the regularization. In this short article I first introduce the residual regularization for the bilinear fermion fields operators, then derive the covariant fermion current divergence anomaly and Gauss law commutator anomaly of the chiral gauge theory through the fixed time method in the canonical quantization procedure, and finally discuss the obtained result.

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This work is done in 1 + 3 dimension. Taking Weyl gauge, where $A_0^a = 0$, the system in which the chiral fermion,

$$\psi_L(x) = \frac{1+\gamma_5}{2}\psi(x),$$

along with the gauge field can be described by the following Hamiltonian

$$H = \int d^3x(\mathscr{H}_{\text{Fermi}} + \mathscr{H}_A) \tag{1a}$$

$$\mathscr{H}_{\text{Fermi}} = \psi_L^+(x)\gamma_4\gamma_i(\partial_i + A_i)\psi_L(x) \tag{1b}$$

$$\mathscr{H}_A = \frac{1}{2}E_i^a(x)E_i^a(x) + \frac{1}{4}F_{ij}^a(x)F_{ij}^a(x) \tag{1c}$$

The canonical quantization requires the basic equal time commutators

$$[E_i^a(x), A_j^b(y)] = i\delta_{ij}\delta^{ab}\delta^3(x-y),$$
$$\{\psi^\alpha(x), \psi^{\beta+}(y)\} = \delta^{\alpha\beta}\delta^3(x-y), \text{ others} = 0. \tag{2}$$

Define the classical current, $j_\mu^a(x) = \psi_L^+(x)\gamma^4\gamma_\mu T^a\psi_L(x)$, here T^a are the generators of the gauge group, and $T^{a+} = -T^a$. It is well known that the products of the fermion fields defined at the same time-space point will lead to singularity after quantization. The cure usually is the regularization. Now we define the fermion current and fermion Hamiltonian operators through the residual regularization

$$j_\mu^a(x) \to \hat{j}_\mu^a(x) = \oint_c \frac{dz}{2\pi i}\bar{\psi}_L(x)\gamma_\mu T^a \frac{1}{z-D^2}\psi_L(x), \tag{3}$$

$$\mathscr{H}_{\text{Fermi}}(x) \to \hat{\mathscr{H}}_{\text{Fermi}}(x) = \oint_c \frac{dz}{2\pi i}\bar{\psi}_L(x)\gamma_i D_i \frac{1}{z-D^2}\psi_L(x), \tag{4}$$

where the Hermitian operator $D = i\gamma_i D_i = i\gamma_i(\partial_i + A_i)$ and the integral path on the complex plane z, C, encloses all the poles of the regulator $1/(z-D^2)$. It is obvious that the regularization is manifestly covariant.

Expanding the fermion field $\psi(x)$ in the eigenstates of the fermion energy operator $H = \gamma_4\gamma_i D_i$,

$$\psi(x) = \sum_{E_n>0}\alpha_n\varphi_n + \sum_{E_n<0}\beta_n^+\varphi_n,$$

we have $\{\alpha_n, \alpha_m^+\} = \{\beta_n, \beta_m^+\} = \delta_{n,m}$; Correspondingly, define the fermion vacuum state under the quantized background configuration $A(x)$ as $|\ \rangle_A$ So that $\alpha_n|\ \rangle_A = \beta_n|\ \rangle_A = 0$.

Consider the Hamiltonian equation of the fermion charge

$$\dot{\hat{j}}_0^a(x) = -i[\hat{j}_0^a(x), \hat{H}_{\text{Fermi}} + \hat{H}_A]. \tag{5}$$

Using the commutator relations (2), it is easy to show the first term of the above commutator

$$i[\hat{j}_0^a(x),\hat{H}_{\rm Fermi}] = (D_i\hat{j}_i(x))^a. \tag{6}$$

On the other hand, noticing there is the electronic field density $E_i^a(x)$ (could be written as $i^\delta/\,\delta A_i^a(x)$) in $\hat{H}_A$ and the regularized charge density dependence on the gauge potential $A_i^a(x)$, it is possible that the second commutator in (5) has nonzero contribution, which will give the current divergence anomaly. Now, let's calculate the vacuum expectation of the commutator between $E_j^b(x)$ and $\hat{j}_0^a(x)$.

$$
\begin{aligned}
{}_A\langle i[j_0^a(x),E_j^b(y)]\rangle_A &= \left\langle -i\frac{\delta}{\delta A_j^b(y)}\oint_c\frac{dz}{2\pi i}\psi_L^\dagger(x)T^a\frac{1}{z-D^2}\psi_L(x)\right\rangle_A \\
&= -i\lim_{h\to x}\oint\frac{dz}{2\pi i}\,\mathrm{Tr}\frac{1+\gamma^5}{2}T^a\frac{\delta}{\delta A_j^b(y)}\frac{1}{z-D_x^2}P_-(x,h),
\end{aligned}
\tag{7}
$$

here Tr denotes the trace over Dirac matrix and the gauge index, and $P_-(x,y)$ is the negative energy project operator

$$P_-(x,y) = \sum_{E_n<0}\varphi_n(x)\varphi_n^\dagger(y) = \int\frac{d^3p}{(2\pi)^3}P_-(p)e^{-ip\cdot(x-y)}, \tag{8}$$

Its Fourier transformation $P_-(p)$ could be perturbatively expanded as

$$P_-(p) = \sum_{i=0}^{\infty}P_-^{(i)}(p) \tag{9a}$$

$$P_-^{(0)}(p) = \frac{1}{2}\left(1+\frac{i\gamma\cdot p\gamma^4}{|p|}\right),\ (\gamma\cdot p = \gamma^i p^i), \tag{9b}$$

$$
\begin{aligned}
P_-^{(1)}(p) = \int\frac{d^3\xi}{(2\pi)^3}e^{-i\xi y}\tilde{A}_k(\xi)&\frac{1}{2(|p|+|p+\xi|)} \\
&\cdot\left(\gamma^k\gamma^4-\frac{p\cdot\gamma\gamma^k(p+\xi)\cdot\gamma\gamma^4}{|p||p+\xi|}\right),
\end{aligned}
\tag{9c}
$$

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Substituting (8) and (9) into (7), moving the factor $e^{-ip\cdot(x-h)}$ to the left of the operator $\frac{1}{z-D_x^2}$, then taking the limit $h\to x$, it leaves an infinite summation in (7)

$$\lim_{h\to x}\frac{1}{z-D_x^2}e^{-ip\cdot(x-h)} = \sum_{n=0}^{\infty}\frac{(D'^2)^n}{(z-p^2)^{n+1}} \tag{10a}$$

$$(D')^2 = 2p_iD_i - D_i^2 - \frac{1}{2}\gamma_i\gamma_jF_{ij}. \tag{10b}$$

Due to the actions of Tr and $\oint_c\frac{dz}{2\pi i}$ in (7), only the first three terms in the infinite summation (10a) have nonzero contributions to (7). After a few calculations, we arrive at

$${}_A\langle i[j_0^a(x),E_j^b(y)]\rangle_A = \frac{1}{16\pi^2}\varepsilon^{j4lm}\mathrm{Tr}F_{lm}\{T^a,T^b\}\delta^3(x-y) \tag{11}$$

From (11), (6) and (5), we get the current divergence anomaly

$$\langle (D_\mu \hat{j}_\mu)^a \rangle_A = -\frac{1}{16\pi^2} \varepsilon^{i0jk} \mathrm{Tr} T^a \{ E_i(x), F_{jk}(x) \}. \tag{12}$$

Now we turn to the Gauss law commutator anomaly. The Gauss law is $G^a(x) = 0$, which is the first class constraint, where the operators $G^a(x)$ satisfy the gauge algebra

$$[G^a(x), G^b(y)] = if^{abc} G^c(x) \delta^3(x-y), \tag{13}$$

are the generators of the gauge group: $G^a(x) = -(D \cdot E(x))^a + \hat{j}_0^a(x)$. After quantization, it is possible that (13) has a center extension denoted by $W^{ab}(x,y)$. Let's calculate the vacuum expectation $\langle W^{ab}(x,y) \rangle_A$, which is the Gauss law commutator anomaly.

From (3), (9) and (10), we get

$$\langle [\hat{j}_0^a(x), \hat{j}_0^b(y)] \rangle_A - if^{abc} \langle \hat{j}_0^c(x) \rangle_A \delta(x-y)$$
$$= -\frac{1}{16\pi^2} \varepsilon^{i0jk} \mathrm{Tr} F_{jk} ((\{T^a, T^b\} \partial_i) + \{T^a, [A_k, T^b]\}) \delta^3(x-y). \tag{14}$$

we also have

$$[(D_i \cdot E_i)^a(x), (D_j E_j)^b(y)] - if^{abc} (D_i \cdot E_i)^c(x) \delta^3(x-y) = 0 \tag{15}$$

It means that there is no anomaly in the algebra of the gauge transformation generators of the gauge fields. And from (11) we find

$$_A\langle [(D_i E_i)^a(x), \hat{j}_0^b(y)] \rangle_A = {}_A\langle D_i^{aa'} [E_i^{a'}(x), \hat{j}_0^b(y)] \rangle_A$$
$$= -\frac{1}{16\pi^2} \varepsilon^{i0jk} \mathrm{Tr} F_{jk} (\{T^a, T^b\} \partial_i + \{[T^a, A_i], T^b\}) \delta^3(x-y). \tag{16}$$

Putting together (14)–(16), in a compact form, we can write the covariant Gauss law commutator anomaly as

$$_A\left\langle \int d^3x d^3y u^a(x) v^b(y) ([G^a(x), G^b(y)] - if^{abc} G^c(x) \delta(x-y)) \right\rangle_A$$
$$= \frac{1}{16\pi^2} \varepsilon^{i0jk} \int d^3x \mathrm{Tr} F_{jk} (u D_i v + D_i v u). \tag{17}$$

Finally, we have two discussions. 1. A comparison with the consistent anomaly. It is clear in the case of the current divergence anomaly that the covariant form and the consistent form are only different by a redefinition of the fermion current. When we take another regular operator in (3), we will come to the consistent form of the fermion current divergence. For the Gauss law anomaly, however, there is no relation between the two forms established. This is because the covariant form is derived in the paper by the canonical formalism and the consistent form is given by the BJL approach, which is a non-canonical formalism and surrounds non-zero commutators between different components of electronic fields, and we don't know how a canonical formalism can go to a non-canonical formalism. 2. There might be a way to analyze the covariant Gauss law commutator anomaly in the language of the differential geometry. Faddeev's 2-cocycle, identified as the consistent Gauss law anomaly, is derived from a 6-form by descending it to a 3-form (in the $D = 4$ space-time), and the 4-form in the series of forms, i.e. 1-cocycle, is the consistent anomaly. Ref.[3] suggested a similarity of the derivation in the covariant case and gave the covariant current divergence anomaly. The series also gives the covariant Gauss law anomaly. We will discuss the issue in detail elsewhere[7].

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